MURRAY & DERMOTT,
« Solar System Dynamics »

7.4 Free and Forced Elements

We have shown that under certain conditions we can construct a secular solution to the motion of two orbiting bodies moving under their mutual gravitational effects; at any time we can obtain the eccentricities, longitudes of pericentre, inclinations, and longitudes of ascending node of both bodies. We can make use of this solution to study the motion of an additional body, of negligible mass, moving under the influence of the central body and perturbed by the other two bodies.

Following the example given in Sect. 7.2 for the secular theory for two bodies, the disturbing function $\mathcal{R}$ for a test particle with orbital elements a, n, e, I, ϖ, and Ω is given by

ho semplificato questa parte assumendo che $a < a_j$ $(j=1,2)$; vedi pagina aggiunta

$$\mathcal{R} = na^2\left[\frac{1}{2}Ae^2 + \frac{1}{2}BI^2 + \sum_{j=1}^{2} A_j e e_j \cos(\varpi - \varpi_j) + \sum_{j=1}^{2} B_j I I_j \cos(\Omega - \Omega_j)\right], \quad (7.54)$$

where

$$A = +n\frac{1}{4}\sum_{j=1}^{2}\frac{m_j}{m_c}\alpha_j\bar{\alpha}_j b_{3/2}^{(1)}(\alpha_j), \quad (7.55)$$

$$A_j = -n\frac{1}{4}\frac{m_j}{m_c}\alpha_j\bar{\alpha}_j b_{3/2}^{(2)}(\alpha_j), \quad (7.56)$$

$$B = -n\frac{1}{4}\sum_{j=1}^{2}\frac{m_j}{m_c}\alpha_j\bar{\alpha}_j b_{3/2}^{(1)}(\alpha_j), \quad (7.57)$$

$$B_j = +n\frac{1}{4}\frac{m_j}{m_c}\alpha_j\bar{\alpha}_j b_{3/2}^{(1)}(\alpha_j) \quad (7.58)$$

ho semplificato questa parte assumendo che $a < a_j$ $(j=1,2)$; vedi pagina aggiunta

and

$$\alpha_j = \begin{cases} a_j/a & \text{if} \quad a_j < a, \\ a/a_j & \text{if} \quad a_j > a, \end{cases} \tag{7.59}$$

$$\bar{\alpha}_j = \begin{cases} 1 & \text{if} \quad a_j < a, \\ a/a_j & \text{if} \quad a_j > a. \end{cases} \tag{7.60}$$

If we now transform to a new set of variables h, k, p, and q for the test particle and h_j, k_j, p_j, and q_j $(j = 1, 2)$ for each perturbing body, where

$$h = e \sin \varpi, \qquad k = e \cos \varpi \tag{7.61}$$

and

$$p = I \sin \Omega, \qquad q = I \cos \Omega \tag{7.62}$$

and the other elements are already defined in Eqs. (7.18) and (7.19), we have

$$\mathcal{R} = na^2 \left[\frac{1}{2} A (h^2 + k^2) + \frac{1}{2} B (p^2 + q^2) + \sum_{j=1}^{2} A_j (h h_j + k k_j) + \sum_{j=1}^{2} B_j (p p_j + q q_j) \right]. \tag{7.63}$$

l'ho chiamata $R^{(sec)}$

The equations of motion are

$$\dot{h} = + \frac{1}{na^2} \frac{\partial \mathcal{R}}{\partial k}, \qquad \dot{k} = - \frac{1}{na^2} \frac{\partial \mathcal{R}}{\partial h}, \tag{7.64}$$

$$\dot{p} = + \frac{1}{na^2} \frac{\partial \mathcal{R}}{\partial q}, \qquad \dot{q}_j = - \frac{1}{na^2} \frac{\partial \mathcal{R}}{\partial p}. \tag{7.65}$$

Substituting for $\mathcal{R}$ from Eq. (7.63) we can write the equations of motion as

$$\dot{h} = +Ak + \sum_{j=1}^{2} A_j k_j, \qquad \dot{k} = -Ah - \sum_{j=1}^{2} A_j h_j, \tag{7.66}$$

$$\dot{p} = +Bq + \sum_{j=1}^{2} B_j q_j, \qquad \dot{q} = -Bp - \sum_{j=1}^{2} B_j p_j, \tag{7.67}$$

where the values of h_j, k_j, p_j, and q_j are derived from the secular solution given in Eqs. (7.28) and (7.29). Substituting from these equations we get

$$\dot{h} = +Ak + \sum_{j=1}^{2} A_j \sum_{i=1}^{2} e_{ji} \cos(g_i t + \beta_i), \tag{7.68}$$

$$\dot{k} = -Ah - \sum_{j=1}^{2} A_j \sum_{i=1}^{2} e_{ji} \sin(g_i t + \beta_i), \tag{7.69}$$

asteroide della fascia principale ($a \in [\approx 2.06\ \mathrm{au}, \approx 3.28\ \mathrm{au}]$)

perturbato da Giove ($\approx 5.2\ \mathrm{au}$) e Saturno ($\approx 9.58\ \mathrm{au}$)

Funzione perturbatrice

$$R = \sum_{i=1}^{2} Gm_i \left(\frac{1}{|r_i - r|} - \frac{r \cdot r_i}{|r_i|^3} \right)$$

Funzione perturbatrice secolare (al secondo ordine in e, e_i, s, s_i, $i = 1, 2$)

$$R^{(sec)} = \sum_{i=1}^{2} \frac{Gm_i}{a_i} \left(\frac{1}{8} \alpha_i b_{3/2}^{(1)} (e^2 + e_i^2) - \frac{1}{2} \alpha_i b_{3/2}^{(1)} (s^2 + s_i^2) - \right.$$

$$\left. - \frac{1}{4} \alpha_i b_{3/2}^{(2)} e e_i \cos(\varpi - \omega_i) + \alpha_i b_{3/2}^{(2)} s s_i \cos(\Omega - \Omega_i) \right)$$

Introduco $\tilde{R}$

$$R^{(sec)} = n a^2 \tilde{R}$$

e gli elementi equinottali h, k, p, q; si ha

$$e^2 + e_i^2 = (h^2 + k^2) + (h_i^2 + k_i^2)$$

$$s_i^2 = \frac{p_i^2 + q_i^2}{1 + p_i^2 + q_i^2} = (p_i^2 + q_i^2)(1 + \mathcal{O}(p_i^2 + q_i^2)) \approx p_i^2 + q_i^2$$

$$e e_i \cos(\varpi - \varpi_i) = h h_i + k k_i$$

$$s s_i \cos(\Omega - \Omega_i) = p p_i + q q_i$$

Risulta (per opportuni coefficienti A, B, A_i, B_i, C_i, D_i)

$$\tilde{R} = \frac{1}{2} A (h^2 + k^2) + \frac{1}{2} B (p^2 + q^2) + \sum_{i=1}^{2} \Big(A_i (h h_i + k k_i) + B_i (p p_i + q q_i)$$

$$+ C_i (h_i^2 + k_i^2) + D_i (p_i^2 + q_i^2) \Big)$$

possono essere trascurati perché $\tilde{R}$ è derivata rispetto agli elementi orbitali dell'asteroide

$$\dot{p} = +Bq + \sum_{j=1}^{2} B_j \sum_{i=1}^{2} I_{ji} \cos(f_i t + \gamma_i), \tag{7.70}$$

$$\dot{q} = -Bp - \sum_{j=1}^{2} B_j \sum_{i=1}^{2} I_{ji} \sin(f_i t + \gamma_i). \tag{7.71}$$

By taking another time derivative of each equation and using Eq. (7.26) again we have

$$\ddot{h} = -A^2 h - \sum_{i=1}^{2} \nu_i (A + g_i) \sin(g_i t + \beta_i), \tag{7.72}$$

$$\ddot{k} = -A^2 k - \sum_{i=1}^{2} \nu_i (A + g_i) \cos(g_i t + \beta_i), \tag{7.73}$$

$$\ddot{p} = -B^2 p - \sum_{i=1}^{2} \mu_i (B + f_i) \sin(f_i t + \gamma_i), \tag{7.74}$$

$$\ddot{q} = -B^2 q - \sum_{i=1}^{2} \mu_i (B + f_i) \cos(f_i t + \gamma_i), \tag{7.75}$$

where

$$\nu_i = \sum_{j=1}^{2} A_j e_{ji} \qquad \text{and} \qquad \mu_i = \sum_{j=1}^{2} B_j I_{ji}. \tag{7.76}$$

The solutions to the uncoupled differential equations in Eqs. (7.72)–(7.75) are

$$\begin{aligned} h &= e_{\text{free}} \sin(At + \beta) + h_0(t), \qquad & k &= e_{\text{free}} \cos(At + \beta) + k_0(t), \\ p &= I_{\text{free}} \sin(Bt + \gamma) + p_0(t), \qquad & q &= I_{\text{free}} \cos(Bt + \gamma) + q_0(t), \end{aligned} \tag{7.77}$$

where e_{free}, I_{free}, β, and γ are constants determined from the boundary conditions and

$$h_0(t) = -\sum_{i=1}^{2} \frac{\nu_i}{A - g_i} \sin(g_i t + \beta_i), \tag{7.78}$$

$$k_0(t) = -\sum_{i=1}^{2} \frac{\nu_i}{A - g_i} \cos(g_i t + \beta_i), \tag{7.79}$$

$$p_0(t) = -\sum_{i=1}^{2} \frac{\mu_i}{B - f_i} \sin(f_i t + \gamma_i), \tag{7.80}$$

$$q_0(t) = -\sum_{i=1}^{2} \frac{\mu_i}{B - f_i} \cos(f_i t + \gamma_i). \tag{7.81}$$

Note that h_0, k_0, p_0, and q_0 are only functions of the (constant) semi-major axis of the particle and do not involve any of its other orbital elements. However, since the values of h_0, k_0, p_0, and q_0 also depend on the secular solution for the two perturbing bodies, they will vary with time.

If we define the quantities

$$e_{\text{forced}} = \sqrt{h_0^2 + k_0^2}, \qquad I_{\text{forced}} = \sqrt{p_0^2 + q_0^2} \tag{7.82}$$

then the solutions given in Eq. (7.77) have a simple geometrical interpretation (Figs. 7.2 and 7.3). In the case of the h–k solution, the values of k and h for the particle define a point in the k–h plane. The vector from the origin to this point has a length e and it makes an angle ϖ with the k axis. In the light of our solution given above, this vector can also be thought of as the vector sum of two other vectors: The first goes from the origin to the point (k_0, h_0); it has a length e_{forced} and makes an angle ϖ_{forced} with the k axis. The second goes from (k_0, h_0) to the point (k, h); it has a length e_{free} and makes an angle $\varpi_{\text{free}} = At + \beta$ with the k axis. This implies that the particle's motion can be thought of as motion around a circle with centre (k_0, h_0) at a constant rate A while this point itself moves in some complicated path determined by the secular solution for the two perturbing bodies. This is illustrated in Fig. 7.2. The quantities e_{forced} and ϖ_{forced} are derived from h_0 and k_0 and they are called the *forced eccentricity* and *forced longitude of pericentre* of the particle. Their values are determined solely by the semi-major axis of the particle and the secular solution for the two perturbing bodies. In contrast, e_{free} and ϖ_{free}, the *free eccentricity* and *free longitude of pericentre* of the particle, are derived from the boundary conditions and denote fundamental orbital parameters of the particle. These quantities are also referred to as the *proper eccentricity* and *proper longitude of pericentre* of the particle's orbit.

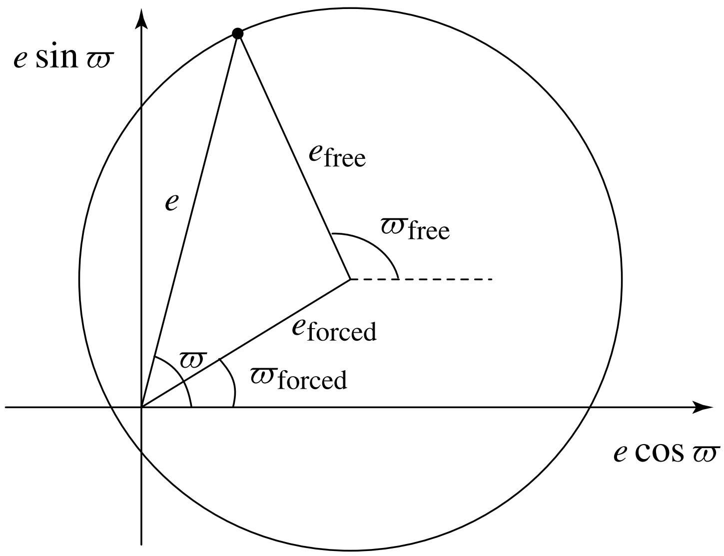

Fig. 7.2. The geometrical relationship among the osculating, free, and forced eccentricities and longitudes of pericentre for the case $e_{\text{free}} > e_{\text{forced}}$.

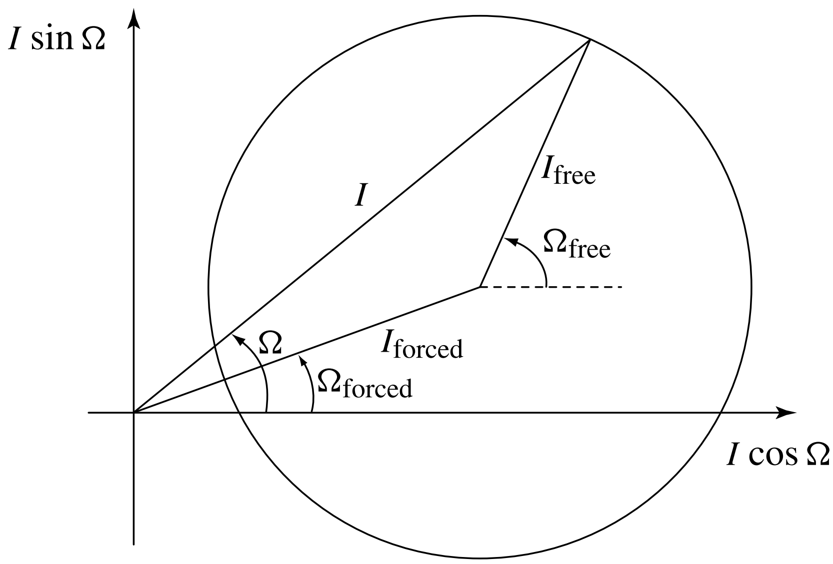

Fig. 7.3. The geometrical relationship among the osculating, free, and forced inclinations and longitudes of ascending node for the case $I_{\rm free} < I_{\rm forced}$.

It is important to note that the circle shown in Fig. 7.2 need not encompass the origin. If $e_{\rm free}$ is small enough or the value of $e_{\rm forced}$ for the particle's semi-major axis is large enough, the motion of the particle around the circle might be such that ϖ or Ω (the osculating longitudes of pericentre or ascending node) vary over some fixed range of angles.

The p–q solution is shown in Fig. 7.3, where $I_{\rm forced}$, $\Omega_{\rm forced}$, $I_{\rm free}$, and $\Omega_{\rm free}$ denote the forced and free inclinations and nodes of the particle. Here we illustrate a situation where the forced inclination is larger than the free one, such that the circle does not enclose the origin. As stated above, this is an outcome of the boundary conditions.

Because the expressions for $e_{\rm forced}$ and $I_{\rm forced}$ given in Eq. (7.82) depend on the definitions of h_0, k_0, p_0, and q_0 given in Eqs. (7.78)–(7.81), it is clear that potentially large values of $e_{\rm forced}$ or $I_{\rm forced}$ can arise if either of the conditions $A - g_i \approx 0$ or $B - f_i \approx 0$ is satisfied. The g_i and f_i are the eigenfrequencies of the system of two interacting bodies whereas, from Eqs. (7.55) and (7.57), the quantities A and B are functions of the semi-major axis of the test particle. This implies that at certain locations in semi-major axis there will be singularities in the forced eccentricities or inclinations. We will consider a specific example of this in Sect. 7.5.

Another important point concerns the limiting values of $e_{\rm forced}$ and $I_{\rm forced}$ as the orbit of either of the two perturbers is approached. Since the A, A_j, B, and B_j in Eqs. (7.55)–(7.58) as well as the definitions of the ν_i and μ_i in Eq. (7.76) involve the Laplace coefficients $b_{3/2}^{(1)}(\alpha_j)$ or $b_{3/2}^{(2)}(\alpha_j)$, which all approach infinity as $\alpha_j \to 1$, it is not obvious that there are finite limiting values for $e_{\rm forced}$ and $I_{\rm forced}$ at the orbits of the perturbers. Let us assume that we are considering the behaviour of $e_{\rm forced}$ in the vicinity of a perturbing body denoted by subscript

$j = l$. In this case

$$b_{3/2}^{(1)}(\alpha_l) \to \infty \qquad \text{and} \qquad b_{3/2}^{(2)}(\alpha_l) \to \infty \qquad \text{as} \qquad \alpha_l \to 1\,. \tag{7.83}$$

At the orbit of the perturber $A \gg g_i$ for $i = 1, 2$ and $A_l \gg A_i$, where $i \neq l$. Therefore

$$A - g_i \approx A \approx \frac{1}{4} n \frac{m_l}{m_c} \alpha_l \bar{\alpha}_l b_{3/2}^{(1)}(\alpha_l) \tag{7.84}$$

and

$$\nu_i = \sum_{j=1}^{2} A_j e_{ji} \approx A_l e_{li}. \tag{7.85}$$

This implies that

$$h_0(t) \approx -\sum_{i=1}^{2} \frac{A_l e_{li}}{A} \sin(g_i t + \beta_i) = +\frac{b_{3/2}^{(2)}(\alpha_l)}{b_{3/2}^{(1)}(\alpha_l)} \sum_{i=1}^{2} e_{li} \sin(g_i t + \beta_i). \tag{7.86}$$

Since the definition of the Laplace coefficient given in Eqs. (6.67) and (6.68) can also be written as

$$\frac{1}{2} b_s^{(j)}(\alpha) = \frac{s(s+1)\ldots(s+j-1)}{j!} \alpha^j F(s, s+j, j+1; \alpha^2), \tag{7.87}$$

where $F(a, b, c; d)$ is the standard hypergeometric function, we have

$$\lim_{\alpha_l \to 1} \frac{b_{3/2}^{(2)}(\alpha_l)}{b_{3/2}^{(1)}(\alpha_l)} = \lim_{\alpha_l \to 1} \left[\frac{5}{4} \alpha_l \frac{F(\frac{3}{2}, \frac{7}{2}, 3, \alpha_l^2)}{F(\frac{3}{2}, \frac{5}{2}, 2, \alpha_l^2)} \right] = 1 \tag{7.88}$$

from the properties of hypergeometric series and their relationship with elliptical integrals. Therefore

$$\lim_{\alpha_l \to 1} h_0(t) = \sum_{i=1}^{2} e_{li} \sin(g_i t + \beta_i) = h_l. \tag{7.89}$$

Similarly

$$\lim_{\alpha_l \to 1} k_0(t) = \sum_{i=1}^{2} e_{li} \cos(g_i t + \beta_i) = k_l. \tag{7.90}$$

Therefore, as the orbit of perturbing body l is approached,

$$e_{\text{forced}} = \sqrt{h_0^2 + k_0^2} \to \sqrt{h_l^2 + k_l^2} = e_l. \tag{7.91}$$

This implies that the forced values of the eccentricity and longitude of pericentre at the orbit of the perturber are equal to the equivalent osculating values of these elements for the perturber. A similar result for the forced values of the inclination and longitude of ascending node can be shown using the same method as given above. However, we have to be careful not to make too many generalisations about the nature of the particle's orbit in the vicinity of the orbit of a perturber. We have already seen in Fig. 3.30 that particles near the L_1 and L_2 points acquire an eccentricity from their encounter with the satellite, despite the fact that the satellite is moving in a circular orbit. Therefore our result concerning the forced eccentricity and inclination is really only valid for particle orbits far from L_1 and L_2 with low values of e and I.

7.5 Jupiter, Saturn, and a Test Particle

We can illustrate the results derived in Sect. 7.4 by calculating the forced orbital elements on a test particle moving under the effects of secular perturbations from Jupiter and Saturn. In Sect. 7.3 we obtained the secular solution for the Jupiter–Saturn system and so we have values of h_j, k_j, p_j, and q_j $(j = 1, 2)$ at any time t. This allows us to calculate the forced elements of a test particle at any location in the solar system.

Before proceeding it is important to point out that this entire analysis ignores the effect of mean motion resonances. We have already seen in Sect. 6.9.2 that at certain locations the particle is subjected to additional perturbations over an above those due to the secular terms in the disturbing function. A more complete analysis of the resonant terms is given in Sect. 8. Here we content ourselves with the warning that our results on forced elements do not take account of the effects of mean motion resonances.

The first step in the secular theory for the particle's motion is to calculate the value of the frequency, A, given in Eq. (7.55). Note that in the case of two bodies orbiting a spherical or point-mass central object, we have $B = -A$. The value of A depends on (i) the masses of the perturbers and (ii) the semi-major axis of the perturbed particle.

Figure 7.4 shows the variation in A from 0 to 30 AU. The singularities close to 5 and 10 AU arise from the fact that the Laplace coefficients tend to infinity as $\alpha_j \to 1$. On the same diagram the three, nonzero eigenfrequencies of the system are denoted by the solid and dashed horizontal lines. Two of them, namely $g_1 = 0.00096^\circ\mathrm{y}^{-1}$ and $g_2 = 0.0061^\circ\mathrm{y}^{-1}$, are the eccentricity–pericentre eigenfrequencies while the third one, $f_2 = -0.0071^\circ\mathrm{y}^{-1}$, is the single nonzero inclination–node eigenfrequency. The intersection of these lines with the curve showing the variation of A identify the semi-major axes where large forced eccentricities or inclinations can be expected.

Since the value of A only depends on the semi-major axes and masses of the planets and the particle, and since all these quantities are constant (recall that there is no secular change in the semi-major axes), the value of A is constant at any given semi-major axis. It is also independent of time.

The values of the forced eccentricity and longitude of pericentre as a function of semi-major axis for a given time can be calculated using Eqs. (7.76) and

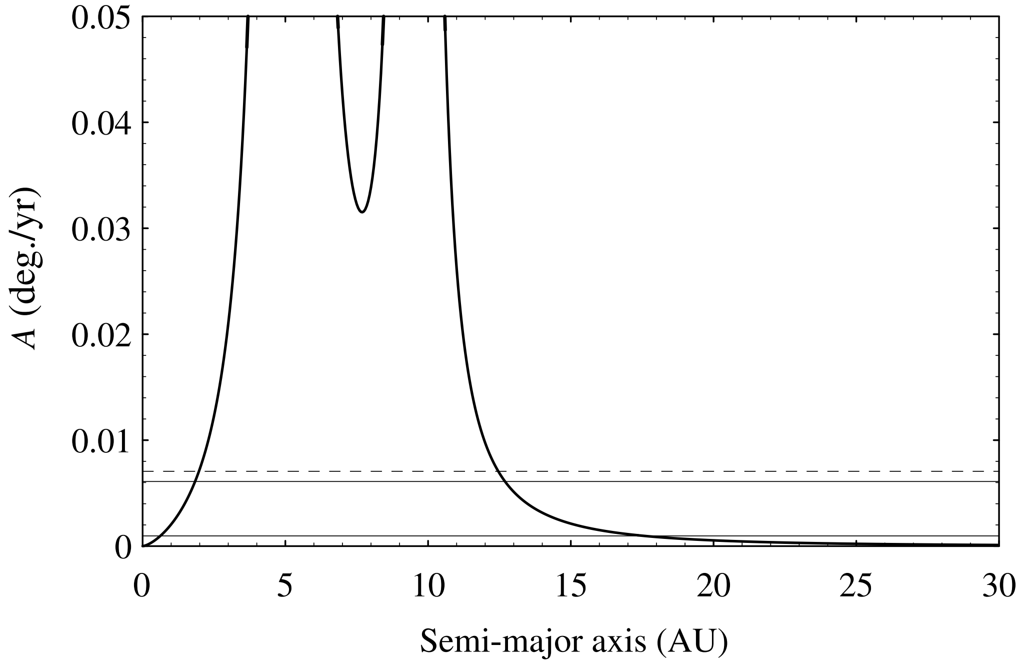

Fig. 7.4. The variation of the frequency A, defined in Eq. (7.55), as a function of semi-major axis, derived from perturbations by Jupiter and Saturn. The horizontal solid lines denote the values of the two eccentricity–pericentre eigenfrequencies, $A = g_1 = 0.00096°\mathrm{y}^{-1}$ and $A = g_2 = 0.0061°\mathrm{y}^{-1}$; the dashed line denotes the value of the nonzero inclination–node eigenfrequency, $A = -B = -f_2 = 0.0071°\mathrm{y}^{-1}$. Singularities in the plot correspond to the orbital semi-major axes of Jupiter and Saturn.

(7.78)–(7.82) where the values of e_{ji} and I_{ji} are given in Sect. 7.3. Figure 7.5 shows the variation of e_{forced} and ϖ_{forced} in the range 0 to 30 AU. Note that the values of e_{forced} and ϖ_{forced} at the orbits of Jupiter and Saturn are equal to the osculating values of the two planets at this time. This phenomenon was discussed in Sect. 7.4. It is important to point out that the curves shown in Fig. 7.5 are for one specific time ($t = 0$). Since the values of the osculating elements for the planets vary according to the secular solution, so too will the shape of the curves.

The singularities in the e_{forced} plot (see Fig. 7.5) close to 0.5, 2, 12.5, and 17.5 AU are a consequence of the small divisors inherent in Eqs. (7.78) and (7.79). At each of these locations the value of A is equal to one of the g_i eigenfrequencies of the perturbing system. The equivalent effect in the forced longitudes of perihelion, ϖ_{forced}, is to cause a sudden shift in the longitudes of 180° (see Fig. 7.5b). Although the shape of both curves will alter with time as the osculating elements of Jupiter and Saturn vary, the singularities will remain at the same locations.

The equivalent set of plots for the forced inclination and forced longitude of ascending node are shown in Fig. 7.6. Since there is only one nonzero eigenfrequency in the system, there will only be two singularities (in this case near 2

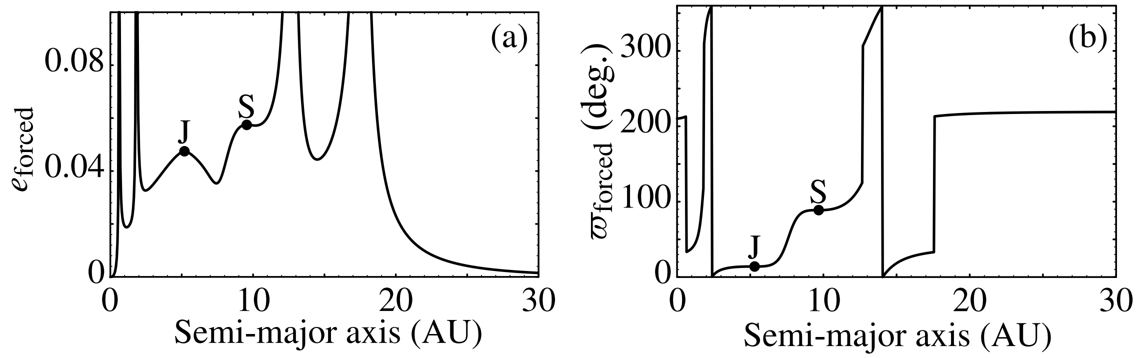

Fig. 7.5. The variation in (a) the forced eccentricity and (b) the forced longitude of perihelion as a function of semi-major axis at time $t = 0$. The letters J and S denote the osculating values for Jupiter and Saturn respectively at their semi-major axes. The singularities near 0.5, 2, 12.5, and 17.5 AU arise from the small divisors $A - g_i$ in Eqs. (7.78)–(7.79).

and 12 AU) in the plots for I_{forced} and Ω_{forced} corresponding to the two points of intersection of A with the dashed line shown in Fig. 7.4.

We can provide an additional illustration of the results derived in Sect. 7.4 by considering the orbital evolution of a number of test particles moving under the effects of perturbations from Jupiter and Saturn. A numerical integration of the full equations of motion was carried out of a system consisting of 250 test particles, Jupiter, and Saturn. The planets perturbed each other and the particles but the test particles were assumed to have negligible mass. The test particles were started with the same semi-major axis (1.8 AU), free eccentricity (0.049), and free inclination (2.12°), but with random initial free longitudes of pericentre and ascending nodes. These values were chosen to minimise any resonant effects (see Sect. 6.9). The initial osculating values of e, I, ϖ, and Ω for each particle were

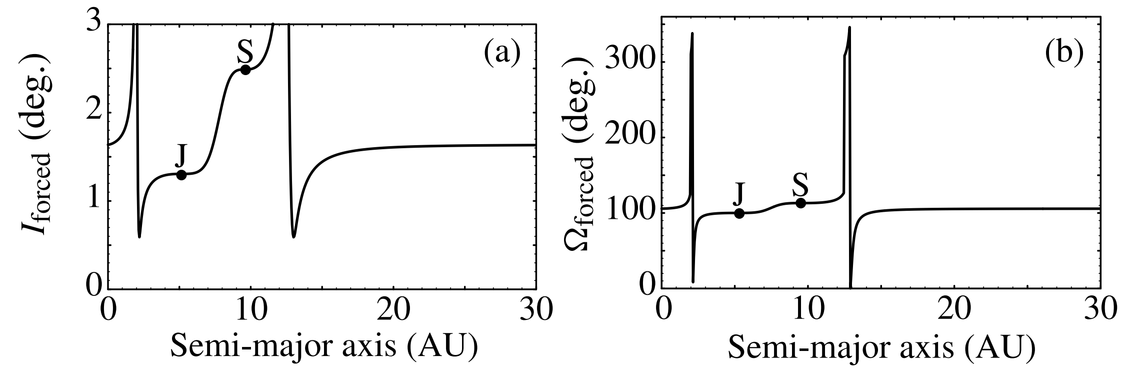

Fig. 7.6. The variation in (a) the forced inclination and (b) the forced longitude of ascending node as a function of semi-major axis at time $t = 0$. The letters J and S denote the osculating values Jupiter and Saturn respectively at their semi-major axes. The singularities near 2 and 12 AU arise from the small divisor $B - f_2$ in Eqs. (7.80) and (7.81).

obtained by first calculating the equivalent forced elements and then carrying out the vector sum of the forced and free components as shown in Figs. 7.2 and 7.3. The starting conditions are equivalent to setting up a ring of particles in the osculating (k, h) plane. The centre of this ring will move with time along a path determined by the secular solution for Jupiter and Saturn.

Figure 7.7a shows the (k, h) values of the particles at time $t = 0$ (corresponding to 1983) and after 30,000 years. In each case a circular ring is clearly discernable. The dashed line denotes the predicted path of the centre of the circle over this time interval and the arrows indicate the direction of motion of the particles in the circle and of the centre of the circle itself. Note that the circle of particles in the (k, h) plane is maintained over the timescale of the integration and that the predicted centre of the circle is in excellent agreement with its observed location. This implies that the results of the numerical integration agree with the predictions of secular perturbation theory and that the motion of a test particle can be described as a combination of uniform rotation in a circle where the centre of the circle is executing some well-defined path in the (k, h) plane. It is clear from Fig. 7.7a that there has been some movement of the particles along the circle and in the radial direction. This can probably be accounted for by the effects of the short-period terms neglected in the secular calculation.

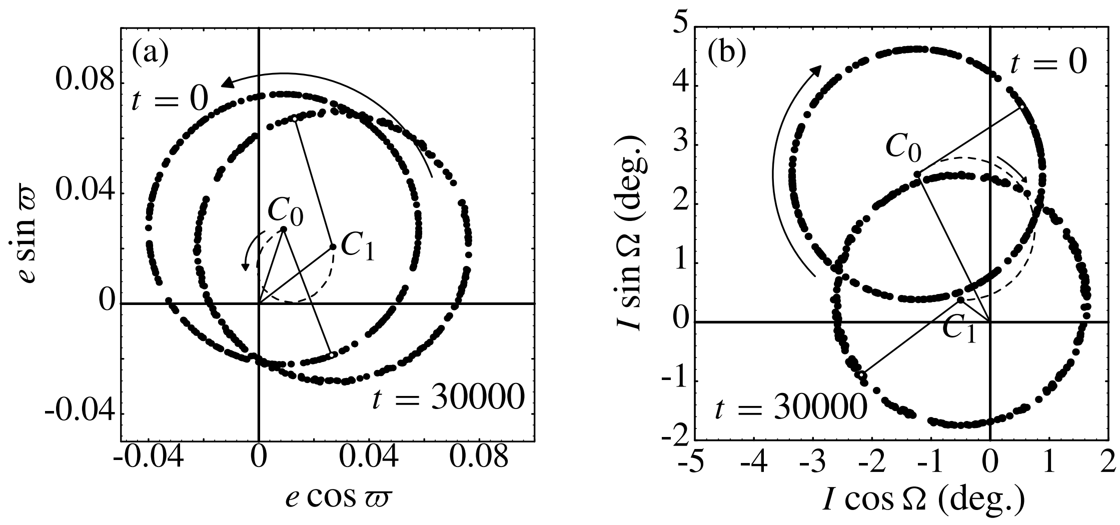

Fig. 7.7. The initial and final locations in (a) the (k, h) plane and (b) the (q, p) plane of 250 test particles started with the same free eccentricity and free inclination but randomised free longitudes of perihelion and free longitudes of ascending node. The orbits of the particles, Jupiter, and Saturn were numerically integrated for 30,000 years. The points C_0 and C_1 denote the coordinates of (k_0, h_0) (and (q_0, p_0)) at times $t = 0$ and $t = 30{,}000$ y respectively. For each circle there is a line (of length equal to the forced eccentricity or inclination) joining the origin to the centre and another line (of length equal to the free eccentricity or inclination) from there to the first test particle, denoted by a small white circle.

The equivalent results for the (q, p) values of the same particles are shown in Fig. 7.7b. Once again the preservation of the circle with time is evident, as well as the fact that the centre of the circle lies close to the calculated location, (q_0, p_0), determined from secular perturbation theory. As with Fig. 7.7a there is also some movement along the circle and in the radial direction caused by the effects of short-period terms.

A further subtlety arises when we consider direct comparisons between secular perturbation theory and full numerical integrations. Any integration of the full equations of motion must, by necessity, include all the infinite number of short-period terms in the expansion of the disturbing function. However, the secular perturbation theory we have derived and the resulting theory of free and forced elements are based on the truncation of the disturbing function and the averaging principle. This difference causes a problem when we have to consider starting values for the orbital elements of Jupiter and Saturn. In order to make the comparison as fair as possible we should numerically integrate the orbits of Jupiter and Saturn using the full equations of motion. The results should then be Fourier analysed to determine an approximation to the secular solution at time $t = 0$. These orbital elements should be used in the secular perturbation theory, which leads to a determination of the forced elements. In fact this procedure was adopted for the comparison shown in Fig. 7.7. This method is also used in the example given in Sect. 8.19 where we have to take account of resonant and secular terms.