$\Lambda = L$

$\xi = \sqrt{-2V}\,\mathrm{sen}\,v \qquad \alpha = \sqrt{-2Z}\,\mathrm{sen}\,z$

$\eta = \sqrt{-2V}\cos v \qquad \beta = \sqrt{-2Z}\cos z$

Chapter 5

SECULAR PERTURBATION THEORY

5.1 The secular perturbation Hamiltonian

As shown in Section 2.6, after removal of the short periodic terms the Hamiltonian would be reduced to the normal form:

$$H' = H'_0 + \epsilon\, H'_1 + \epsilon^2\, H'_2 + \mathcal{O}(\epsilon^3) \ ,$$

where all the H'_j are functions of the mean elements.

If we assume the Hamiltonian in mean elements is expressed by means of the mean Poincaré variables $(\boldsymbol{\Lambda}', \boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\lambda}', \boldsymbol{\eta}', \boldsymbol{\beta}')$ then

$$H' = H'_0(\boldsymbol{\Lambda}') + \epsilon\, H'_1(\boldsymbol{\Lambda}', \boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}') + \epsilon^2\, H'_2(\boldsymbol{\Lambda}', \boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}') + \mathcal{O}(\epsilon^3) \ .$$

This series contains the mean mean longitudes $\boldsymbol{\lambda}'$ only in the $\mathcal{O}(\epsilon^3)$ term. Thus, if we truncate the series after the $\mathcal{O}(\epsilon^2)$ term, we obtain a **secular Hamilton function**, not depending at all on the variables $\boldsymbol{\lambda}'$, that is such that all the variables $\boldsymbol{\Lambda}'$ are integrals at this level of approximation. In other words, we can claim that if we were able to compute mean variables $\boldsymbol{\Lambda}'$ from which all the short periodic perturbations are removed[1], then they would be constant, thus each corresponding mean a'_i would be also an **analytic proper semimajor axis**. Of course this goal can actually be achieved only up to some level of approximation; it needs to be explicitly described by the rules by which the terms in the determining function $\chi = \epsilon\,\chi_1 + \epsilon^2\,\chi_2$ are selected to be explicitly computed.

As a consequence, the components of $\boldsymbol{\Lambda}'$ are not anymore dynamical variables, but they appear just as parameters in H', which can be reinterpreted as the Hamiltonian providing the equations of motion in the space of the $4N + 4$ variables $(\boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}')$.

This would be a big step in the direction of computing an approximate solution for the equation of motion for $N + 1$ planets (asteroid included) problem, but there is a problem. In such a

[1]As pointed out in Section 2.10, secular perturbations in the semimajor axes (thus also $\boldsymbol{\Lambda}'$) $\mathcal{O}(\epsilon^2)$ are introduced by beats between short periodic perturbations $\mathcal{O}(\epsilon)$, and these need to be removed to compute accurate integrals.

context, the order zero portion $H'_0(\mathbf{\Lambda}')$ is just a constant and has no dynamical effect. Thus we can use as Hamilton function just

$$H' = \epsilon\, H'_1 + \epsilon^2\, H'_2 \tag{5.1}$$

which cannot be described as a "perturbed" Hamiltonian, that is it does not appear as the sum of an integrable portion plus a perturbation containing a small parameter. Indeed, ϵ is not a small parameter, it just sets the order of magnitude for the entire function H'.

This is best expressed as a change in scale for the time: indeed, if the Hamiltonian (5.1) is divided by ϵ, this is equivalent to a change of the time coordinate by a factor ϵ, that is by setting $\tau = \epsilon\, t$, the **slow time**, the Hamiltonian equations with τ as independent variable have as Hamiltonian $H'/\epsilon = H'_1 + \epsilon\, H'_2$:

$$\begin{aligned} \frac{d\xi'}{d\tau} &= -\frac{\partial(H'/\epsilon)}{\partial\eta'} \quad , \quad \frac{d\eta'}{d\tau} = \frac{\partial(H'/\epsilon)}{\partial\xi'} \\ \frac{d\alpha'}{d\tau} &= -\frac{\partial(H'/\epsilon)}{\partial\beta'} \quad , \quad \frac{d\beta'}{d\tau} = \frac{\partial(H'/\epsilon)}{\partial\alpha'} \ . \end{aligned} \tag{5.2}$$

Anyway the problem is: how can we use a perturbation approach to solve the problem defined by the secular Hamilton function? To answer to this question we can go back to the third D'Alembert rule, and to the approach already used in Section 2.9: we use $e_i, \sin I_i$ as small parameters, all considered to be $\mathcal{O}(\epsilon^{1/r})$ for some $r > 0$, and expand into homogeneous functions of these small parameters, e.g.,

$$H'_1 = K_0^{(0)}(;\mathbf{\Lambda}') + K_1^{(2)}(\boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}'; \mathbf{\Lambda}') + K_1^{(4)}(\boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}'; \mathbf{\Lambda}') + \mathcal{O}(\epsilon^{6/r}) \ ,$$

where $K_1^{(k)}$ is the portion of H'_1 homogeneous of degree k in the small parameters $e_i, \sin I_i$; odd k terms are not allowed by D'Alembert rules. * Now the integrable portion of H' has appeared again, because a quadratic homogeneous Hamiltonian implies linear Hamilton equations, and those can be integrable. As for $K_1^{(0)}$ it does not matter, unless the time evolution of $\mathbf{\Lambda}'$ has to be computed: indeed, the notation $;\mathbf{\Lambda}'$ indicates that the variables connected with the mean semimajor axes appear just as non-dynamical parameters.

An interesting property of the secular Hamiltonian H'_1 is that it does not contain any contribution from the indirect part of the perturbing Hamiltonian H_1. If heliocentric canonical coordinates are used, this implies that T_1 of equation (1.51) is such that its Fourier series expansion in Delaunay variables has no secular part, that is $\overline{T}_1 = 0$. For a classical proof see [Brouwer and Clemence 1961][page 508]. [but for the heliocentric canonical coordinates this should be easier... or maybe not?] in H'_2 there are terms arising from beats between indirect short periodic terms, see Section 5.6.

* infatti $j_5 + j_6$ è pari e $j_3 + j_4 + j_5 + j_6 = 0$

5.2 Linear secular perturbation theory

As dictated by the third D'Alembert rule, $K_1^{(2)}$ contains only expressions of the form $\xi_i\,\xi_j + \eta_i\,\eta_j$ and $\alpha_i\,\alpha_j + \beta_i\,\beta_j$, where i, j are indexes of the planets (and asteroids), not excluding $i = j$. The

$\alpha_i \alpha_j + \beta_i \beta_j = 2\sqrt{Z_i Z_j}\cos(z_i - z_j)$, $\xi_i \xi_j + \eta_i \eta_j = 2\sqrt{V_i V_j}\cos(v_i - v_j)$

per esempio il termine $\xi_i \eta_j$ può comparire solo sommato a $\eta_i \eta_j$

$\xi_i \eta_j = 2\sqrt{V_i V_j}\,\mathrm{sen}\, v_i \cos v_j = \sqrt{V_i V_j}\,(\mathrm{sen}(\underline{v_i + v_j}) + \mathrm{sen}(v_i - v_j))$

questo argomento non può rimanere in H'_1

coefficients of these expressions are just functions of the variables Λ_i, Λ_j, that is

$$\begin{aligned} K_1^{(2)}(\boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}') &= \frac{1}{2} \sum_{i,j=1}^{N+1} \left[A_{ij}(\Lambda_i', \Lambda_j')\,(\xi_i'\,\xi_j' + \eta_i'\,\eta_j') + E_{ij}(\Lambda_i', \Lambda_j')\,(\alpha_i'\,\alpha_j' + \beta_i'\,\beta_j') \right] = \\ &= \frac{1}{2} \left[\boldsymbol{\xi}' \cdot A\,\boldsymbol{\xi}' + \boldsymbol{\eta}' \cdot A\,\boldsymbol{\eta}' + \boldsymbol{\alpha}' \cdot E\,\boldsymbol{\alpha}' + \boldsymbol{\beta}' \cdot E\,\boldsymbol{\beta}' \right] \end{aligned} \tag{5.3}$$

where $A(\boldsymbol{\Lambda}') = (A_{ij})$, $E(\boldsymbol{\Lambda}') = (E_{ij})$ $i,j = 1, N+1$ are real symmetric matrices. Then the Hamilton equations defined by the Hamiltonian $K_1^{(2)}$ are, for each $i = 1, N+1$

si scrive t ma si intende τ

$$\frac{d\xi_i'}{dt} = -\sum_{j=1}^{N+1} A_{ij}\,\eta_j' \quad , \quad \frac{d\eta_i'}{dt} = \sum_{j=1}^{N+1} A_{ij}\,\xi_j'$$

$$\frac{d\alpha_i'}{dt} = -\sum_{j=1}^{N+1} E_{ij}\,\beta_j' \quad , \quad \frac{d\beta_i'}{dt} = \sum_{j=1}^{N+1} E_{ij}\,\alpha_j'$$

In vector form

$$\begin{aligned} \frac{d\boldsymbol{\xi}'}{dt} &= -A\,\boldsymbol{\eta}' \quad , \quad \frac{d\boldsymbol{\eta}'}{dt} = A\,\boldsymbol{\xi}' \\ \frac{d\boldsymbol{\alpha}'}{dt} &= -E\,\boldsymbol{\beta}' \quad , \quad \frac{d\boldsymbol{\beta}'}{dt} = E\,\boldsymbol{\alpha}' \,, \end{aligned} \tag{5.4}$$

or equivalently as second order equations

$$\begin{aligned} \frac{d^2\boldsymbol{\xi}'}{dt^2} &= -A^2\,\boldsymbol{\xi}' \quad , \quad \frac{d^2\boldsymbol{\eta}'}{dt^2} = -A^2\,\boldsymbol{\eta}' \\ \frac{d^2\boldsymbol{\alpha}'}{dt^2} &= -E^2\,\boldsymbol{\alpha}' \quad , \quad \frac{d^2\boldsymbol{\beta}'}{dt^2} = -E^2\,\boldsymbol{\beta}' \,. \end{aligned}$$

Note that the two sets of equations (5.4) are decoupled, thus we can solve separately first for the ξ_i', η_i', then for the α_j', β_i'. The solutions can be expressed by means of matrix exponentials: by assembling all the equations for ξ_i', η_i'

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{\xi}' \\ \boldsymbol{\eta}' \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -A \\ A & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}' \\ \boldsymbol{\eta}' \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \boldsymbol{\xi}'(t) \\ \boldsymbol{\eta}'(t) \end{bmatrix} = \exp\begin{bmatrix} \mathbf{0} & -At \\ At & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}'(0) \\ \boldsymbol{\eta}'(0) \end{bmatrix} .$$

Note that the matrix of coefficients of the system is antisymmetric, thus the exponential is a rotation: all the lenghts of the vectors are preserved. Exactly the same applies to the solutions for $\boldsymbol{\alpha}', \boldsymbol{\beta}'$.

Let R, S be matrices such that $R^T\,A\,R = -diag[g_1, \ldots, g_{N+1}]$ and $S^T\,E\,S = -diag[s_1, \ldots, s_{N+1}]$ are diagonal; the coordinate change is orthogonal, that is $R^T = R^{-1}$, $S^T = S^{-1}$; in particular we can assume that R, S are rotations. The eigenvalues of the matrices A, E are the frequencies g_i, s_i, respectively, and are secular, in that they appear in the solutions of the equations (5.2) with as independent variable the slow time τ; in practice, periods are of the order of tens of thousands to millions of years. Then by changes of coordinates

$$\boldsymbol{\xi}'' = R^T\,\boldsymbol{\xi}' \quad , \quad \boldsymbol{\eta}'' = R^T\,\boldsymbol{\eta}' \quad , \quad \boldsymbol{\alpha}'' = S^T\,\boldsymbol{\alpha}' \quad , \quad \boldsymbol{\beta}'' = S^T\,\boldsymbol{\beta}'$$

we can decouple the differential equations, that is the double-primed variables represent the **proper modes** of oscillation, solutions of uncoupled harmonic oscillators:

$$\frac{d\boldsymbol{\xi}''}{dt} = diag[g_1, \dots, g_{N+1}]\,\boldsymbol{\eta}'' \quad , \quad \frac{d\boldsymbol{\eta}''}{dt} = -diag[g_1, \dots, g_{N+1}]\,\boldsymbol{\xi}''$$

$$\frac{d\boldsymbol{\alpha}''}{dt} = diag[s_1, \dots, s_{N+1}]\,\boldsymbol{\beta}'' \quad , \quad \frac{d\boldsymbol{\beta}''}{dt} = -diag[s_1, \dots, s_{N+1}]\,\boldsymbol{\alpha}'' \ .$$

Then we can define action-angle variables V_i'', v_i'' and Z_i'', z_i'' such that

$$\xi_i'' = \sqrt{-2V_i''}\,\sin v_i'' \quad , \quad \eta_i'' = \sqrt{-2V_i''}\,\cos v_i''$$

$$\alpha_i'' = \sqrt{-2Z_i''}\,\sin z_i'' \quad , \quad \beta_i'' = \sqrt{-2Z_i''}\,\cos z_i'' \ ,$$

and the actions are integrals for the linear equations, while the angles circulate with frequencies g_i, s_i. Indeed, the expression for $K_1^{(2)}$ as a function of the new variables is

$$K_1^{(2)} = \frac{1}{2}\sum_i (-g_i)\,(\xi_i''\,\xi_i'' + \eta_i''\,\eta_i'') + \frac{1}{2}\sum_i (-s_i)\,(\alpha_i''\,\alpha_i'' + \beta_i''\,\beta_i'') = \sum_i (g_i\,V_i'' + s_i\,Z_i'')$$

and the Hamilton equations in the double primed variables are

$$\frac{dV_i''}{dt} = 0 \quad , \quad \frac{dZ_i''}{dt} = 0$$

*

$$\frac{dv_i''}{dt} = g_i \quad , \quad \frac{dz_i''}{dt} = s_i$$

and $(V_i'', Z_i'', v_i'', z_i'')$ are action-angle variables for the proper mode i. Thus the linear portion of the secular Hamilton equations is integrable, not only in the sense of having a solution expressed by elementary analytic functions, but also in the sense of having action-angle variables. These double primed variables can be called **linear proper elements**.

However, there is an important difference in the system of linear differential equations for $\boldsymbol{\alpha}', \boldsymbol{\beta}'$, with respect to the one for $\boldsymbol{\xi}', \boldsymbol{\eta}'$: namely one of the eigenvalues of E must be zero, for all values of $\boldsymbol{\Lambda}'$. [Brouwer and Clemence 1961][page 518] show this result for two planets only, by the argument that one possible solution needs to be the one in which the two planets have the same inclination and nodes, $I_1 = I_2$, $\Omega_1 = \Omega_2$, then the coplanar orbits remain coplanar. [B&C claim that no general proff is available at that time (1961)]

Our way of showing the same property for an arbitrary number of planets is to argue that the components of the angular momentum c_x, c_y in the reference plane are integrals, thus since this property applies for all e_i, I_i then it must apply already at the lowest order, that is

$$\left\{c_x, K_1^{(2)}\right\} = 0 \quad , \quad \left\{c_y, K_1^{(2)}\right\} = 0 \ .$$

The contribution to c_x from the body i is

$$c_x^{(i)} = C_i\,\sin I_i\,\sin\Omega_i = C_i\,2\,\sin\frac{I_i}{2}\,\cos\frac{I_i}{2}\,\sin\Omega_i$$

* la soluzione nelle variabili $\xi', \eta', \alpha', \beta'$ è

$$\xi_i' = \sum_{j=1}^{N+1} R_{ij}\sqrt{-2V_j}\,\text{sen}(g_j t + \varpi''_{j,0}) \qquad \eta_i' = \sum_{j=1}^{N+1} R_{ij}\sqrt{-2V_j}\,\cos(g_j t + \varpi''_{j,0})$$

$$\alpha_i' = \sum_{j=1}^{N+1} S_{ij}\sqrt{-2Z_j}\,\text{sen}(s_j t + \Omega''_{j,0}) \qquad \beta_i' = \sum_{j=1}^{N+1} S_{ij}\sqrt{-2Z_j}\,\cos(s_j t + \Omega''_{j,0})$$

and by using $\sqrt{(1-\cos I_i)} = \sqrt{2}\sin\frac{I_i}{2}$ and the relationships with α_i', β_i' eq.(2.44)

$$c_x^{(i)} = \sqrt{C_i'}\cos\frac{I_i'}{2}\,\alpha_i' = \sqrt{L_i'}\,\alpha_i' + \mathcal{O}((e_i')^2, (I_i')^2)$$

$$c_y^{(i)} = -\sqrt{C_i'}\cos\frac{I_i'}{2}\,\beta_i' = -\sqrt{L_i'}\,\beta_i' + \mathcal{O}((e_i')^2, (I_i')^2)$$

Then we can compute the lowest order (in e_i', I_i') Poisson brackets $\sqrt{\Lambda'} = (\sqrt{L_1}, \dots, \sqrt{L_{N+1}})^T$

$$\left\{c_x, K_1^{(2)}\right\} = -\frac{\partial c_x}{\partial \alpha'}\cdot\frac{\partial K_1^{(2)}}{\partial \beta'} = -\sqrt{\Lambda'}\cdot E\beta' = \beta'\cdot E\sqrt{\Lambda'} = 0$$

and similarly for $\left\{c_y, {}_1^{(2)}\right\}$. Because the equation above must be true for all possible α', it follows that the vector forming a scalar product with $\boldsymbol{\alpha}'$ is $\mathbf{0}$, thus

$$E^T\sqrt{\boldsymbol{\Lambda}'} = E\sqrt{\boldsymbol{\Lambda}'} = \mathbf{0}\ ,$$

that is the symmetric matrix E has one zero eigenvalue, and the matrix of the linear dynamical system for $\boldsymbol{\alpha}', \boldsymbol{\beta}'$ has two zero eigenvalues [note: matrices linear Hamilton equations have an even equation for eigenvalue, being of the form JB with B symmetric; prove here?]. This implies that one of the frequencies s_k is zero; conventionally, this frequency is associated to Jupiter, that is $s_5 = 0$.

If the reference system for which the computations are performed is such that the reference plane (x, y) is orthogonal to the total angular momentum vector, then it coincides with **Laplace invariable plane** and $c_x = c_y = 0$. In this case the linear proper action Z_5'' corresponding to the frequency s_5 is zero. Then the amplitudes of all the terms containing the angle variable z_5'' is also zero. From this a new D'Alembert rule arises, namely, the Fourier terms of K_1, at all degrees, cannot contain z_5''; this applies to the transformed series appearing in Section 5.4.

Thus the solutions belong to invariant tori of dimension $N+1$ in the space of $\boldsymbol{\xi}', \boldsymbol{\eta}'$ and to invariant tori of dimension N in the space of $\boldsymbol{\alpha}', \boldsymbol{\beta}'$. [reference to next chapter, also a more extensive discussion is needed on why there are neither perturbative terms nor secular resonances containing s_5.]

Osservazione

$$\xi_i = \sqrt{-2V_i}\,\mathrm{sen}\,v_i = \sqrt{2L_i}\sqrt{1-\sqrt{1-e_i^2}}\,\mathrm{sen}\,\varpi_i = \sqrt{2L_i}\,h_i\,(1/\sqrt{2} + \mathcal{O}(e_i^2)) \approx \sqrt{L_i}\,h_i$$

$$\eta_i \approx \sqrt{L_i}\,k_i$$

$$\alpha_i = \sqrt{-2Z_i}\,\mathrm{sen}\,z_i = \sqrt{2C_i}\sqrt{1-\cos I_i}\,\mathrm{sen}\,\Omega_i = \sqrt{C_i}\,2\,\mathrm{sen}\frac{I_i}{2}\,\mathrm{sen}\,\Omega_i$$

$$= 2\sqrt{L_i}\,(1+\mathcal{O}(e_i^2))\,p_i\,(1+\mathcal{O}(p_i^2+q_i^2)) \approx 2\sqrt{L_i}\,p_i$$

$$\beta_i \approx 2\sqrt{L_i}\,q_i$$