UTOPIÆ

Uncertainty Treatment and Optimisation in Aerospace Engineering

THE FRAMEWORK PROGRAMME FOR RESEARCH AND INNOVATION

HORIZON 2020

MARIE CURIE ACTIONS

Uncertainty Quantification in Orbital Mechanics

Massimiliano Vasile,

Aerospace Centre of Excellence,

Department of Mechanical & Aerospace Engineering, University of Strathclyde

UTOPIAE is a **€3.9M** research and training network supported by the European Commission to focus on uncertainty treatment and optimisation

UTOPIAE will last 4 years starting 01 January 2017

Coordinated by Strathclyde University

15 partners over Europe

11 full partners + 4 associate partners

7 universities

3 companies

5 national research centres

1 university research centre

15 researchers will be recruited within UTOPIAE

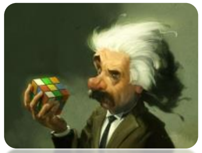
8 major training & outreach events organised within UTOPIAE

- Who's who in the network?
- Where are they?

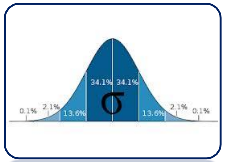
Open Positions

- University of Durham (Department of Mathematics and Statistics): Marie Curie fellowship position on the Imprecise Probabilities applied to large scale dynamic decision processes. Closing date: end of September.

- University of Strathclyde (Department of Mechanical & Aerospace Engineering): PhD position in Artificial Intelligence for Space Mission Design. Closing date: end of September.
- University of Strathclyde: Global Talent and Chancellor's fellowship schemes.
  - Faculty positions at different levels from Lecturer to Professor. Closing date: 24th of September.



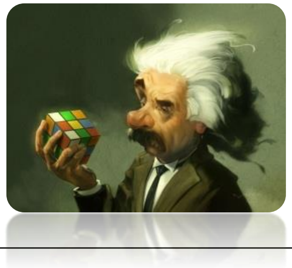
What is UQ?



Types of Uncertainty

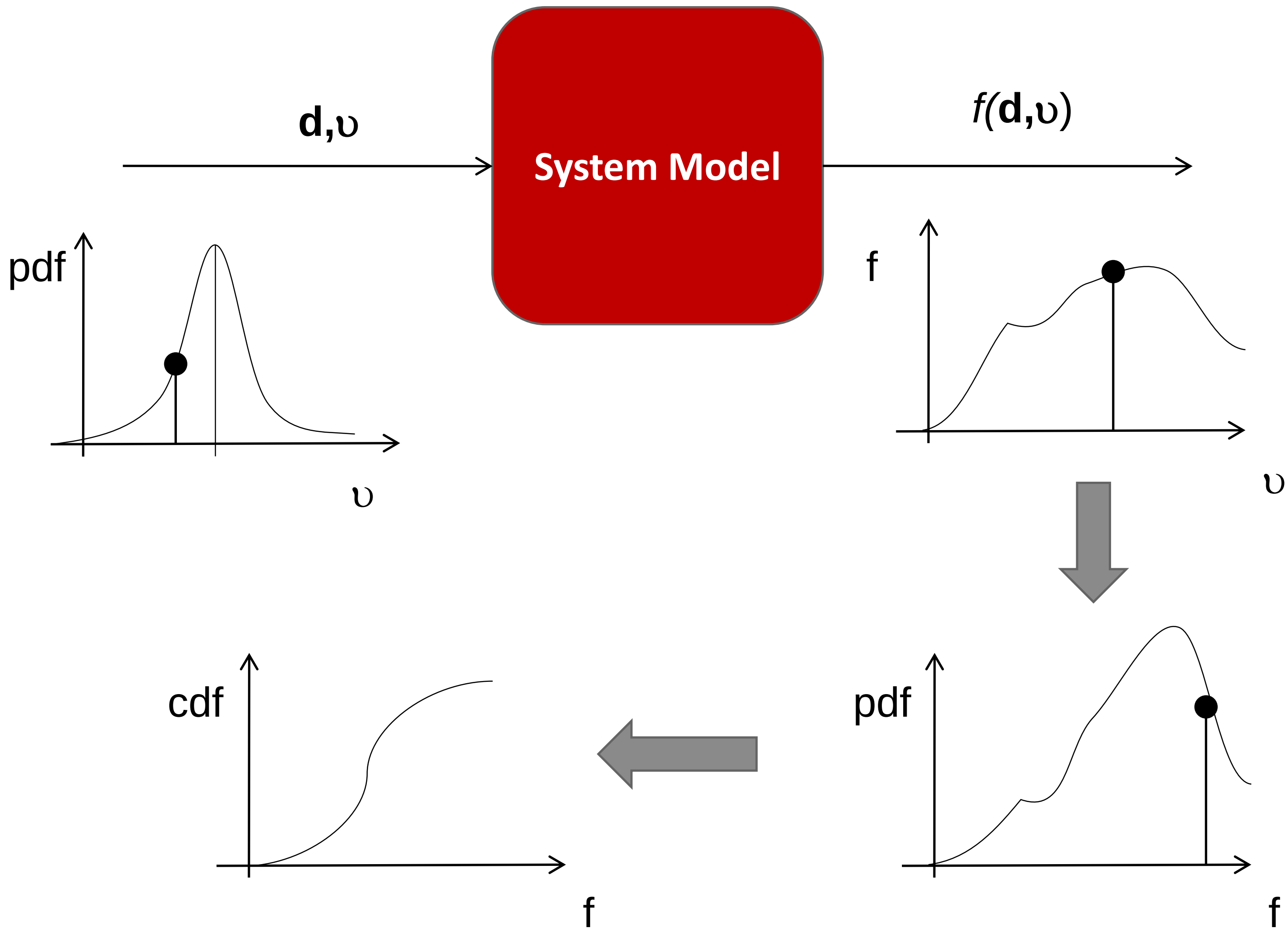
Some UQ Methods

Model Uncertainty

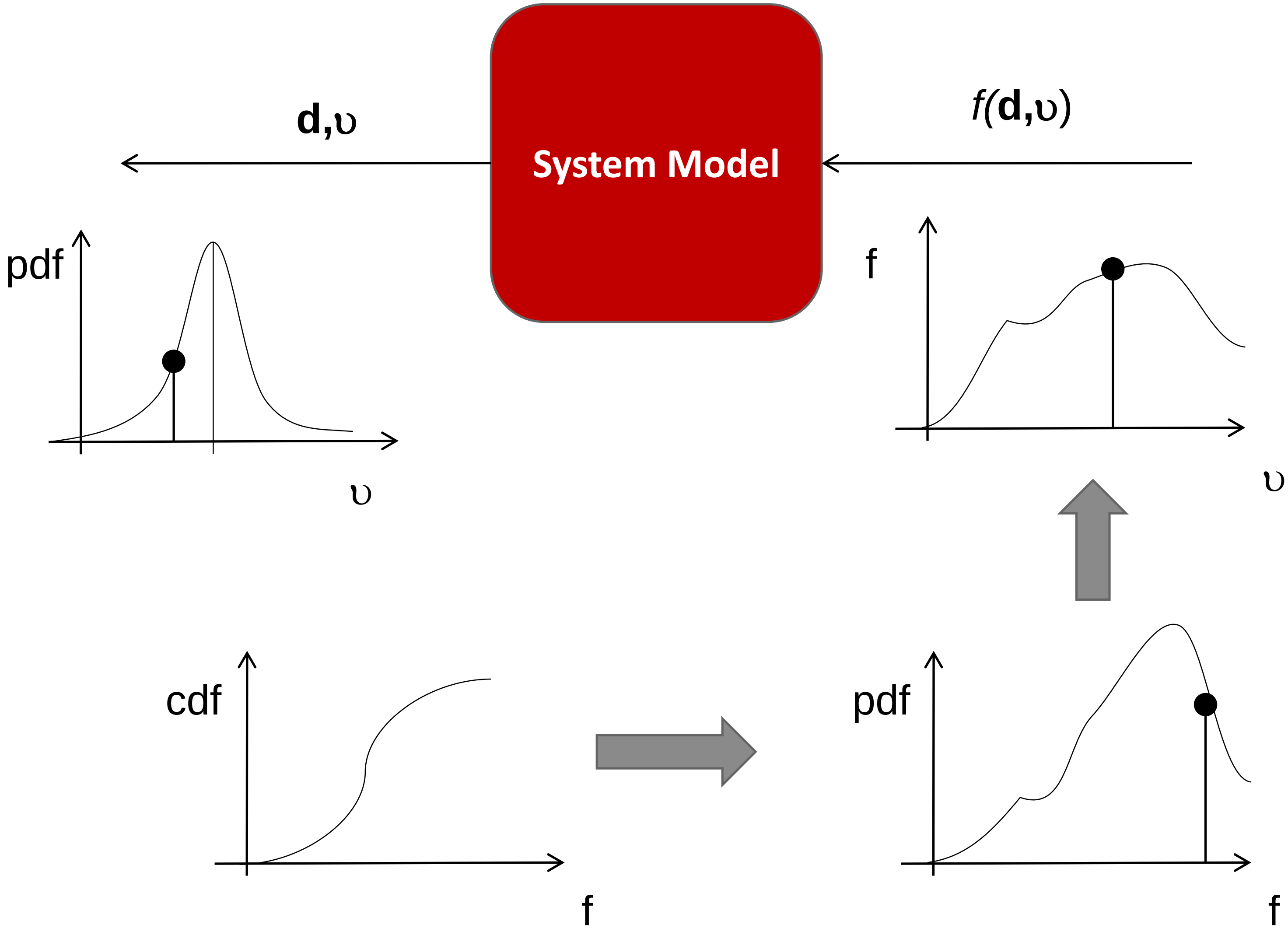


What is UQ?

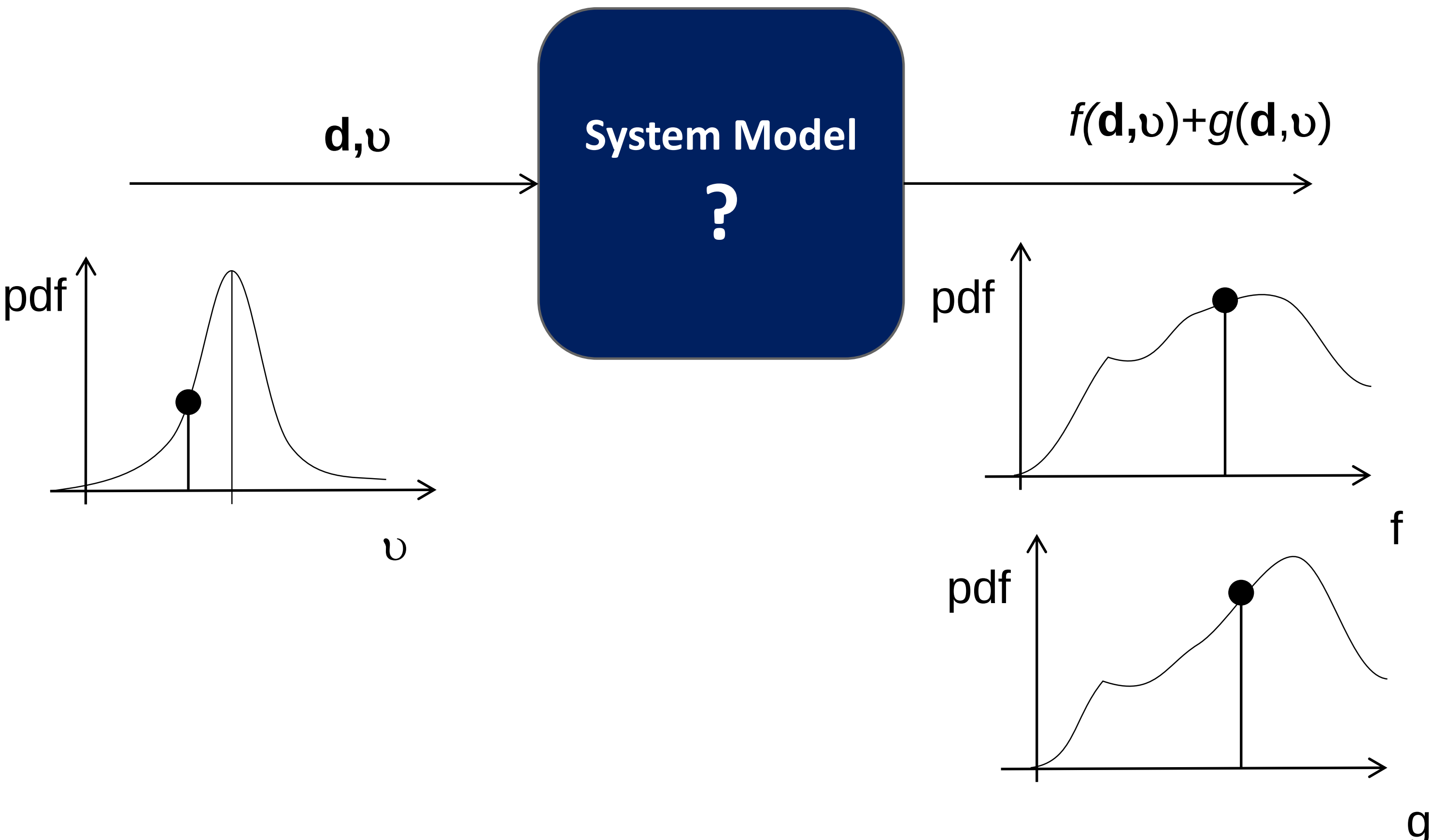
What is UQ? – Direct Problem



What is UQ? – Inverse Problem



What is UQ? – Model Uncertainty



What is UQ in Orbital Mechanics?

- In Orbital Mechanics we are concerned with the following problem:

$$\ddot{s} = f(s,\upsilon,t)h(s,q,t) + g(s,p,t)$$
$$s(t=0) = s_0$$

- Where *p,q* and ν are uncertain parameter vectors, *h* and *g* are uncertain functions and s_0 is an uncertain initial condition vector.

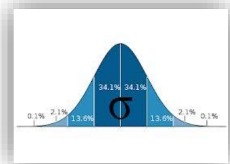
WHAT IS UQ?

- Direct UQ problem
  - Given a quantification of the uncertainty in *q,p, ν, h* and *g* find:
    - the spatial distribution of *s* at a future time
    - the probability associated to a quantity of interest or an event dependent on *s*
- Inverse UQ problem
  - Given the spatial distribution of *s* and the probability associated to an event dependent on *s* find:
    - *q,p, ν, h* and *g*

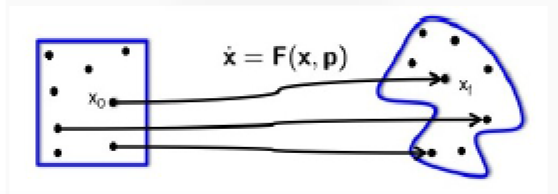
UQ – Basic Ingredients

- The overall UQ process is made of three fundamental elements:

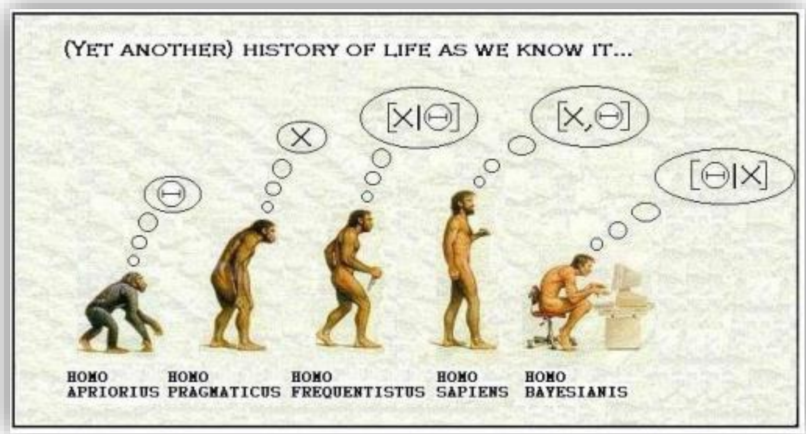
  - An uncertainty model

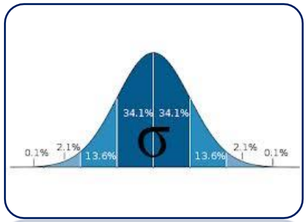


  - A propagation method



  - An inference process





Types of Uncertainty

EPISTEMIC VS. ALEATORY: WHAT IS THE DIFFERENCE?

- **Aleatory** uncertainties are non-reducible uncertainties that depend on the very nature of the phenomenon under investigation. They can generally be captured by well defined probability distributions as one can apply a frequentist approach. E.g. measurement errors.

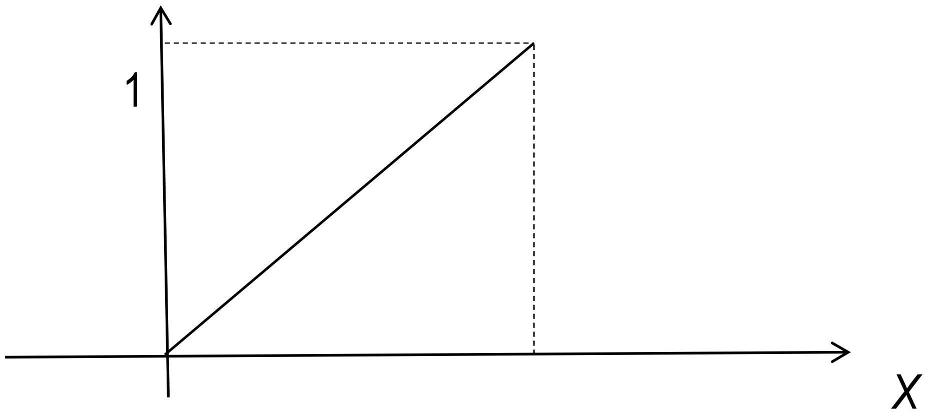
- **Epistemic** uncertainties are reducible uncertainties and are due to a lack of knowledge. Generally they cannot be quantified with a well defined probability distribution and a more subjectivist approach is required. Two classes:
  - a lack of knowledge on the distribution of the stochastic variables or…
  - a lack of knowledge of the model used to represent the phenomenon under investigation.

EPISTEMIC VS. ALEATORY: DOES IT MATTER?

- Suppose that one has no knowledge of the distribution of variable *X*.
- One might be tempted to use a uniform distribution.
- Let's compute the probability of *X* or the expectation of the indicator of *X*:

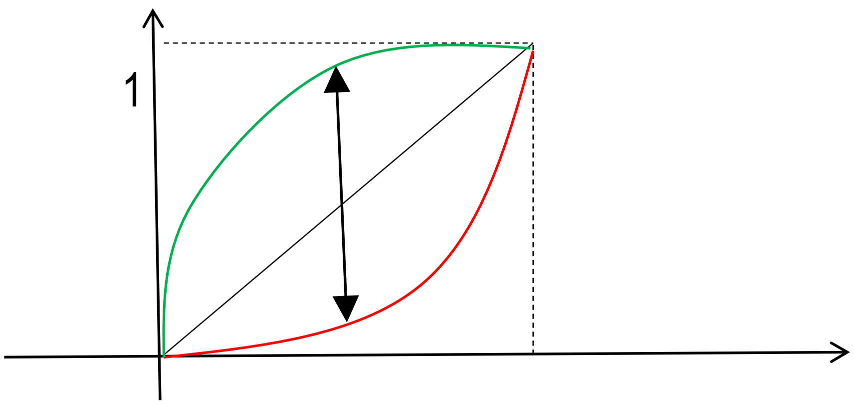
$$P_r(X < \nu) = E(I(X)) = \int_{\Omega} I(X)\,p(X)\,dX$$

- In 1D and for $p(X)$ uniform over a finite set Ω, one would get:



Epistemic vs. Aleatory: does it matter?

- Suppose now that p(X) is a family of two parameter beta distributions.
- Consider the upper and lower expectation on the same finite set:

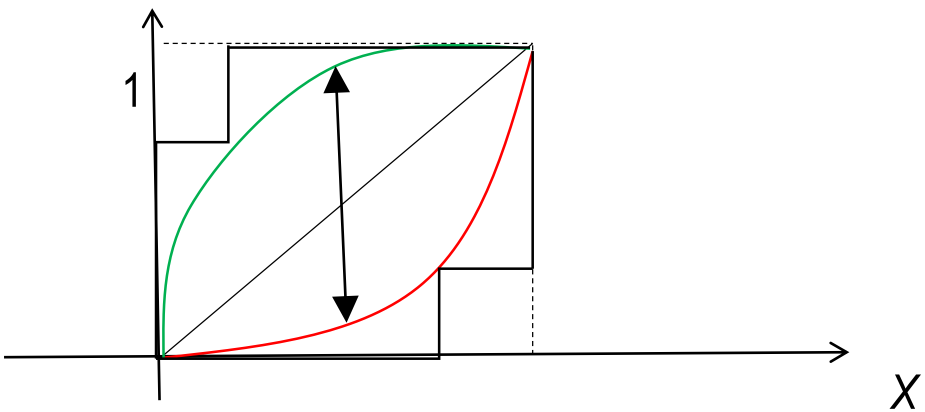


- The gap between upper and lower expectations is our degree of ignorance on the actual probability of X.

- The uniform distribution actually sits in the middle giving a very precise quantification.

EPISTEMIC VS. ALEATORY: DOES IT MATTER?

- Suppose now we have no information on the possible family of probability distributions.
- Then all we can say is if X belongs to a subset of Ω or not:



$$\max_{X \in \omega} X \leq \nu \qquad \longrightarrow \qquad \omega \subseteq \Omega$$

$$\min_{X \in \omega} X \leq \nu \qquad\qquad m(\omega): 2^{\Omega} \rightarrow [0,1]$$

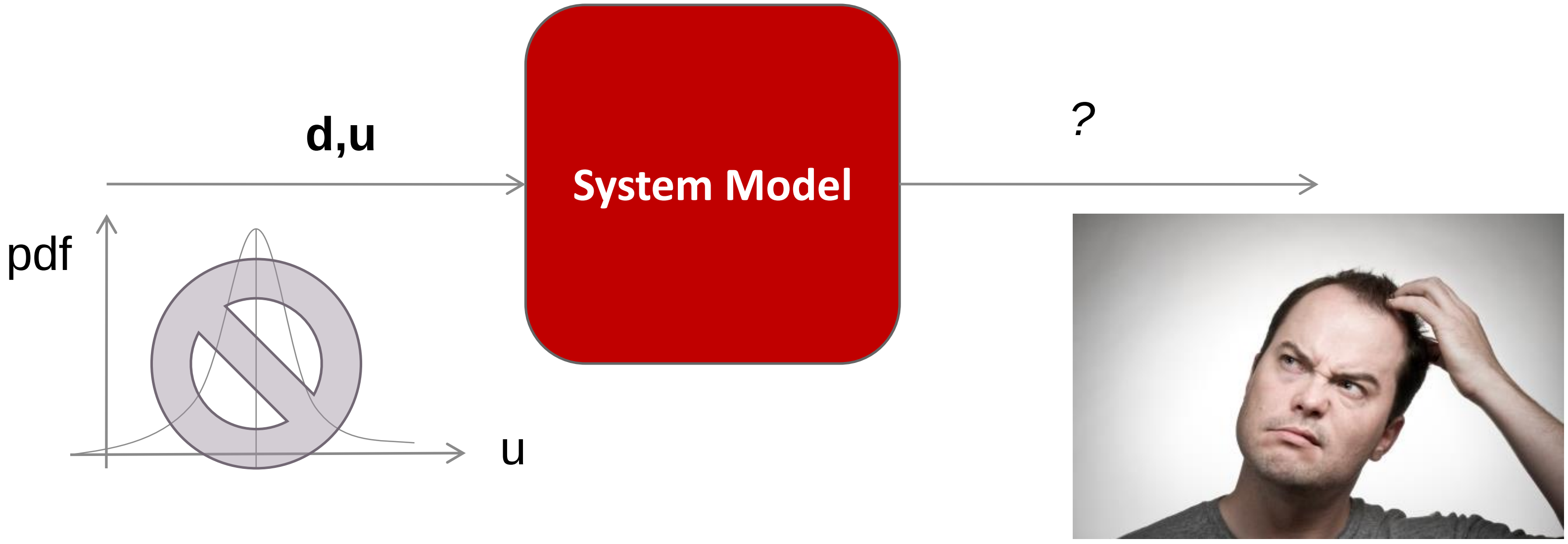
General Classification

- **Structural (or model) uncertainty** is a form of epistemic uncertainty on our ability to correctly model natural phenomena, systems or processes. If we accept that the only exact model of Nature is Nature itself, we also need to accept that every mathematical model is incomplete. One can then use an incomplete (and often much simpler and tractable) model and account for the missing components through some model uncertainty.
- **Experimental uncertainty** is aleatory. It is probably the easiest to understand and model, if enough data are available on the exact repeatability of measurements.
- **Geometric uncertainty** is a form of aleatory uncertainty on the exact repeatability of the manufacturing of parts and systems.
- **Parameter uncertainty** can be either aleatory or epistemic and refers to the variability of model parameters and boundary conditions.

General Classification

- **Numerical (or algorithmic) uncertainty**, also known as numerical errors, refers to different types of uncertainty related to each particular numerical scheme, and to the machine precision (including clock drifts).
- **Human uncertainty** is difficult to capture as it has both aleatory and epistemic elements and is dependent on our conscious and unconscious decisions and reactions. It includes the possible variability of goals and requirements due to human decisions.

Epistemic Uncertainty and Imprecision



- What is the expected value if **u** is expressed as an **opinion** without a distribution function (EPISTEMIC **uncertainty**)?

Imprecision and Multivalued Mapping

- Sets (e.g. focal elements) instead of crisp numbers:

$$\mathbf{u} \in \theta \subseteq U$$

- No a priori distribution function:

$$\Phi = \left\{ \varphi \mid \theta \in U, \underline{\varphi}(\theta) \leq \varphi(\theta) \leq \overline{\varphi}(\theta) \right\}$$

- Propositions in the form:

$$A = \left\{ \mathbf{u} \mid f(\overline{\mathbf{d}}, \mathbf{u}) < v, \overline{\mathbf{d}} \in D, \mathbf{u} \in U \right\}$$

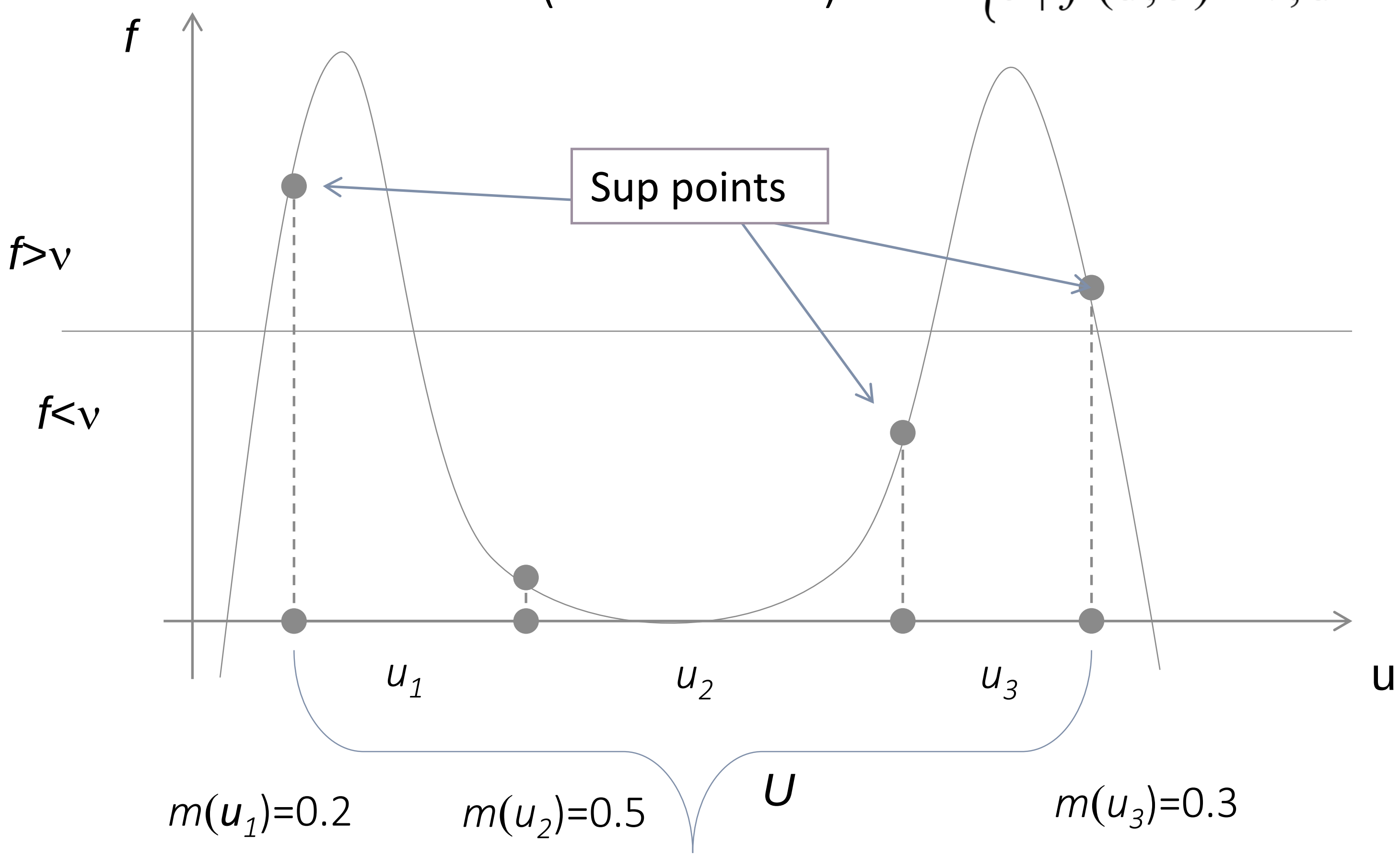
- Hence a multivalued mapping:

$$A \rightarrow \Phi$$

- Aggregation rules for conflicting and incomplete information

Simple Example with Evidence Quantification

- Given the statement (in set form): $A = \{\mathbf{u} \mid f(\bar{\mathbf{d}}, \mathbf{u}) < \nu, \bar{\mathbf{d}} \in D, \mathbf{u} \in U\}$



Simple Example with Evidence Quantification

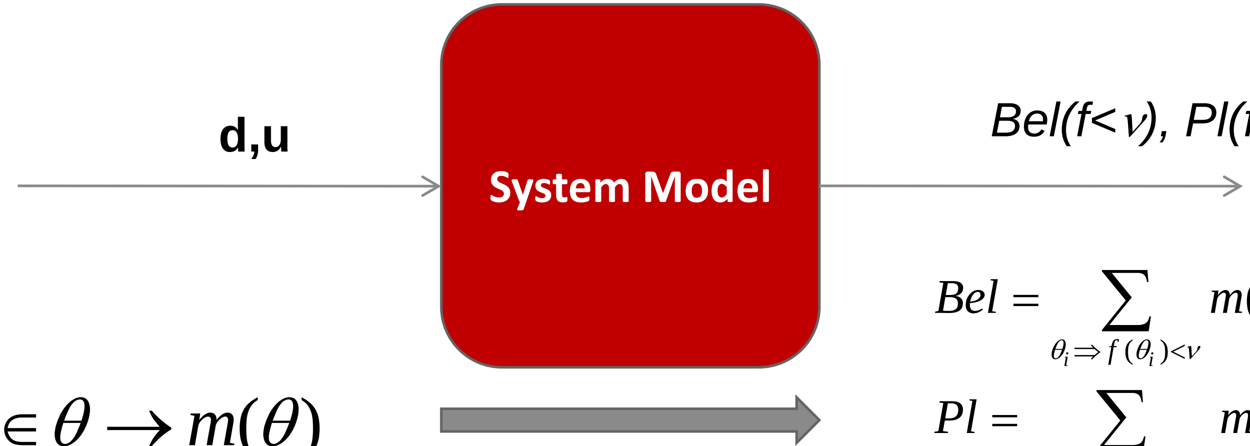
$$Bel(f<\nu) = m(u_2) = 0.5$$

$$Pl(f<\nu) = m(u_1) + m(u_2) + m(u_3) = 1$$

- The Belief *Bel* in the proposition $f<\nu$ represents the lower bound on the expectation that $f<\nu$ is true given the current information.

- The Plausibility *Pl* in the proposition $f<\nu$ represents the upper bound on the expectation that $f<\nu$ is true given the current information.

Evidence-Based Quantification



$$u \in \theta \rightarrow m(\theta)$$

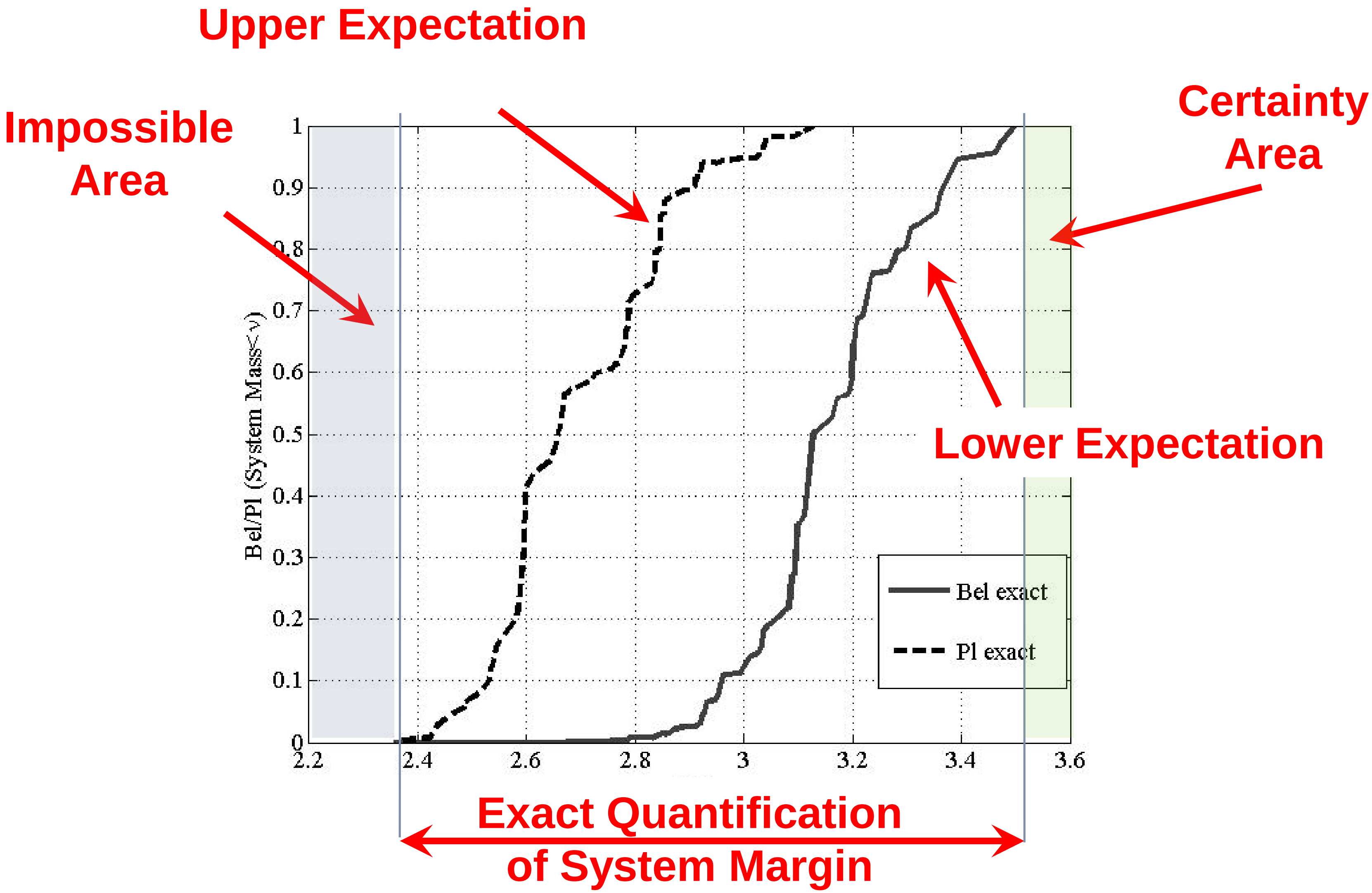
$$Bel = \sum_{\theta_i \Rightarrow f(\theta_i) < \nu} m(\theta_i)$$

$$Pl = \sum_{\theta_i \Rightarrow f(u \in \theta_i) < \nu} m(\theta_i)$$

- Both epistemic and **aleatory** uncertainty are treated in the same way and the output is the cumulative belief and plausibility given by all the pieces of evidence that support the statement:

$$f < \nu$$

Evidence-Based Quantification



Some UQ Methods

INTRUSIVE VS. NON-INTRUSIVE – WHAT DOES IT MEAN?

- Common terminology in the UQ community that fundamentally indicates two classes of algorithms/methods.

- **Intrusive methods** – the system/process model is not a black box and can be accessed to, for example, modify the algebra or compute derivatives, etc.

- **Non-intrusive methods** – the system/process model is a black box that cannot be accessed and can be interrogate only through sampling (oracle model).

NON-EXHAUSTIVE LIST TO NON-INTRUSIVE METHODS

- **Monte Carlo Sampling** - The most common and widely known.
- **Unscented Transformation** – A non-intrusive method in disguise related to orthogonal sampling methods.
- **Polynomial Chaos Expansions and Stochastic Collocation**– Popular alternatives to MCS, based on the Karhunen–Loève theorem.
- **Gaussian Mixture** Representation – Related to Kernel based approaches it represents complex distributions with a sum of basic Kernels
- **High Dimensional Model Representation** – Decomposition approach to reduce the dimensionality of the problem
- **Chebyshev Interpolation** – Example of interpolation approach

NON-EXHAUSTIVE LIST OF INTRUSIVE METHODS

- **Taylor expansion** of the quantity of interest – simple expansion of the quantity of interest through automatic differentiation or analytical derivatives.
- **State Transition Matrix** – first order method related to Taylor expansions of the quantity of interest to the first order.
- **State Transition Tensor**– higher order method related to Taylor expansions of the quantity of interest to the higher orders.
- **Intrusive PCEs** – embedding of the Polynomial Chaos Expansion in the system model and propagation through operations among polynomials.
- **Taylor Algebra** – similar to intrusive PCEs with real algebra replaced by operations among Taylor polynomials.
- **Generalised Algebra** - similar to intrusive PCEs with real algebra replaced by operations among general polynomials.

LINEAR VS. NON-LINEAR – WHAT DOES IT MEAN?

- We distinguish between linear approximation of the equations of motion and linear approximation of the distribution

- **Linear approximation of the equations of motion** – the equations of motion are expanded in Taylor series and only the first order terms are retained.

$$\delta\ddot{x} = J(x,t)\delta x$$

- **Linear approximation of the distribution** – only mean and covariance are of interest.

LINEAR VS. NON-LINEAR – WHAT DOES IT MEAN?

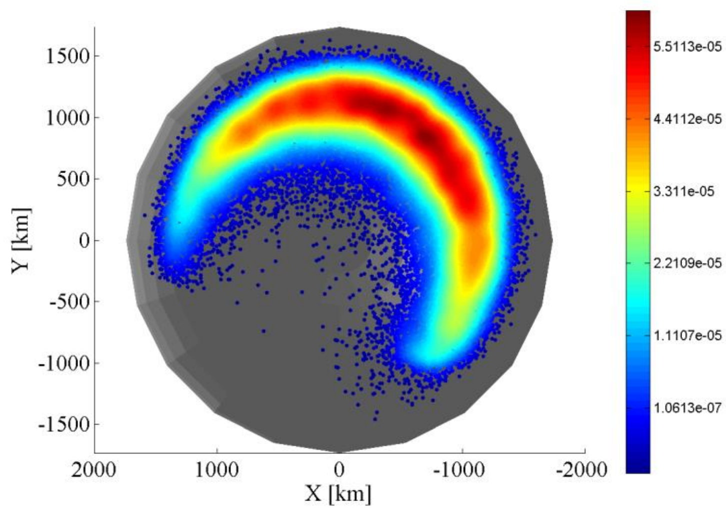
- It was demonstrated, in the second half of the '90 and more recently, that a change of formulation from Cartesian to orbital elements can recover the quasi-linearity of the motion.

- Junkins, J. L., Akella, M. R., and Alfriend, K. T. "Non-Gaussian Error Propagation in Orbital Mechanics." Journal of Astronautical Sciences, Vol. 44, No. 4, pp. 541–563, OctoberDecember 1996
- J. M. Aristoff, J. T. Horwood, N. Singh, and A. B. Poore, "Nonlinear uncertainty propagation in orbital elements and transformation to Cartesian space without loss of realism," in Proceedings of the 2014 AAS/AIAA Astrodynamics Specialist Conference, San Diego, CA, August 2014 (Paper AIAA-2014-4167)

- These methods require a reparameterisation of the equations of motion, typically in the form of orbital elements, and then uses a linear distribution model.

Probability distribution or Spatial distribution?

- There is a difference between the spatial distribution of the quantity of interest and the distribution of the probability mass.
- For example, the spatial distribution of particles in the configuration space can be derived deterministically by propagation of the initial conditions but does not say what the probability is that a given particle is at a given location.



Impact probability on the Moon for LPO disposal (Vetrisano and Vasile 2013)

METHODS BASED ON GLOBAL QUANTITIES

- One could borrow from statistical mechanics using for example Boltzmann equation.
- Nazarenko in 1992 proposed to study the evolution of the density of particles assuming a continuous distribution:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \sum_{k=1}^{N} \psi_k + n_+ + n_-$$

- In later work, in 1997, he introduced the dependency on the orbital elements and associated probability distribution functions.

- The probability of an event is simply the integral over a given control volume.
- In recent times other authors followed a similar approach, see Colombo et al. 2015 for example.

Monte Carlo Simulations (MC)
(not an uncertainty propagation method)

- Build a significant statistics by collecting a **sufficient number** of outcomes of the simulations. Commonly used to solve multidimensional integrals.
- By the central limit theorem, the expectation E of a random variable X belongs with probability ε to:

$$E(X) \in \left[\bar{X}_n - \frac{c\bar{\sigma}_n}{\sqrt{n}}, \bar{X}_n + \frac{c\bar{\sigma}_n}{\sqrt{n}} \right] \qquad \varepsilon = \frac{1}{2\pi} \int_{-c}^{c} e^{-\frac{x^2}{2}} dx$$

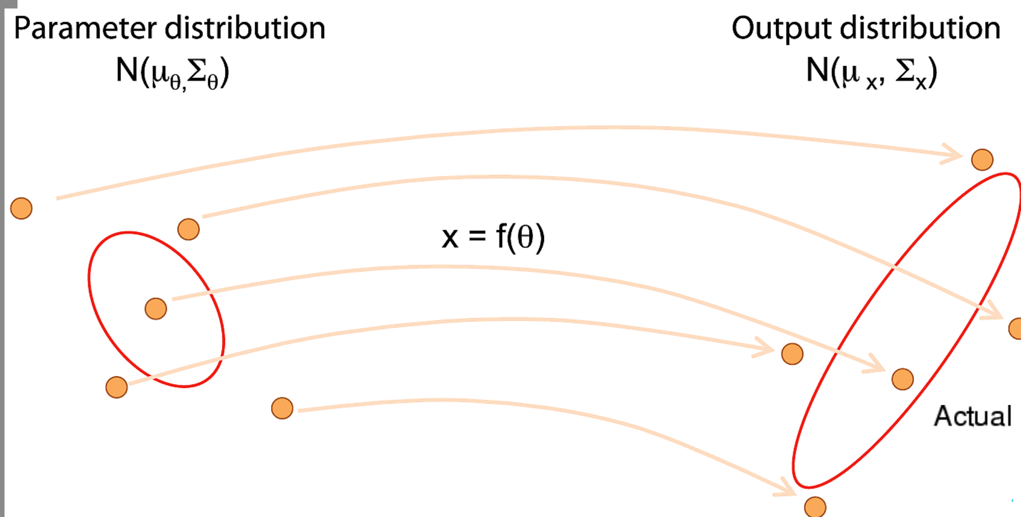
- with

$$\bar{X}_n = \frac{1}{n} \sum_i^n X_i$$

$$\bar{\sigma}_n^2 = \frac{1}{n-1} \sum_{i=1}^n \left(X_i - \bar{X}_n \right)^2$$

- This does not say a) how the distribution converges and b) if the distribution is unimodal
- The mean value might not ‘exist’!
- The hypothesis on the generation of the samples is very important!!!!
- Yields the spatial distribution and the probability distribution.

Unscented Transformation

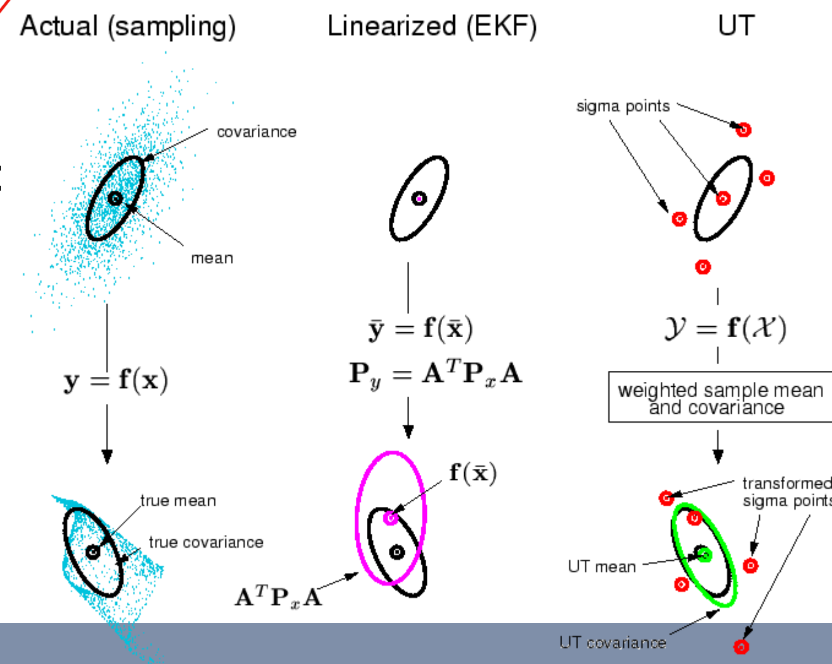


$$\boldsymbol{\chi}_i = \begin{cases} \tilde{\mathbf{x}}_k & i = 0 \\ \tilde{\mathbf{x}}_k + \left(\sqrt{(n+\kappa_{ukf})\mathbf{P}_k + \mathbf{Q}_k}\right)_i & i = 1,2,\ldots,n \\ \tilde{\mathbf{x}}_k - \left(\sqrt{(n+\kappa_{ukf})\mathbf{P}_k + \mathbf{Q}_k}\right)_i & i = n+1,\ldots,2n \end{cases}$$

Prior distribution of states and measurements:

$$\boldsymbol{\chi}_{i,k+1} = f(\boldsymbol{\chi}_{i,k}, \mathbf{u}_k)$$

$$\mathbf{Y}_i = h\left(\boldsymbol{\chi}_{i,k}\right)$$



Unscented Transformation and UQ

- Builds the covariance matrix of state and measurements assuming a known covariance of process **Q** and measurement **R** noise (linear Bayesian model hypothesis) .

$$\tilde{\mathbf{x}}_k^- = \sum_{i=0}^{2n} W_i^{(m)} \boldsymbol{\chi}_{k|k-1}^i$$

$$\mathbf{P}_k^- = \sum_{i=0}^{2n} W_i^{(c)} \left[\boldsymbol{\chi}_{k|k-1}^i - \tilde{\mathbf{x}}_k^- \right] \left[\boldsymbol{\chi}_{k|k-1}^i - \tilde{\mathbf{x}}_k^- \right]^T + \mathbf{Q}_k$$

$$\mathbf{Y}_{k|k-1}^i = h\left(\boldsymbol{\chi}_{k|k-1}^i \right) \longrightarrow \tilde{\mathbf{y}}_k^- = \sum_{i=0}^{2n} W_i^{(m)} \mathbf{Y}_{k|k-1}^i$$

$$\mathbf{P}_{y,k} = \sum_{i=0}^{2n} W_i^{(c)} \left[\mathbf{Y}_{k|k-1}^i - \tilde{\mathbf{y}}_k^- \right] \left[\mathbf{Y}_{k|k-1}^i - \tilde{\mathbf{y}}_k^- \right]^T + \mathbf{R}_k$$

Unscented Transformation and UQ

- Cross correlation of states and measurements and builds the posterior estimation based on the new measurement **y**.

$$\mathbf{P}_{xy,k} = \sum_{i=0}^{2n} W_i^{(c)} \left[\boldsymbol{\chi}^{i}_{k|k-1} - \tilde{\mathbf{x}}_k^- \right] \left[\mathbf{Y}^{i}_{k|k-1} - \tilde{\mathbf{y}}_k^- \right]^T$$

- State estimation: $\tilde{\mathbf{x}}_k = \tilde{\mathbf{x}}_k^- + \mathbf{K}\left(\mathbf{y}_k - \tilde{\mathbf{y}}_k^-\right) \longleftarrow \mathbf{K} = \mathbf{P}_{xy,k}\mathbf{P}_{y,k}^{-1}$
- Posterior distribution (UNCERTAINTY):

$$(\mathbf{P}_k^+)^{-1} = (\mathbf{P}_k^-)^{-1} + (\mathbf{P}_k^-)^{-1}\mathbf{P}_{xy,k}\mathbf{R}_k^{-1}[(\mathbf{P}_k^-)^{-1}\mathbf{P}_{xy,k}]^T - \vartheta_k \mathbf{I}_d$$

- Max estimated uncertainty on the covariance

$$\vartheta_k^{-1} = \xi \max\left(eig\left((\mathbf{P}_k^-)^{-1} + (\mathbf{P}_k^-)^{-1}\mathbf{P}_{xy,k}\mathbf{R}_k^{-1}[(\mathbf{P}_k^-)^{-1}\mathbf{P}_{xy,k}]^T \right)^{-1} \right)$$

Polynomial Chaos Expansion

- Response function representation on the quantity of interest:

$$R = a_0 B_0 + \sum_{i_1=1}^{\infty} a_{i_1} B_1(\chi_{i_1}) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2} B_2(\chi_{i_1},\chi_{i_2}) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3} B_3(\chi_{i_1},\chi_{i_2},\chi_{i_3}) + \ldots = \sum_{j=0}^{P} \alpha_j \Psi_j(\boldsymbol{\chi})$$

- Basis functions chosen to represent the input distribution:

$$B_n(\chi_{i_1},\chi_{i_2},\ldots,\chi_{i_n}) = e^{\frac{1}{2}\boldsymbol{\chi}^T\boldsymbol{\chi}}(-1)^n \frac{\partial^n}{\chi_{i_1},\ldots,\chi_{i_n}} e^{-\frac{1}{2}\boldsymbol{\chi}^T\boldsymbol{\chi}}$$

- The coefficients can be recovered with a least square approach or exploiting the orthogonality of the basis functions:

$$\alpha_j = \frac{\langle R, \Psi_j \rangle}{\langle \Psi_j^2 \rangle} = \frac{1}{\langle \Psi_j^2 \rangle} \int_\Omega \mathbf{R}\boldsymbol{\Psi}_j \rho(\boldsymbol{\chi}) d\boldsymbol{\chi} \qquad \int_\Omega R\Psi_j \rho(\chi) d\chi \cong \sum_{i=1}^{ngrid} R(\xi_i)\Psi_j(\chi_i) w(\chi_i)$$

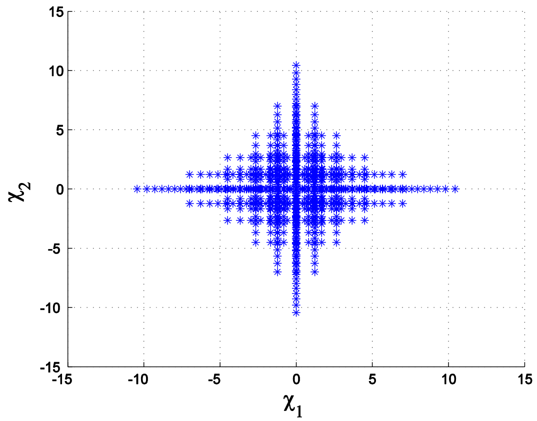
- Analytical expressions of statistical moments:

$$\boldsymbol{\mu}_R = \langle \mathbf{R} \rangle \cong \sum_{j=0}^{P} \boldsymbol{\alpha}_j \langle \Psi_j \rangle = \boldsymbol{\alpha}_0$$

$$\mathbf{P}_R = \langle (\mathbf{R} - \boldsymbol{\mu}_R)^2 \rangle \cong \sum_{j=0}^{P} \boldsymbol{\alpha}_j (\boldsymbol{\alpha}_j)^T \langle \Psi_j^2 \rangle$$

Polynomial Chaos Expansion

- Different ways to reduce the number of samples required to calculate the coefficients.
- Smolyak sparse grids to approximate integrals:



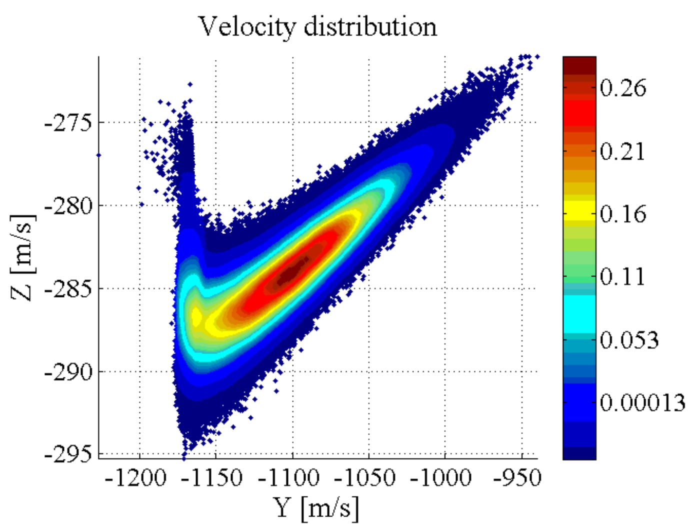
$$\int_{\Omega} R\Psi_j \rho(\chi) d\chi \cong \sum_{i=1}^{ngrid} R(\xi_i)\Psi_j(\chi_i) w(\chi_i)$$

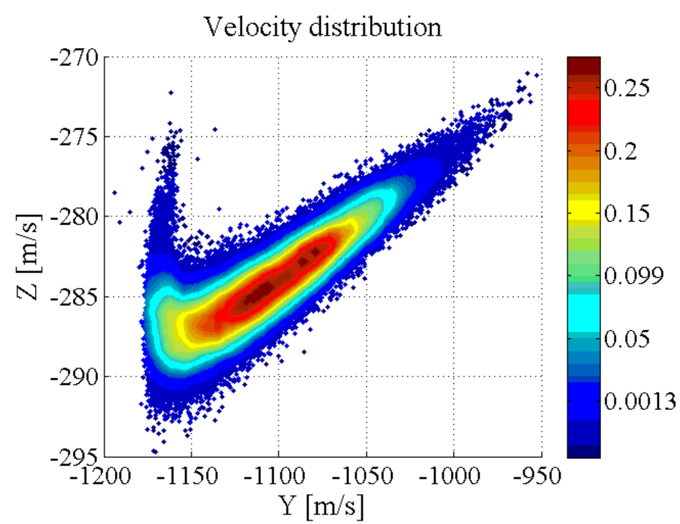
- Compressive Sampling is another option to reduce the number of samples (Jons et all 2015).

EXAMPLE: DISPOSAL TRAJECTORY FROM L2 TO THE MOON

Vetrisano and Vasile, ASR 2016 Analysis of Spacecraft Disposal Solutions from LPO to the Moon with High Order Polynomial Expansions

- Trajectory from L2 of the Earth-Sun system to the Moon in a full ephemerides model





- Monte Carlo Simulation with 1e6 samples vs. PCE degree 6 with 26,000 samples.

Gaussian Mixture

- Introduced by Garmier et al. and by Terejanu et al. in 2008 for uncertainty propagation was then developed further by Giza et al. and De Mars et al. with specific application to space debris.
- The idea is to represent the distribution of the quantity of interest with a weighted sum of Gaussians:

$$p(\mathbf{x}_{k+1}, t_{k+1}) = \sum_{i=1}^{N} w_{k+1}^{i} N(\mathbf{x}_{k+1} \mid \mu_{k+1}^{i}, \mathbf{P}_{k+1}^{i})$$

- The covariance and mean value are recovered from the updating step of an Unscented Kalman Filter.

FROM GAUSSIAN MIXTURE TO KRIGING MODELS

- One can use a weighted sum of Kernels to build a surrogate of the PDF of the quantity of interest using a Kriging type of approach.

$$z(x_{k+1}, t_{k+1}) = z_{k+1}^0 + \sum_{i=1}^{N} a_{k+1}^i e^{-\sum_{l=1}^{d} \theta_l \left|x - x_{k+1}\right|^{p_l}}$$

$$\max -\frac{n}{2}\log\left(\sigma_{k+1}^2\right) - \frac{1}{2}\log\left(\left|\frac{\mathbf{P}_{k+1}}{\sigma_{k+1}^2}\right|\right)$$

- The hyper-parameters of the Kriging model are then derived from the solution of a maximum likelihood problem:

HDMR

- HDMR allows for a direct cheap reconstruction of the quantity of interest and for analyses similar to an ANOVA (Analysis Of Variance) decomposition.
- HDMR decomposes the function response, *f*(x), in a sum of the contributions given by each variable and each one of their interactions through the model.

- If one considers the contribution of each variable as a variation with respect to an anchored value f_c (anchored-HDMR) then the decomposition becomes:

$$f(x) = f_c + \sum_{i}^{n} dF_i + \sum_{1 \le i < j \le n} dF_{ij} + \cdots + dF_{1 \dots n}$$

- Important point:

As for PCE the decomposition allows for the identification of the interdependency among variables and the order of the dependency of the quantity of interest on the uncertain parameters

INTRUSIVE POLYNOMIAL CHAOS EXPANSIONS

- Embed the Polynomial Chaos Expansion in the differential equations:

$$\frac{dy}{dt} = -py$$

$$y = \sum_{i=0}^{n} y_i \Phi_i ; p = \sum_{i=0}^{n} p_i \Phi_i$$

- After embedding the expansion in the differential equations one gets:

$$\sum_{i=0}^{n} \frac{dy_i}{dt} \Phi_i = -\sum_{i=0}^{n} \sum_{j=0}^{n} \Phi_i \Phi_j p_i y_j$$

- We multiply times Φ_l and exploit the orthogonality of the basis with the probability distribution. The result is n differential equations to be integrated:

Yields the spatial distribution and the probability distribution

State Transition Tensor

- The **local** dynamics described by applying a Taylor series expansion

- $$\delta \mathbf{x}(t) = \phi(t, \mathbf{x}_0 + \delta \mathbf{x}_0; t_0) - \phi(t, \mathbf{x}_0; t_0)$$ Φ solution flow flow from t_0 to t.

- State transition tensors STT are the higher-order partials of the solution

$$\delta x^i(t) = \sum_p^s \frac{1}{p!} \phi_{(t,t_0)}^{i,\gamma_1 \cdots \gamma_p} \delta x_0^{\gamma_1} \cdots \delta x_0^{\gamma_p} \qquad \phi_{(t,t_0)}^{i,\gamma_1 \cdots \gamma_p}(t; \mathbf{x}_0; t_0) = \left. \frac{\partial^p \phi_{(t,t_0)}^i (t; \boldsymbol{\xi}_0; t_0)}{\partial \xi_0^{\gamma_1} \cdots \partial \xi_0^{\gamma_p}} \right|_{\xi_0^{\gamma_j} = x_0^{\gamma_j}}$$

- Set of non-linear dynamics equations for STT (order 3)

$$\begin{aligned}
\dot{\phi}^{i,a} &= f^{i,\alpha} \phi^{\alpha,a} \\
\dot{\phi}^{i,ab} &= f^{i,\alpha} \phi^{\alpha,ab} + f^{i,\alpha\beta} \phi^{\alpha,a} \phi^{\beta,b} \\
\dot{\phi}^{i,abc} &= f^{i,\alpha} \phi^{\alpha,abc} + f^{i,\alpha\beta} (\phi^{\alpha,a} \phi^{\beta,bc} \\
&\quad + \phi^{\alpha,ab} \phi^{\beta,c} + \phi^{\alpha,ac} \phi^{\beta,b}) + f^{i,\alpha\beta\delta} \phi^{\alpha,a} \phi^{\beta,b} \phi^{\delta,c}
\end{aligned}$$

$$f^{i,\gamma_1 \cdots \gamma_p} = \left. \frac{\partial^p f^i (t; \boldsymbol{\xi}_0; t_0)}{\partial \xi_0^{\gamma_1} \cdots \partial \xi_0^{\gamma_p}} \right|_{\xi_0^{\gamma_j} = x_0^{\gamma_j}}$$

- Analytical expressions for mean **m** and covariance matrix **P** for Gaussian distribution

$$\vartheta(\mathbf{u}) = E[e^{j\mathbf{u}^T \mathbf{x}}] = \exp\left(j\mathbf{u}^T \mathbf{m} - \frac{1}{2} \mathbf{u}^T \mathbf{P} \mathbf{u} \right)$$

$$E[\mathbf{x}^{\gamma_1} \mathbf{x}^{\gamma_2} \cdots \mathbf{x}^{\gamma_p}] = j^{-p} \left. \frac{\partial^p \vartheta(\mathbf{u})}{\partial \mathbf{u}^{\gamma_1} \partial \mathbf{u}^{\gamma_2} \cdots \partial \mathbf{u}^{\gamma_p}} \right|_{\mathbf{u}=0}$$

$$\mathbf{m}_{k+1}^i = \phi^i(t_{k+1}; \mathbf{m}_k) + \delta \mathbf{m}_{k+1}^i =$$

$$\phi^i(t_{k+1}; \mathbf{m}_k) + \sum_{p=1}^s \frac{1}{p!} \phi_{(t_{k+1},t_k)}^{i,\gamma_1 \cdots \gamma_p} E[\delta x_k^{\gamma_1} \cdots \delta x_k^{\gamma_p}]$$

$$(\mathbf{P}_{k+1})^{ij} = \sum_{p=1}^s \sum_{q=1}^s \frac{1}{p!q!} \phi_{(t_{k+1},t_k)}^{i,\gamma_1 \cdots \gamma_p} \phi_{(t_{k+1},t_k)}^{i,\varsigma_1 \cdots \varsigma_q}$$

$$E[\delta x_k^{\gamma_1} \cdots \delta x_k^{\gamma_p} \delta x_k^{\varsigma_1} \cdots \delta x_k^{\varsigma_q}] - \delta m_{k+1}^i \delta m_{k+1}^j$$

POLYNOMIAL ALGEBRA

- 1982 (Epstein) Ultra Arithmetic
- 1986 (Berz) Taylor Differential Algebra
- 1997 (Berz) Taylor Models
- 2003 (Berz) Taylor Models and Other Validated Functional Inclusion Methods
- 2004 (Debusschere et al.) Intrusive PCE and Taylor expansions
- 2005-2015 (Armellin, DiLizia) application of Taylor algebra to orbital mechanics
- 2010 (Joldes) Formal comparison between Taylor, Chebyshev, Newton Models
- 2014 (Jai Rajyaguru et al.) Chebyshev models for ODEs
- 2015 (Riccardi et al.) Chebyshev polynomial expansion for orbital mechanics

Polynomial Algebra

Algebra Definition

$(\mathcal{P}_{n,d}, \otimes)$ is the Algebra on the space of Tchebycheff polynomials s. t. if $P_{f(x)}$ and $P_{g(x)}$ are the polynomial approximation of $f(x)$ and $g(x)$, respectively, then

$$P_{f(x)\oplus g(x)} = P_{f(x)} \otimes P_{g(x)} \,,$$

where $\oplus \in \{+, -, *, /\}$ and $\otimes$ is the corresponding operation in the algebra.

- $S = \dim(\mathcal{P}_{n,d}, \otimes) = \binom{n+d}{d} = \frac{(n+d)!}{n!d!}$,
- $P_{f(x)} \in (C_{n,d}, \otimes) \leftrightarrow \mathbf{c} \in \{c_{\boldsymbol{\alpha}} : |\boldsymbol{\alpha}| \le n\} \in \mathbb{R}^S$ such that

$$\mathcal{P}_{f(x)} = \sum_{\boldsymbol{\alpha}, |\boldsymbol{\alpha}| \le n} c_{\boldsymbol{\alpha}} T_{\boldsymbol{\alpha}}(x) \,.$$

Polynomial Algebra

Expansion of the flow of an autonomous ODE

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$$

Initialize x_0 as an element of the algebra

$$P_{x_0}(x_0) = (P_{x_1}(x_0), \ldots, P_{x_d}(x_0)) \in (\mathcal{P}_{n,d}, \otimes)^d \ .$$

Forward Euler scheme:

- Real Algebra: $x_1 = x_0 + dt\, f(x_0)$
- Tchebycheff Algebra: $P_{x_1}(x_0) = P_{x_0}(x_0) + dt\, f(P_{x_0}(x_0)) \in (\mathcal{P}_{n,d}, \otimes)^d$

Polynomial Expansion of the Flow

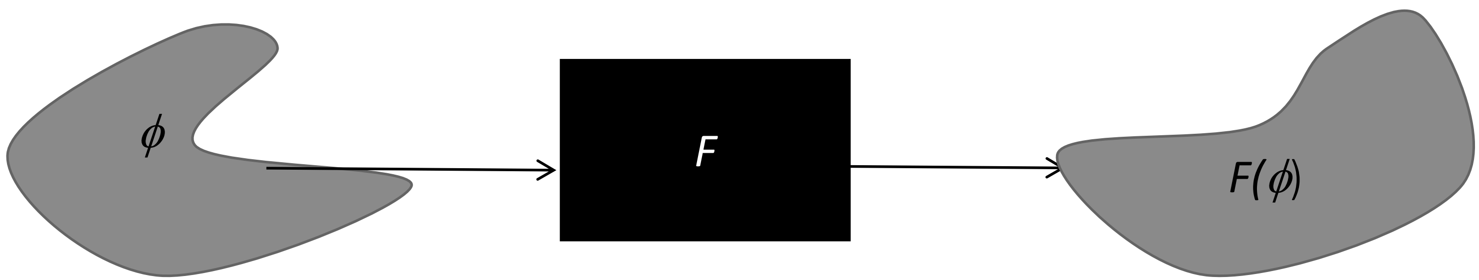
At the k-th iteration in the Tchebycheff algebra environment we have that

$$P_{x_k}(x_0) = P_{x_{k-1}}(x_0) + dt\, f(P_{x_{k-1}}(x_0)) \in (\mathcal{P}_{n,d}, \otimes)^d$$

GENERALISED POLYNOMIAL ALGEBRA

(RICCARDI, TARDIOLI, VASILE 2015)

Consider the wider class of problems, typical in Viability Theory, where a level set ϕ is propagated through a model function *F (equations of motion).*

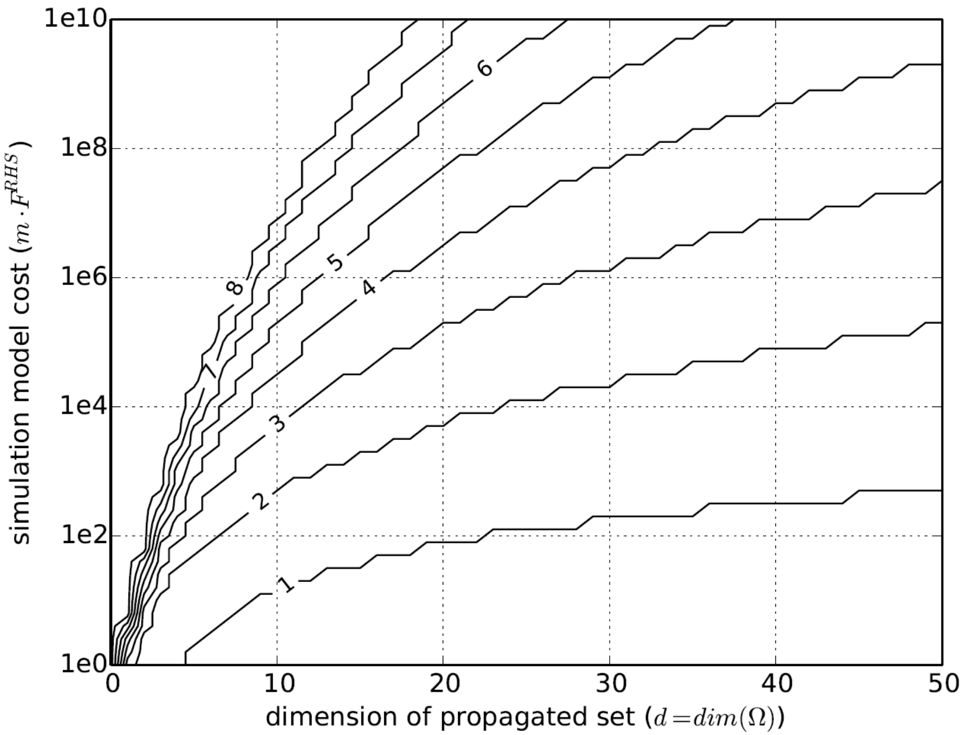


For any *n* dimensional manifold that can be represented with a polynomial expansion, one can obtain its image through *F*

Yields the spatial distribution

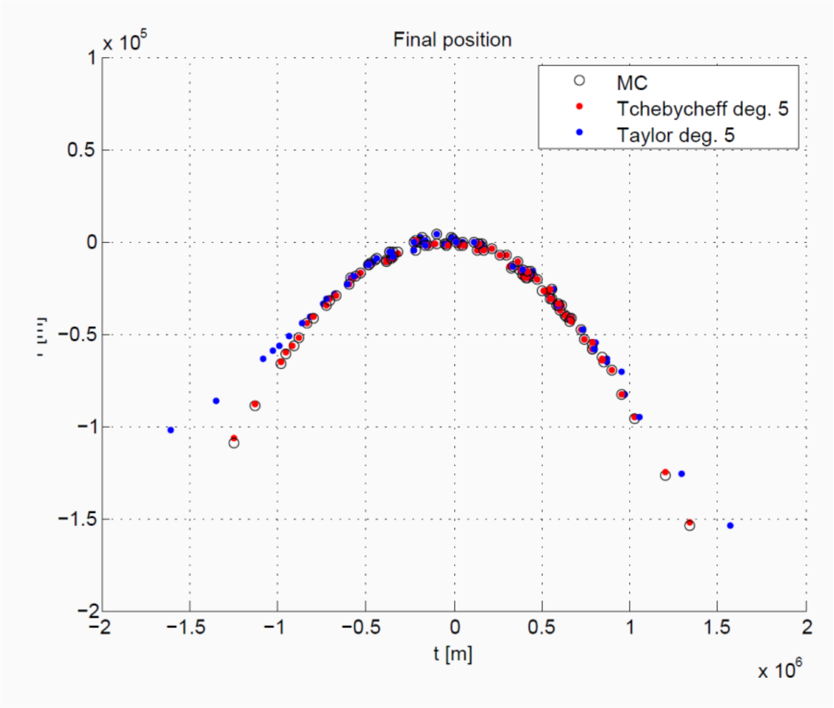
COMPUTATIONAL COMPLEXITY

- The computational complexity of an algebra compared to a non-intrusive method can be theoretically derived regardless of the implementation (Ortega, Vasile, Riccardi, Tardioli 2016).

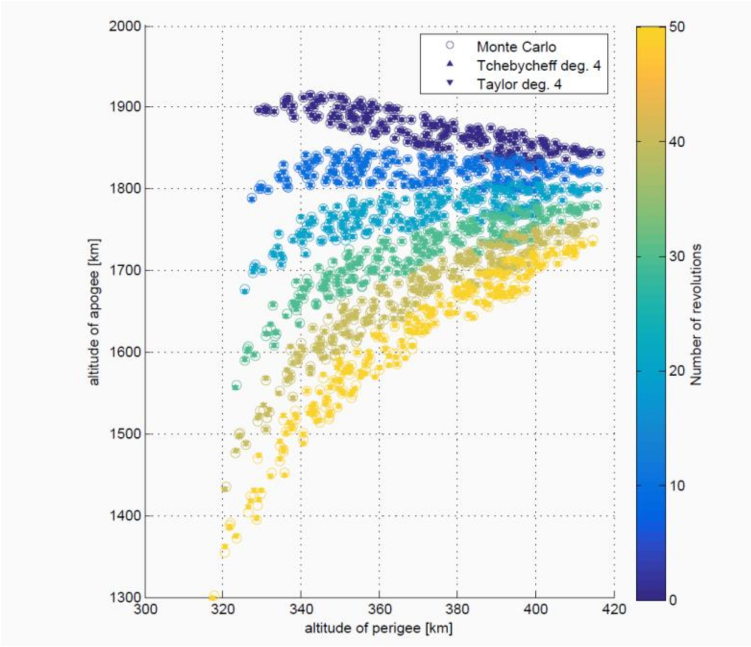


EXAMPLE: RE-ENTRY OF GOCE AND HAMR FRAGMENTS (ORTEGA, VASILE, RICCARDI, SERRA 2016)

- De-orbiting of GOCE



- Evolution of a cloud of HAMR fragments



- Single integration with the algebra vs. full MC simulation

Model Uncertainty

Dynamics with Unknown Components

- There is an underlying process υ that is dependant on the state *s* and on some unknown parameters *b*:

$$\dot{s} = f(s, p, t) + \upsilon(s, b, t)$$

- The uncertainty component υ can be expressed as a polynomial expansion of the states and of *b:*

$$Q_h(s, b) \simeq c_0 + \sum_i^{2N} c(b)_i \zeta_i(s_i) + \sum_i^{2N} \sum_j^{2N} c(b)_{ij} \zeta_{ij}(s_i, s_j) + \sum_i^{2N} \sum_j^{2N} \sum_k^{2N} c(b)_{ijk} \zeta_{ijk}(s_i, s_j, s_k) + \ldots$$

Matching Predictions

- Sparse data points are available $s_o : (\Gamma, \mathscr{L}, \mathscr{M}) \longrightarrow \mathbb{R}^n$

$$Pr(s_o \in \Sigma) > \varepsilon$$

- The problem needs to be reformulated assuming *c* are stochastic and *s* belongs to a confidence interval:

$$\begin{aligned} &\min_{c \in C} J(s, c) \\ &s.t. \\ &s(t_i) \in \Sigma \quad i = 0, \ldots, N_o \end{aligned}$$

Orbital Dynamics with Unknown Drag Component

- Let's assume that the true dynamics are:

$$
\begin{aligned}
\dot{v}_r &= -\frac{\mu}{r^2} + \frac{v_t^2}{r} - \frac{1}{2}\rho C_d v v_r \\
\dot{v}_t &= -\frac{v_t v_r}{r} - \frac{1}{2}\rho C_d v v_t \\
\dot{r} &= v_r \\
\dot{\theta} &= \frac{v_t}{r}
\end{aligned}
$$

- But the expected dynamics does not contain drag terms
- The observations however do not match the predicted state

Orbital Dynamics with Unknown Drag Component

- If the orbit has low eccentricity, a Taylor expansion of the drag terms up to the first order is telling us that the solution should be in the form:

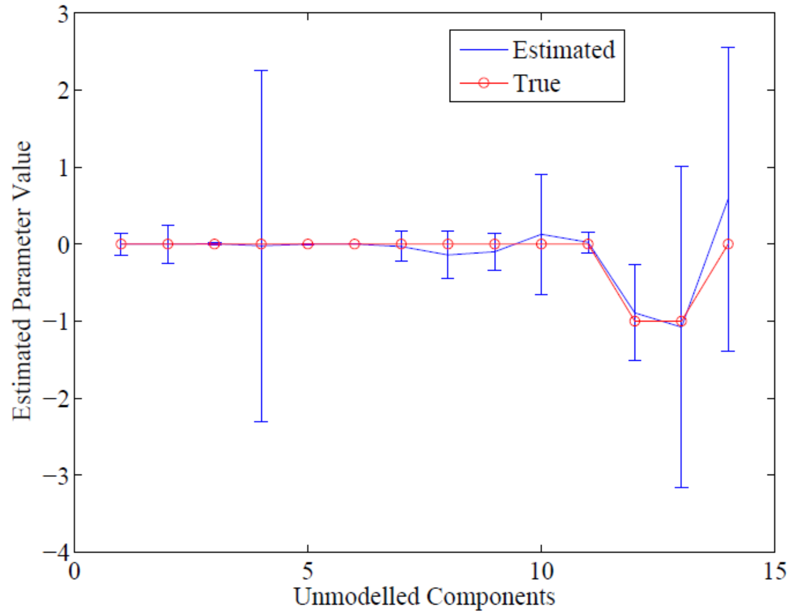
$$
\begin{aligned}
\dot{v}_r &= -\frac{\mu}{r^2} + \frac{v_t^2}{r} + c_{13} v_r v_t \\
\dot{v}_t &= -\frac{v_t v_r}{r} + c_{12} v_t^2 \\
\dot{r} &= v_r \\
\dot{\theta} &= \frac{v_t}{r}
\end{aligned}
$$

- We can then expand the uncertainty function as:

$$
\begin{aligned}
\dot{v}_r = -\frac{\mu}{r^2} + \frac{v_t^2}{r} + c_1 + c_3 r + c_5 r^2 + \\
c_7 r\theta + c_9 v_r + c_{11} v_r^2 + c_{13} v_r v_t \\
\dot{v}_t = -\frac{v_t v_r}{r} + c_2 + c_4 \theta + c_6 \theta^2 + \\
c_8 r\theta + c_{10} v_t + c_{12} v_t^2 + c_{14} v_r v_t \\
\dot{r} = v_r \\
\dot{\theta} = \frac{v_t}{r}
\end{aligned}
$$

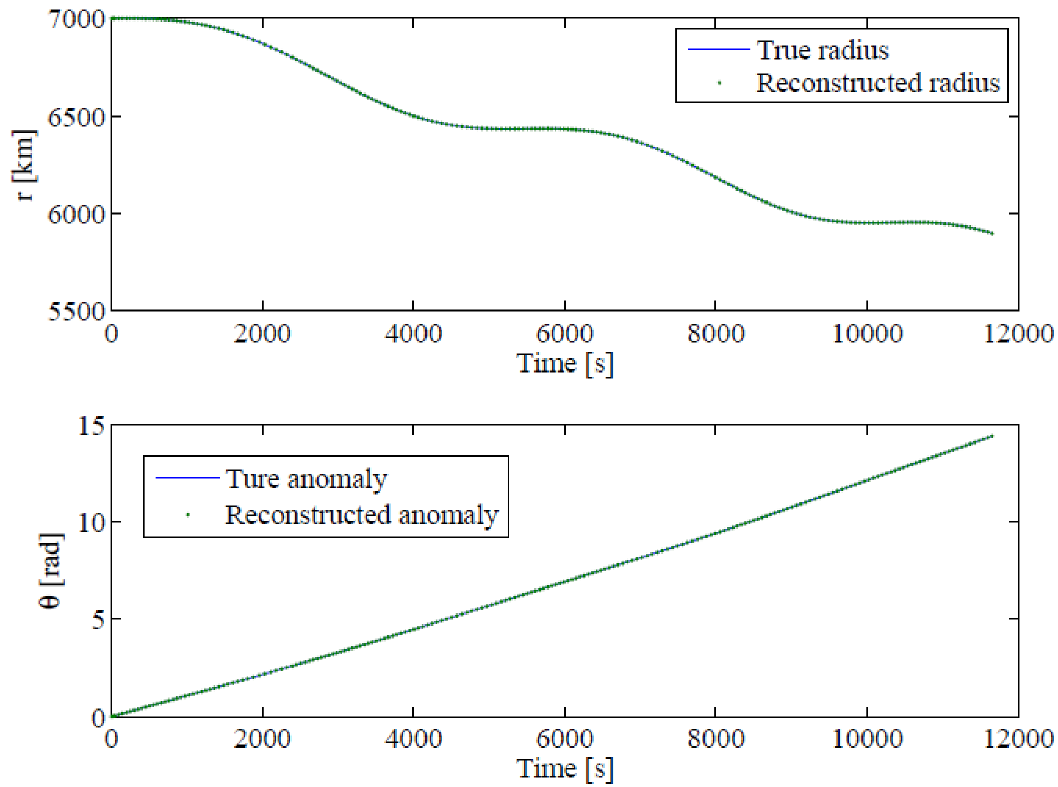
Orbital Dynamics with Unknown Drag Component

- Assume 2 sets of measurements at t=[T, T/2], for a total of 8 constraints and 14 unknowns
- We use the distance metric $$J = c^T c$$
- The initial conditions are also uncertain with uniform distribution within a confidence interval



Orbital Dynamics with Unknown Drag Component

- This estimation allows us to extrapolate the prediction over a time span that is 2 times the one over which the measurements are available



Orbital Dynamics with Unknown Drag Component

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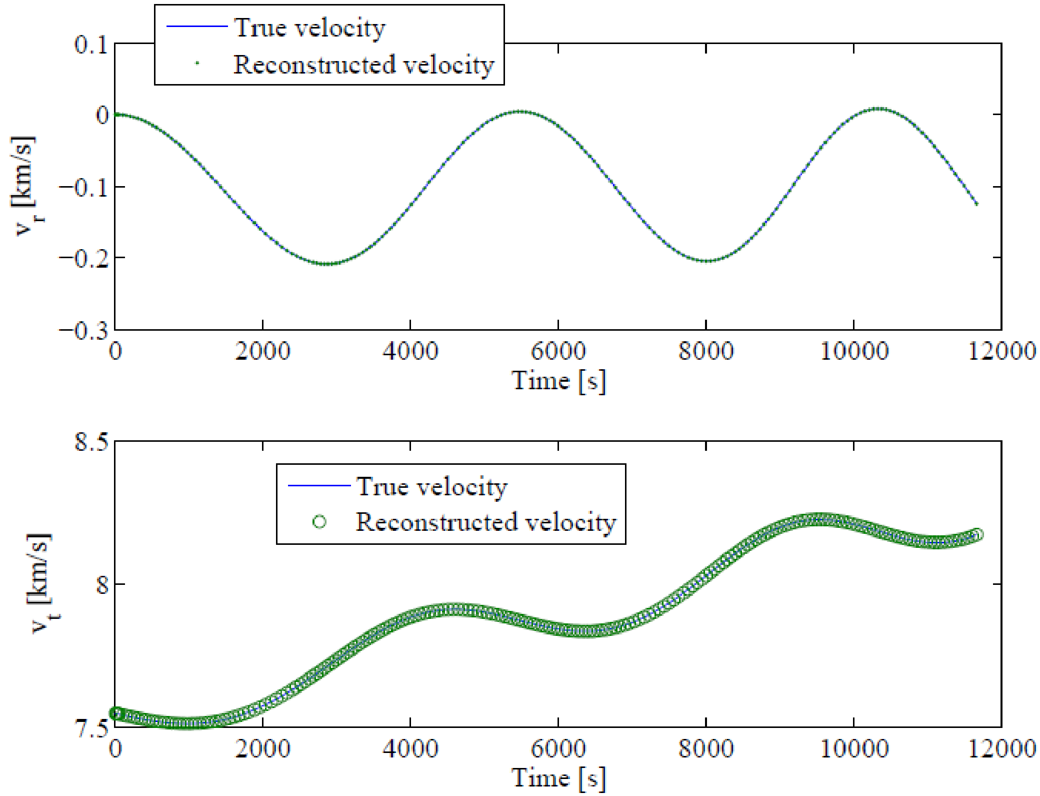

Handling the unknown at the edge of tomorrow

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http://utopiae.eu

twitter.com/utopiae_network

info@utopiae.eu