Space Debris: from LEO to GEO

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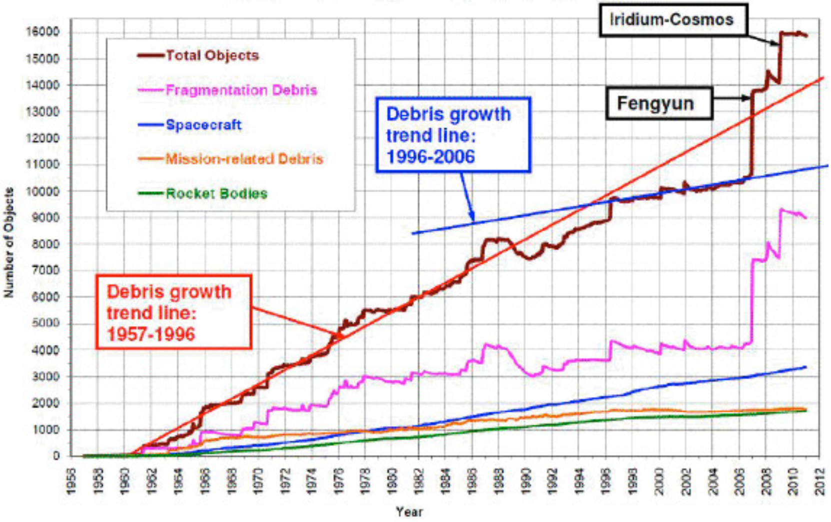
Plan

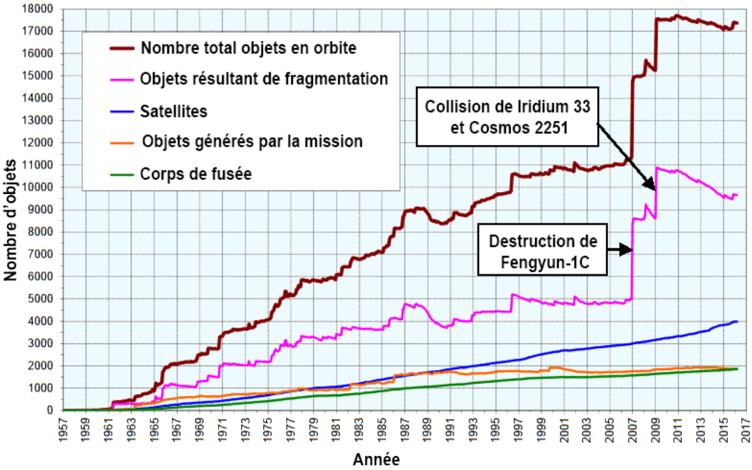
- Space debris problematic
- Forces
- Gravitational resonances
- Solar radiation pressure (SRP)
- Shadowing effects
- Lunisolar resonances
- Numerical integrations
- Chaos
- Atmospheric drag
- Other aspects : rotation, Yarkovsky, synthetic population

Post-doc : Deleflie and Casanova, and Phd : Valk, Delsate, Hubaux, Petit and Murawiecka

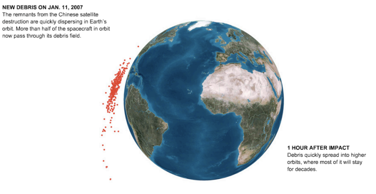
Number of debris

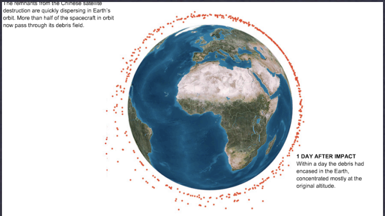


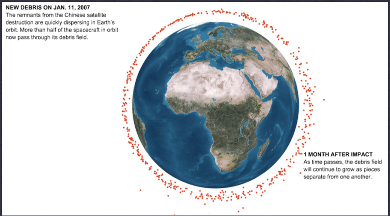
Number of debris



New York Times
Explosion of the chinese satellite
Fengyun FY-1C on January 11, 2007

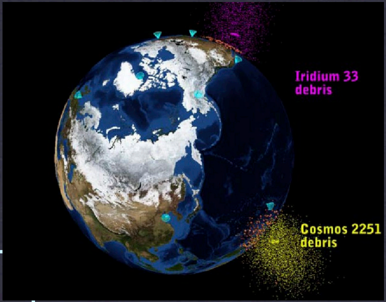






Recent collision : Cosmos - Iridium 2009

Collision



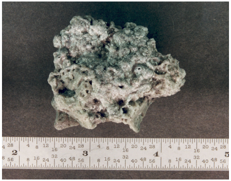
- Iridium 33 (active American telecommunication satellite)
- Cosmos 2251 (non active military Russian satellite)
- Date : February 10, 2009
- Speed : 11.7 km/second

What are Orbital Space Debris?

Definition

Orbital debris refers to material on orbit resulting from space missions but no longer serving any function.

- Launch vehicle upper stages
- Abandoned satellites
- Lens caps
- Momentum flywheels
- Core of nuclear reactors
- Objects breakup
- Paint flakes
- Solid-fuel fragments



Current debris population

- There are about 18 000 objects larger than 10 cm
  TLE Catalogue
- About 350 000 objects larger than 1 cm
- More than 3×10^8 objects larger than 1 mm

Catalogued objects (NASA)

- 6 % Operational spacecrafts
- 24% Non-operational spacecrafts
- 17% Upper stages of rockets
- 13% Mission related debris
- 40% Debris mostly generated by explosions & collisions

Computer generated images

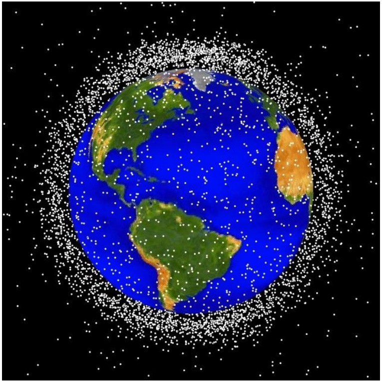


Figure: LEO image

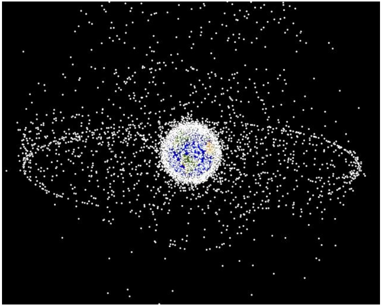
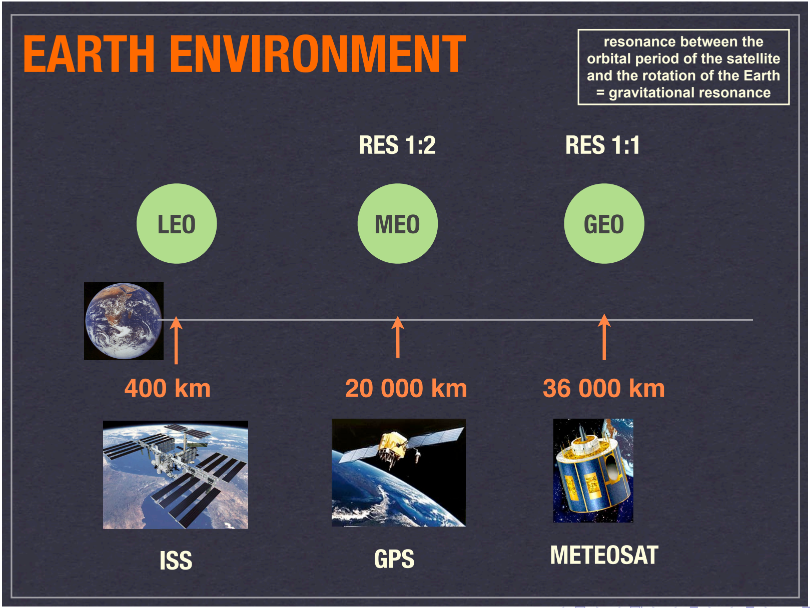


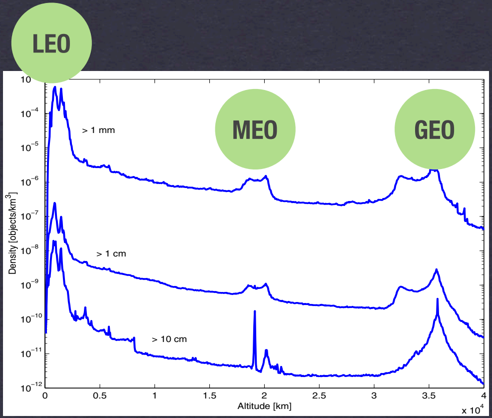
Figure: GEO image

LEO-MEO-GEO


EARTH ENVIRONMENT
resonance between the orbital period of the satellite and the rotation of the Earth = gravitational resonance
RES 1:2
RES 1:1
LEO
MEO
GEO
400 km
20 000 km
36 000 km
ISS
GPS
METEOSAT

Rossi et al (2005)

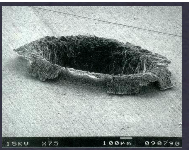
Number of debris



3 PEAKS IN DENSITY
MOST POPULAR ORBITS

Rossi et al., Proceedings of the IAU Colloquium, No. 197, 2005 and MASTER 2009

Situation and solutions



Size (r)	Characteristics	Protection	Number
r < 0.01 cm	cumulative effects surface erosion	not necessary	
0.01 < r < 1 cm	significant damages perforation	armor plating	170 000 000 objects
1 < r < 10 cm	important damages	no solution	670 000 objects
r > 10 cm	catastrophic events catalogued (TLE)	manoeuvres	< 20 000 objects

Analogy with natural bodies

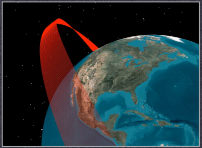
Natural and artificial objects

Natural	Artificial	Debris
existing orbits	chosen orbits	existing orbits
no control	control	no control
long times	short times	long times
model and observations	huge numerical integrations	model and observations
stability	precision	stability

Natural cleaning

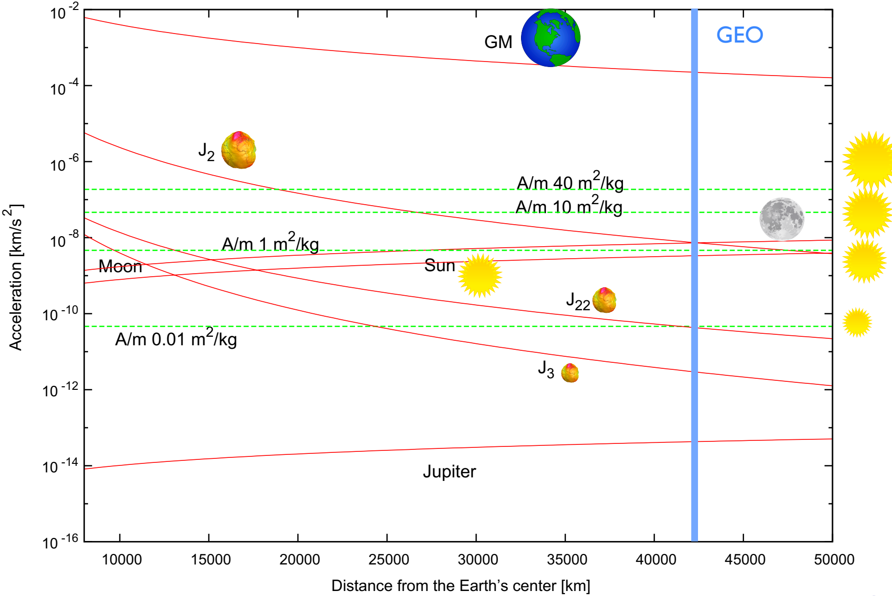
Long term dynamics

ESTIMATION OF LIFETIMES FOR USUAL OBJECTS



300 km	1 month
400 km	1 year
500 km	10 years
700 km	50 years
900 km	1 century
1200 km	1 millennium

The forces for MEO and GEO



First contribution : forces

Dynamics of a debris

= Keplerian orbit around the Earth

+ rotation of the Earth

+ shape of the Earth (geopotential - J_2)

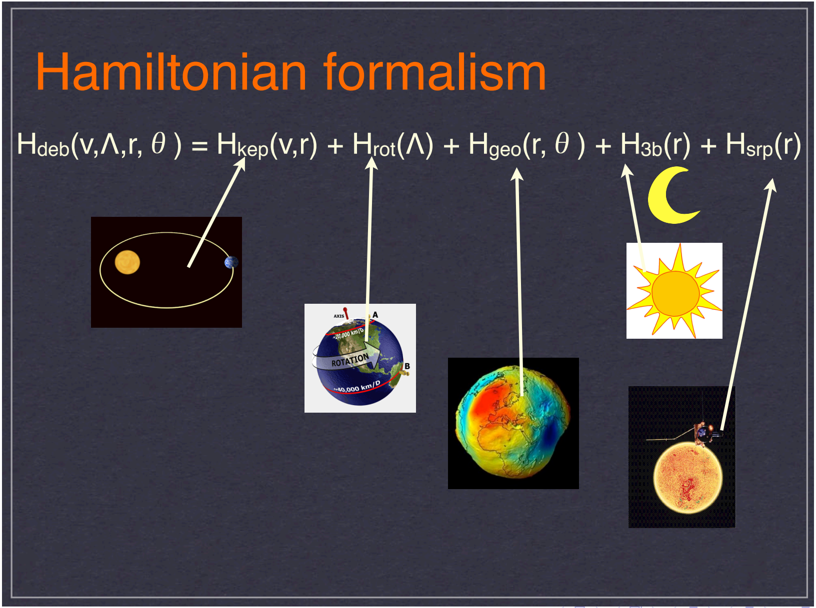
+ third body perturbations (Moon and Sun)

+ solar radiation pressure

+ shadowing effects

+ atmospheric drag (LEO) : cleaner

The Hamiltonian formulation



The geopotential

$$U(\boldsymbol{r}) = \mu \int_V \frac{\rho(\boldsymbol{r_p})}{\|\boldsymbol{r} - \boldsymbol{r_p}\|} \, dV \, , \qquad \mu = G \, m_\oplus$$

$$\begin{array}{rclcrcl} x & = & r \cos\phi \cos\lambda & & x_p & = & r_p \cos\phi_p \cos\lambda_p \\ y & = & r \cos\phi \sin\lambda & & y_p & = & r_p \cos\phi_p \sin\lambda_p \\ z & = & r \sin\phi & & z_p & = & r_p \sin\phi_p \end{array}$$

$$U(r,\lambda,\phi) = -\frac{\mu}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R_e}{r}\right)^n \mathcal{P}_n^m(\sin\phi)(C_{nm} \cos\, m\lambda + S_{nm} \sin\, m\lambda)$$

R_e : the equatorial Earth's radius

$$C_{nm} = \frac{2-\delta_{0m}}{M_\oplus} \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r_p}{R_e}\right)^n \mathcal{P}_n^m(\sin\phi_p) \cos(m\lambda_p) \, \rho(\boldsymbol{r_p}) \, dV$$

$$S_{nm} = \frac{2-\delta_{0m}}{M_\oplus} \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r_p}{R_e}\right)^n \mathcal{P}_n^m(\sin\phi_p) \sin(m\lambda_p) \, \rho(\boldsymbol{r_p}) \, dV$$

The geopotential

$$J_2 = -C_{20} = \frac{2C - B - A}{2\, M_\oplus\, R_e^2} \qquad \text{and} \qquad C_{22} = \frac{B - A}{4\, M_\oplus\, R_e^2}$$

$$U(r,\lambda,\phi) = -\frac{\mu}{r} + \frac{\mu}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{R_e}{r}\right)^n \mathcal{P}_n^m(\sin\phi)\, J_{nm} \cos m(\lambda - \lambda_{nm})$$

$$C_{nm} = -J_{nm} \cos(m\lambda_{nm})$$

$$S_{nm} = -J_{nm} \sin(m\lambda_{nm})$$

$$J_{nm} = \sqrt{C_{nm}^2 + S_{nm}^2}$$

$$m\,\lambda_{nm} = \arctan\left(\frac{-S_{nm}}{-C_{nm}}\right).$$

The geopotential: Kaula formulation

$$U = -\frac{\mu}{r} - \sum_{n=2}^{\infty} \sum_{m=0}^{n} \sum_{p=0}^{n} \sum_{q=-\infty}^{+\infty} \frac{\mu}{a} \left(\frac{R_e}{a} \right)^n F_{nmp}(i)\, G_{npq}(e)\, S_{nmpq}(\Omega, \omega, M, \theta)$$

$$\begin{aligned} S_{nmpq}(\Omega, \omega, M, \theta) &= \begin{bmatrix} +C_{nm} \\ -S_{nm} \end{bmatrix}_{n-m\,\text{odd}}^{n-m\,\text{even}} \cos \Theta_{nmpq}(\Omega, \omega, M, \theta) \\ &+ \begin{bmatrix} +S_{nm} \\ +C_{nm} \end{bmatrix}_{n-m\,\text{odd}}^{n-m\,\text{even}} \sin \Theta_{nmpq}(\Omega, \omega, M, \theta) \end{aligned}$$

Kaula gravitational argument, θ the sidereal time :

$$\Theta_{nmpq}(\Omega, \omega, M, \theta) = (n-2p)\,\omega + (n-2p+q)\,M + m(\Omega - \theta)$$

The luni-solar perturbations

The acceleration :

$$\ddot{\boldsymbol{r}} = -\mu_i \left(\frac{\boldsymbol{r} - \boldsymbol{r_i}}{\|\boldsymbol{r} - \boldsymbol{r_i}\|^3} + \frac{\boldsymbol{r_i}}{\|\boldsymbol{r_i}\|^3} \right) .$$

The potential (i=1 for the Sun, i=2 for the Moon):

$$\mathcal{R}_i = \mu_i \left(\frac{1}{\|\boldsymbol{r} - \boldsymbol{r_i}\|} - \frac{\langle \boldsymbol{r} \,.\, \boldsymbol{r_i} \rangle}{\|\boldsymbol{r_i}\|^3} \right) .$$

$$\mathcal{R}_i = \frac{\mu_i}{r_i} \sum_{n \geq 2} \left(\frac{r}{r_i} \right)^n \mathcal{P}_n(\cos\psi)$$

r_i the geocentric distance
ψ the geocentric angle between the third body and the satellite
$\mathcal{P}_n$ the Legendre polynomial of degree n.

The luni-solar perturbations

- The three components (x, y, z) of the position vector $\boldsymbol{r}$ expressed in Keplerian elements $(a, e, i, \Omega, \omega, f)$
- The Cartesian coordinates X_i, Y_i and Z_i of the unit vector pointing towards the third body.
- Usual developments of f and $\frac{r}{a}$ in series of e, $\sin\frac{i}{2}$ and M

$$\mathcal{R}_i = \frac{\mu_i}{r_i} \sum_{n=2}^{+\infty} \sum_{k,l,j_1,j_2,j_3} \left(\frac{a}{r_i}\right)^n A^{(n)}_{k,l,j_1,j_2,j_3}(X_i, Y_i, Z_i)\, e^{|k|+2j_2} \left(\sin\frac{i}{2}\right)^{|l|+2j_3} \cos\Phi$$

$$\Phi = j_1\,\lambda + j_2\,\varpi + j_3\,\Omega, \quad \lambda = M + \omega + \Omega, \quad \varpi = \omega + M$$

Poincaré variables

Delaunay canonical momenta associated with λ, ϖ and Ω :

$$L = \sqrt{\mu\, a}, \qquad G = \sqrt{\mu\, a(1 - e^2)}\,, \qquad H = \sqrt{\mu\, a(1 - e^2)}\, \cos i$$

Non singular Delaunay elements, keeping L and λ :

$$\begin{array}{ll} P = L - G & \qquad p = -\omega - \Omega \\ Q = G - H & \qquad q = -\Omega \end{array}$$

Poincaré variables :

$$\begin{array}{ll} x_1 = \sqrt{2P}\, \sin p & \qquad x_4 = \sqrt{2P}\, \cos p \\ x_2 = \sqrt{2Q}\, \sin q & \qquad x_5 = \sqrt{2Q}\, \cos q \\ x_3 = \lambda = M + \Omega + \omega & \qquad x_6 = L \end{array}$$

Dimensionless Poincaré variables

$$U = \sqrt{\frac{2P}{L}} \qquad V = \sqrt{\frac{2Q}{L}}$$

$$e = U\left(1 - \frac{U^2}{4}\right)^{\frac{1}{2}} = U - \frac{1}{8}U^3 - \frac{1}{128}U^5 + \mathcal{O}(U^7)$$

$$2\,\sin\frac{i}{2} = V\left[1 - \frac{U^2}{2}\right]^{-\frac{1}{2}} = V + \frac{1}{4}\,VU^2 + \frac{3}{32}\,VU^4 + \mathcal{O}(U^6)$$

Non canonical dimensionless cartesian coordinates

$$\xi_1 = U\,\sin p \qquad \eta_1 = U\,\cos p$$
$$\xi_2 = V\,\sin q \qquad \eta_2 = V\,\cos q$$

Hamiltonian

$$
\begin{aligned}
\mathcal{H}_{pot} &= \mathcal{H}_{2b} + \dot{\theta}\,\Lambda + \sum_{n=2}^{n_{max}} \mathcal{R}_{pot}^{(n)} + \sum_{i=1}^{2} \mathcal{H}_i \\
&= -\frac{\mu^2}{2\,L^2} + \dot{\theta}\,\Lambda + \sum_{n=2}^{n_{max}} \frac{1}{L^{2n+2}} \sum_{j=1}^{N_n} \mathcal{A}_j^{(n)}(\xi_1, \eta_1, \xi_2, \eta_2)\, \mathcal{B}_j^{(n)}(\lambda, \theta) \\
&+ \sum_{i=1}^{2} \sum_{n=2}^{n_{max}} \frac{L^{2n}}{r_i^{n+1}} \sum_{j=1}^{N_n} \mathcal{C}_j^{(n)}(\xi_1, \eta_1, \xi_2, \eta_2, X_i, Y_i, Z_i)\, \mathcal{D}_j^{(n)}(\lambda)
\end{aligned}
$$

Dynamical system

$$
\dot{\xi}_i = \frac{1}{L}\frac{\partial \mathcal{H}}{\partial \eta_i} \qquad \dot{\eta}_i = -\frac{1}{L}\frac{\partial \mathcal{H}}{\partial \xi_i} \qquad i = 1, 2
$$

$$
\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial L} - \frac{1}{2L}\left[\sum_{i=1}^{2} \frac{\partial \mathcal{H}}{\partial \xi_i}\xi_i + \sum_{i=1}^{2} \frac{\partial \mathcal{H}}{\partial \eta_i}\eta_i\right] \qquad \dot{L} = -\frac{\partial \mathcal{H}}{\partial \lambda}
$$

Semi-analytical averaged method

- Use of a series manipulator

λ	θ	ξ_1	η_1	ξ_2	η_2	L	X	Y	Z	r	$X_\odot$	$Y_\odot$	$Z_\odot$	$r_\odot$	Coefficient
cos (0	0)	(0	0	0	0	-6	0	0	0	0	0	0	0	0)	0.12386619D-04
cos (0	0)	(0	0	0	2	-6	0	0	0	0	0	0	0	0)	-0.18579928D-04
cos (0	0)	(0	0	0	4	-6	0	0	0	0	0	0	0	0)	0.46449822D-05

- Averaging process over the fast variable : λ
- Semi-analytical averaged solution

Number of terms

Perturbation	Number of terms			
n-order expansion				
$\xi_1^{i_1}\,\eta_1^{i_2}\,\xi_2^{i_3}\,\eta_2^{i_4}$ with $i_1 + i_2 + i_3 + i_4 \leq n$	$n = 2$	$n = 4$	$n = 6$	$n = 8$
Geopotential				
$\mathcal{H}_{J_2}$	5	15	31	53
	(33)	(145)	(410)	(895)
External Body - Sun & Moon				
up to degree 2	27	86	197	390
	(205)	(836)	(2374)	(5480)
up to degree 3	73	250	611	1227
	(645)	(2642)	(7854)	(18380)

See also STELA (Deleflie - CNRS)

The geopotential: Kaula formulation

$$U = -\frac{\mu}{r} - \sum_{n=2}^{\infty} \sum_{m=0}^{n} \sum_{p=0}^{n} \sum_{q=-\infty}^{+\infty} \frac{\mu}{a} \left(\frac{R_e}{a}\right)^n F_{nmp}(i)\, G_{npq}(e)\, S_{nmpq}(\Omega, \omega, M, \theta)$$

$$\begin{aligned} S_{nmpq}(\Omega, \omega, M, \theta) &= \begin{bmatrix} +C_{nm} \\ -S_{nm} \end{bmatrix}_{n-m \text{ odd}}^{n-m \text{ even}} \cos \Theta_{nmpq}(\Omega, \omega, M, \theta) \\ &+ \begin{bmatrix} +S_{nm} \\ +C_{nm} \end{bmatrix}_{n-m \text{ odd}}^{n-m \text{ even}} \sin \Theta_{nmpq}(\Omega, \omega, M, \theta) \end{aligned}$$

Kaula gravitational argument, θ the sidereal time :

$$\Theta_{nmpq}(\Omega, \omega, M, \theta) = (n-2p)\,\omega + (n-2p+q)\,M + m(\Omega - \theta)$$

Gravitational resonances : resonances with the Earth rotation

- $\frac{P_\oplus}{P_{obj}} = \frac{q_1}{q_2}$
- $P_\oplus$: Earth's rotational period : $2\pi/n_\oplus = 1$ day ($n_\oplus = \dot\theta$)
- P_{obj} : body orbital period : $2\pi/n = P_{obj}$ day ($n = \dot M$)
- 1/1 for GEO and 2/1 for MEO
- $\Theta_{nmpq}(\Omega, \omega, M, \theta) = (n-2p)\,\omega + (n-2p+q)\,M + m(\Omega - \theta)$
- $\dot\Theta_{nmpq}(\dot\Omega, \dot\omega, \dot M, \dot\theta) = (n-2p)\,\dot\omega + (n-2p+q)\,\dot M + m(\dot\Omega - \dot\theta) \simeq 0$
- $q = 0$: $\frac{\dot M}{\dot\theta} \simeq \frac{\dot\lambda}{\dot\theta} \simeq \frac{q_1}{q_2}$
- Resonant Hamiltonian $\mathcal{H}_{J_{22}}$

Geostationary model of resonance

- Cartesian Hamiltonian coordinates for e, i, ϖ, Ω : ξ_i and η_i
- $\mathcal{H} = \mathcal{H}_{J_{22}}(\xi_1, \eta_1, \xi_2, \eta_2, \Lambda, \lambda, L, \theta) + \dot{\theta}\,\Lambda$
- Resonant angle : $\sigma = \lambda - \theta$
- Corrected momentum : $L' = L, \qquad \theta' = \theta, \qquad \Lambda' = \Lambda + L$
- $\mathcal{H} = \mathcal{H}_{J_{22}}\left(\xi_1, \eta_1, \xi_2, \eta_2, \sigma, L', \theta\right) + \dot{\theta}\,\left(\Lambda' - L'\right)$

Resonant averaging

$$\mathcal{H}_{J_{22}}\left(\xi_1, \eta_1, \xi_2, \eta_2, L, \Lambda, \theta, \lambda\right)$$
$$\downarrow$$
$$\mathcal{H}_{J_{22}}\left(\xi_1, \eta_1, \xi_2, \eta_2, L', \Lambda', \theta', \sigma\right)$$
$$\downarrow$$
$$\overline{\mathcal{H}}_{J_{22}}\left(\bar{\xi}_1, \bar{\eta}_1, \bar{\xi}_2, \bar{\eta}_2, \bar{L}', \bar{\Lambda}', -, \bar{\sigma}\right)$$

Resonant averaged hamiltonian

Perturbation	Number of terms			
n-order expansion				
$\xi_1^{i_1}\eta_1^{i_2}\xi_2^{i_3}\eta_2^{i_4}$ with $i_1+i_2+i_3+i_4 \leq n$	$n=2$	$n=4$	$n=6$	$n=8$
Resonant disturbing function				
$\mathcal{H}_{J_{22}} = \mathcal{H}_{C_{22}} + \mathcal{H}_{S_{22}}$	10	40	104	206
	(94)	(468)	(1392)	(3178)

σ	θ	ξ_1	η_1	ξ_2	η_2	L	X	Y	Z	r	$X_\odot$	$Y_\odot$	$Z_\odot$	$r_\odot$	Coefficient
cos (2	0)	(0	0	0	0	-6	0	0	0	0	0	0	0	0)	0.1077767255D-06
cos (2	0)	(0	0	0	0	-6	0	0	0	0	0	0	0	0)	0.1080907167D-06
sin (2	0)	(0	0	0	0	-6	0	0	0	0	0	0	0	0)	-0.6204881922D-07

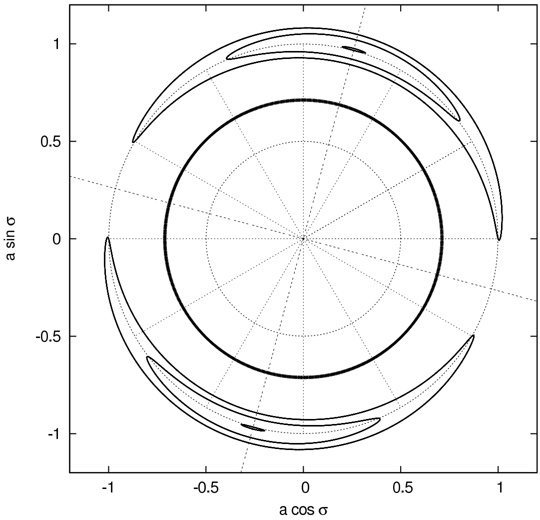
Simple resonant model

- $\mathcal{H}(L, \sigma, \Lambda) = -\frac{\mu^2}{2L^2} + \dot{\theta}(\Lambda - L) + \frac{1}{L^6}\left[\alpha_1 \cos 2\sigma + \alpha_2 \sin 2\sigma\right]$
- $\alpha_1 \simeq 0.1077 \times 10^{-6}, \qquad \alpha_2 \simeq -0.6204 \times 10^{-7}$
- Equilibria : $\frac{\partial \mathcal{H}}{\partial L} = 0 = \frac{\partial \mathcal{H}}{\partial \sigma}$
- Two stable equilibria $(\sigma^*_{11}, L^*_{11})$, $(\sigma^*_{12}, L^*_{12})$
- Two unstable equilibria $(\sigma^*_{21}, L^*_{21})$, $(\sigma^*_{22}, L^*_{22})$ are found to

$$\sigma^*_{11} = \lambda^* \qquad \sigma^*_{12} = \lambda^* + \pi$$
$$\sigma^*_{21} = \lambda^* + \frac{\pi}{2} \qquad \sigma^*_{22} = \lambda^* + \frac{3\pi}{2},$$

- $L^*_{11} = L^*_{12} = 0.99999971, \qquad L^*_{21} = L^*_{22} = 1.00000029,$
- $L = 1$ corresponds to 42 164 km.
- $\lambda^* \simeq 75.07°$

Resonant phase space



Resonant period

- $x = \sqrt{2L}\cos\sigma$, $y = \sqrt{2L}\sin\sigma$ and consequently x^*, y^*.
- Taylor series around (x^*, y^*)
- $X = (x - x^*), \qquad Y = (y - y^*)$
- $\mathcal{H}^*(X, Y, \Lambda) = \dot{\theta}\,\Lambda + \frac{1}{2}(aX^2 + 2bXY + cY^2) + \cdots$
- Rotation : $X = \ p\cos\Psi + q\sin\Psi$ and $Y = -p\sin\Psi + q\cos\Psi$
- Choice of Ψ : $(a - c)\sin 2\Psi + 2b\cos 2\Psi = 0$
- $\mathcal{H}^*(p, q, \Lambda) = \dot{\theta}\,\Lambda + \frac{1}{2}\left[A\,p^2 + C\,q^2\right]$
- Scaling : $p = \alpha\,p'$ and $q = \frac{1}{\alpha}\,q'$ by $A\,\alpha^2 = \dfrac{C}{\alpha^2}$,
- $\mathcal{H}(J, \phi, \Lambda) = \dot{\theta}\,\Lambda + \sqrt{AC}\,J$
- Action-angle (J, ϕ) : $p' = \sqrt{2J}\cos\phi\,, \quad q' = \sqrt{2J}\cos\phi\,.$
- $\nu_f = \frac{\partial\mathcal{H}}{\partial J} = \sqrt{AC} = 7.674 \times 10^{-3}/d$, period of 818.7 days.

Resonant motion

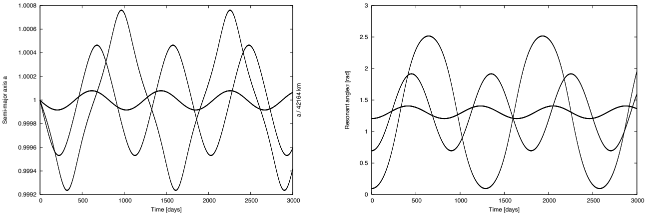


Fig. 6. Semi-major axis a [left] and resonant angle $\sigma = \lambda - \theta$ [right] of several geosynchronous space debris [$a_0 = 42164\ km$, $e_0 = 0$, $i_0 = 0$] the initial longitude of which are $\lambda_0 = 5°, 35°, 75°$.

Resonant motion

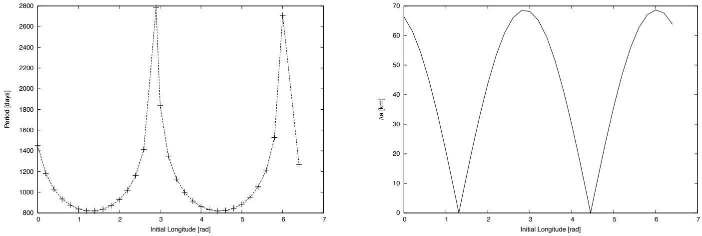


Fig. 7. Libration periods of 32 virtual space debris the initial longitude λ_0 of which varied from 0 to 2π.

Width of the resonant zone

- Hamiltonian level curve corresponding to one of the unstable equilibria L_u and σ_u

$$\mathcal{H}(L_u, \sigma_u, \Lambda) = -\frac{\mu^2}{2L^2} + \dot{\theta}(\Lambda - L) + \frac{1}{L^6}\left[\alpha_1 \cos 2\sigma + \alpha_2 \sin 2\sigma\right]$$

- Maxima and minima of this “banana curve”, corresponding to the stable equilibria
- Quadratic approximation about L_u : the width Δ of the resonant zone

$$\Delta = \sqrt{\frac{\gamma^2 + 8\delta\beta}{\beta^2}} \quad \delta = \frac{\alpha_1}{L_u^6 \cos 2\sigma_u} \quad \beta = -\frac{3}{2}\frac{\mu^2}{L_u^4} \quad \gamma = \frac{\mu^2}{L_u^3} - \dot{\theta}$$

- The numerical value is of the order of 69 km.

Generalization

- Similar approach : Rossi on MEO (resonance 2:1) CM&DA
- Paper of Celletti and Gales : On the Dynamics of Space Debris: 1:1 and 2:1 Resonances (JNS) 2014
- Very complete paper :

Celest Mech Dyn Astr (2015) 123:203–222
DOI 10.1007/s10569-015-9636-1

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ORIGINAL ARTICLE

Dynamical investigation of minor resonances for space debris

Alessandra Celletti[1] · Cătălin Galeş[2]

Resonant motion

Table 2 Value of the semimajor axis corresponding to several resonances

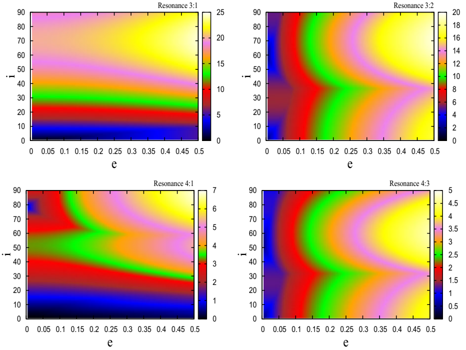
$j:\ell$	a (km)	$j:\ell$	a (km)
1:1	42164.2	4:3	34805.8
2:1	26561.8	5:1	14419.9
3:1	20270.4	5:2	22890.2
3:2	32177.3	5:3	29994.7
4:1	16732.9	5:4	36336

Resonant motion

Table 3 Terms whose sum provides the expression of $R_{earth}^{res\ j:\ell}$ up to the order N

$j : \ell$	N	Terms
3:1	4	$\mathcal{T}_{330\text{-}2}, \mathcal{T}_{3310}, \mathcal{T}_{3322}, \mathcal{T}_{431\text{-}1}, \mathcal{T}_{4321}$
3:2	4	$\mathcal{T}_{330\text{-}1}, \mathcal{T}_{3311}, \mathcal{T}_{430\text{-}2}, \mathcal{T}_{4310}, \mathcal{T}_{4322}$
4:1	6	$\mathcal{T}_{441\text{-}1}, \mathcal{T}_{4421}, \mathcal{T}_{541\text{-}2}, \mathcal{T}_{5420}, \mathcal{T}_{5432}, \mathcal{T}_{642\text{-}1}, \mathcal{T}_{6431}$
4:3	5	$\mathcal{T}_{440\text{-}1}, \mathcal{T}_{4411}, \mathcal{T}_{540\text{-}2}, \mathcal{T}_{5410}, \mathcal{T}_{5422}$
5:1	6	$\mathcal{T}_{551\text{-}2}, \mathcal{T}_{5520}, \mathcal{T}_{5532}, \mathcal{T}_{652\text{-}1}, \mathcal{T}_{6531}$
5:2	6	$\mathcal{T}_{551\text{-}1}, \mathcal{T}_{5521}, \mathcal{T}_{651\text{-}2}, \mathcal{T}_{6520}, \mathcal{T}_{6532}$
5:3	6	$\mathcal{T}_{550\text{-}2}, \mathcal{T}_{5510}, \mathcal{T}_{5522}, \mathcal{T}_{651\text{-}1}, \mathcal{T}_{6521}$
5:4	6	$\mathcal{T}_{550\text{-}1}, \mathcal{T}_{5511}, \mathcal{T}_{650\text{-}2}, \mathcal{T}_{6510}, \mathcal{T}_{6522}$

Fig. 2 The amplitude of the resonances for different values of the eccentricity (within 0 and 0.5 on the x axis) and the inclination (within $0°$ and $90°$ on the y axis) for $\omega = 0°$, $\Omega = 0°$; the *color bar* provides the measure of the amplitude in kilometers. In order from *top left* to *bottom right*: 3:1, 3:2, 4:1, 4:3, 5:1, 5:2, 5:3, 5:4



Solar Radiation pressure

- Solar radiation pressure is a quite complicated force with different components
- *Theory of Orbit determination* : Milani and Gronchi - ch 14
- *New solar Radiation Pressure Force Model for navigation* : McMahon and Scheeres - 2010
- Direct radiation pressure acceleration
- Starting point : simplified models

Solar Radiation pressure with high A/M

Scheeres and Rosengren : Averaged model, based on e and angular momentum

Long-term Dynamics of HAMR Objects in HEO

Aaron Rosengren,* Daniel Scheeres†
University of Colorado at Boulder, Boulder, CO 80309

Gachet, Celletti, Pucacco, Efthymiopoulos : Complete perturbation theory with planetary motion

Celest Mech Dyn Astr (2017) 128:149–181
DOI 10.1007/s10569-016-9746-4
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ORIGINAL ARTICLE

Geostationary secular dynamics revisited: application to high area-to-mass ratio objects

Fabien Gachet[1] · Alessandra Celletti[1] · Giuseppe Pucacco[3] · Christos Efthymiopoulos[2]

Direct radiation pressure acceleration

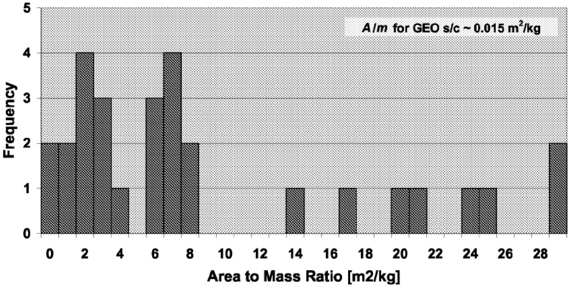
The acceleration due to the direct radiation pressure can be written in the form:

$$\mathbf{a_{rp}} = C_r \, P_r \left[\frac{a_\odot}{\|\mathbf{r} - \mathbf{r}_\odot\|} \right]^2 \frac{A}{m} \, \frac{\mathbf{r} - \mathbf{r}_\odot}{\|\mathbf{r} - \mathbf{r}_\odot\|},$$

- C_r is the non-dimensional reflectivity coefficient $(0 < C_r < 2)$,
- $P_r = 4.56 \cdot 10^{-6} \; N/m^2$ is the radiation pressure per unit of mass for an object located at a distance of $a_\odot = 1 \; AU$,
- $\mathbf{r}$ is the geocentric position of the space debris; $\mathbf{r}_\odot$ is the geocentric position of the Sun,
- A is the exposed area to the Sun of the space debris,
- m is the mass of the space debris.

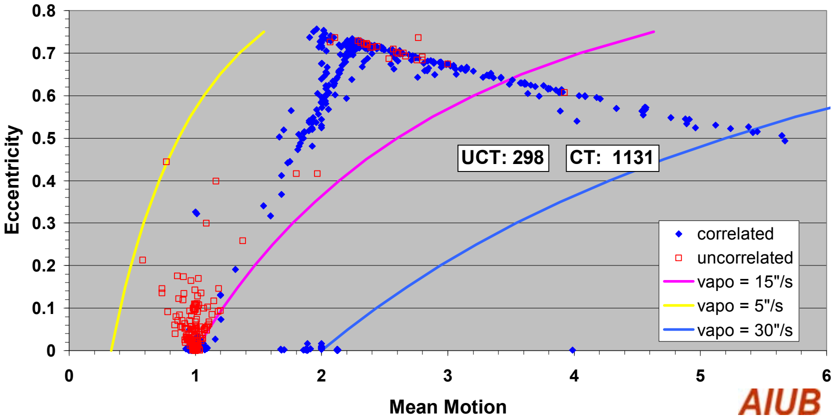
Non-gravitational influence

A/m distribution



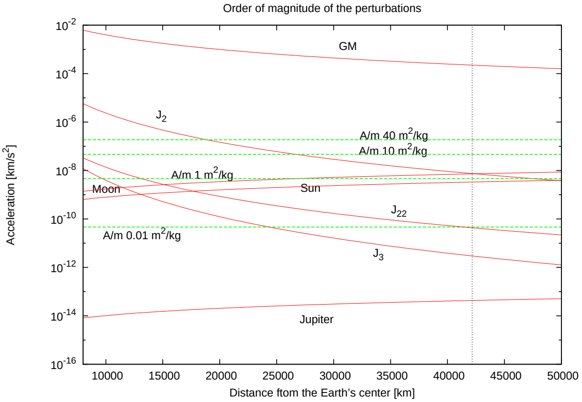
Object	A/m m^2/kg
Lageos 1 and 2	0.0007
Starlette	0.001
GPS (Block II)	0.02
Moon	$1.3 \cdot 10^{-10}$
Space debris	$0 < A/m <$?

GEO debris with very high eccentricity



Schildknecht et al, 2010

Order of magnitude of radiation pressure



Chao 2009

Hamiltonian formulation

$$\mathcal{H}(\mathbf{v},\mathbf{r}) = \mathcal{H}_{kepl}(\mathbf{v},\mathbf{r}) + \mathcal{H}_{srp}(\mathbf{r})$$

fixed inertial equatorial geocentric frame

$\mathbf{r}$	$=$	geocentric position of the satellite
$\mathbf{v}$	$=$	velocity of the satellite
$\mathcal{H}_{kepl}(\mathbf{v},\mathbf{r})$	$=$	attraction of the Earth
$\mathcal{H}_{srp}(\mathbf{r})$	$=$	direct solar radiation pressure potential

$$\mathcal{H}_{kepl} = \frac{\|\mathbf{v}\|^2}{2} - \frac{\mu}{\|\mathbf{r}\|}$$

$$\mathcal{H}_{srp} = -C_r \frac{1}{\|\mathbf{r}-\mathbf{r}_\odot\|} P_r \frac{A}{m} a_\odot^2$$

$\mu = \mathcal{G}M_\oplus$, $C_r \simeq 1$, $\mathbf{r}_\odot$ position of the Sun, $P_r = 4.56 \times 10^{-6}\ N/m^2$, A/m area-to-mass ratio, $a_\odot = 1$ AU.

Polynômes de Legendre : first order

The toy model

$$\mathcal{H} = -\frac{\mu^2}{2L^2} + C_r\, P_r\, \frac{A}{m}\, r\, \bar{r}_\odot\, \cos(\phi)$$

ϕ the angle between $\mathbf{r}$ and $\mathbf{r}_\odot$, $\quad L = \sqrt{\mu a}$, $\quad \bar{r}_\odot = \frac{r_\odot}{a_\odot}$.

$$\begin{aligned} \mathcal{H} &= -\frac{\mu^2}{2L^2} + C_r\, P_r\, \frac{A}{m}\, a\, (u\, \xi + v\, \eta) \\ &= H(L, G, H, M, \omega, \Omega, r_\odot) \end{aligned}$$

Debris orbital motion : $u = \cos E - e \quad$ and $\quad v = \sin E\, \sqrt{1 - e^2}$.

Debris orbit orientation and Sun orbital motion :

$$\begin{aligned} \xi &= \xi_1\, \bar{r}_{\odot,1} + \xi_2\, \bar{r}_{\odot,2} + \xi_3\, \bar{r}_{\odot,3} \\ \eta &= \eta_1\, \bar{r}_{\odot,1} + \eta_2\, \bar{r}_{\odot,2} + \eta_3\, \bar{r}_{\odot,3} \end{aligned}$$

$$\begin{aligned} \xi_1 &= \cos\Omega\, \cos\omega - \sin\Omega\, \cos i\, \sin\omega & \eta_1 &= -\cos\Omega\, \sin\omega - \sin\Omega\, \cos i\, \cos\omega \\ \xi_2 &= \sin\Omega\, \cos\omega + \cos\Omega\, \cos i\, \sin\omega & \eta_2 &= -\sin\Omega\, \sin\omega + \cos\Omega\, \cos i\, \cos\omega \\ \xi_3 &= \sin i\, \sin\omega & \eta_3 &= \sin i\, \cos\omega \end{aligned}$$

Averaging over the short periods : 1 day

Periods : 1 day (Orbital motion E) and 1 year (Sun $\overline{r}_{\odot,i}$)
Averaging over the fast variable (M the mean anomaly) :

$$
\begin{aligned}
\overline{\mathcal{H}} &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}\, dM \\
&= -\frac{\mu^2}{2\overline{L}^2} + \frac{1}{2\pi}\, C_r\, P_r\, \frac{A}{m}\, \overline{a} \int_0^{2\pi} (u\,\xi + v\,\eta)\, dM
\end{aligned}
$$

$dM = (1 - e\cos E)\, dE$

$$
\begin{aligned}
\overline{\mathcal{H}} &= -\frac{\mu^2}{2\overline{L}^2} - \frac{3}{2}\, C_r\, P_r\, \frac{A}{m}\, \frac{\overline{L}^2}{\mu}\, \overline{e}\,\xi \\
&= \overline{\mathcal{H}}(\overline{L}, \overline{G}, \overline{H}, -, \overline{\varpi}, \overline{\Omega}, r_\odot)
\end{aligned}
$$

The development

$$\overline{\mathcal{H}} = -\frac{\mu^2}{2L^2} - \frac{3}{2}\, C_r\, P_r\, \frac{A}{m}\, \frac{L^2}{\mu}\, e\, \xi$$

Poincaré variables :

$$\begin{array}{lcl} p & = & -\varpi \\ q & = & -\Omega \\ x_1 & = & \sqrt{2P}\, \sin p \\ x_2 & = & \sqrt{2Q}\, \sin q \end{array} \qquad \begin{array}{lcl} P & = & L - G \\ Q & = & G - H \\ y_1 & = & \sqrt{2P}\, \cos p \\ y_2 & = & \sqrt{2Q}\, \cos q \end{array}$$

Approximations : $e \simeq \sqrt{\frac{2P}{L}}$, $\cos^2 \frac{i}{2} = 1 - \frac{Q}{2L}$, $\sin \frac{i}{2} \simeq \sqrt{\frac{Q}{2L}}$
Circular orbit for the Sun (obliquity ϵ)

$$\begin{array}{lcl} \bar{r}_{\odot,1} & = & \cos \lambda_\odot \\ \bar{r}_{\odot,2} & = & \sin \lambda_\odot \cos \epsilon \\ \bar{r}_{\odot,3} & = & \sin \lambda_\odot \sin \epsilon \end{array}$$

with $\lambda_\odot = n_\odot t + \lambda_{\odot,0}$.

The truncated Hamiltonian in e and i

$$
\begin{aligned}
\mathcal{H} &= \mathcal{H}(x_1, y_1, x_2, y_2, \lambda_\odot) \\
&\simeq -n_\odot\, \kappa\, \bar{r}_{\odot,1}\, (x_1\, R_2 + y_1\, R_1) \\
&+ \; n_\odot\, \kappa\, \bar{r}_{\odot,2}\, (x_1\, R_3 + y_1\, R_2) \\
&+ \; n_\odot\, \kappa\, \bar{r}_{\odot,3}\, (x_1\, R_5 - y_1\, R_4)
\end{aligned}
$$

$\kappa = \frac{3}{2}\, C_r\, P_r\, \frac{A}{m}\, \frac{a}{\sqrt{L}}$

$R_i(x_2, y_2)$ are second degree polynomials in x_2 and y_2.

Dynamical system associated :

$$
\begin{aligned}
\dot{x}_1 &= \frac{\partial \mathcal{H}}{\partial y_1} & \qquad \dot{y}_1 &= -\frac{\partial \mathcal{H}}{\partial x_1} \\
\dot{x}_2 &= \frac{\partial \mathcal{H}}{\partial y_2} & \qquad \dot{y}_2 &= -\frac{\partial \mathcal{H}}{\partial x_2}.
\end{aligned}
$$

The eccentricity - pericenter motion : x_1 and y_1

$$\boxed{x_2 = 0 = y_2}$$

$$\begin{aligned} \dot{x}_1 &= -n_\odot \kappa \, \bar{r}_{\odot,1} \\ \dot{y}_1 &= -n_\odot \kappa \, \bar{r}_{\odot,2} \end{aligned}$$

Solution explicitly given by

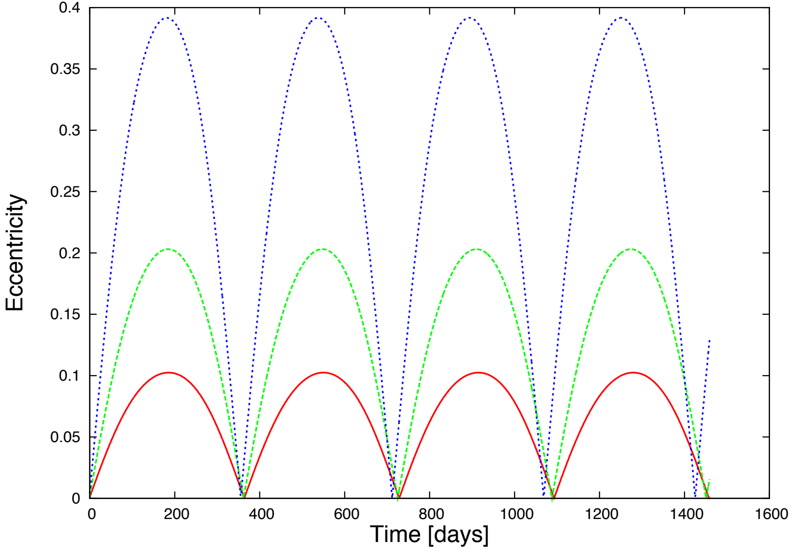
$$\begin{aligned} x_1 &= -\kappa \, \sin \lambda_\odot + C_x &&= -\kappa \, (\sin \lambda_\odot - D_x) \\ y_1 &= \kappa \, \cos \lambda_\odot \, \cos \epsilon + C_y &&= \kappa \, (\cos \lambda_\odot \, \cos \epsilon + D_y). \end{aligned}$$

e and ϖ : a periodic motion (1 year)
κ increases, e_{max} increases

Explanation of the behavior of GEO space debris (high e)

The eccentricity - pericenter motion : 1 year

$A/m = 5\ m^2/kg$ $A/m = 10\ m^2/kg$ $A/m = 20\ m^2/kg$



The inclination - node motion : x_2 and y_2

$$\boxed{x_2 \neq 0 \neq y_2}$$

$$\mathcal{H} = \mathcal{H}(x_1(\lambda_\odot), y_1(\lambda_\odot), R_i(x_2, y_2), \lambda_\odot)$$

Averaged equations over $\lambda_\odot$: system of mean linear equations

$$\begin{aligned} \dot{\bar{x}}_2 &= \nu\,\bar{y}_2 - \rho \\ \dot{\bar{y}}_2 &= -\nu\,\bar{x}_2 \end{aligned}$$

$\nu = n_\odot\,\kappa^2\,\cos\epsilon\,\frac{1}{2L}, \quad \rho = n_\odot\,\kappa^2\,\sin\epsilon\,\frac{1}{2\sqrt{L}}$

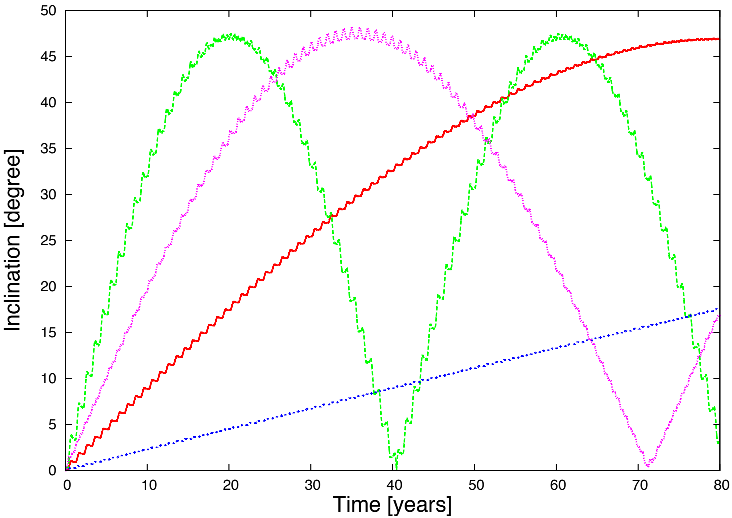
Solution : $\left\{ \begin{array}{lcl} \bar{x}_2 &=& \mathcal{A}\sin\psi \\ \bar{y}_2 &=& \mathcal{A}\cos\psi - \frac{\rho}{\nu} = \mathcal{A}\cos\psi - \tan\epsilon\sqrt{L} \end{array} \right.$

$\psi = \nu\,t + \psi_0$

i and Ω : a periodic motion (dozens of years) with $i_{max} \simeq 2\epsilon$
κ increases, ν increases and the period decreases.

The inclination - node motion : dozens of years

$A/m = 5\ m^2/kg$ $A/m = 10\ m^2/kg$ $A/m = 20\ m^2/kg$ $A/m = 40\ m^2/kg$



The inclination and eccentricity combined motion

Back to the averaging process

$$\begin{aligned}\mathcal{K} &= \mathcal{H}_0(x_1(\lambda_\odot), y_1(\lambda_\odot), R_i(x_2, y_2), \lambda_\odot) + n_\odot \Lambda_\odot \\ &= \mathcal{K}_0(x_2, y_2, \Lambda_\odot) + \mathcal{K}_1(x_2, y_2, \lambda_\odot) \\ &= n_\odot\, \Lambda_\odot - n_\odot\, \kappa^2\, f_0(x_2, y_2) - n_\odot\, \kappa^2\, f_1(x_2, y_2, \lambda_\odot)\end{aligned}$$

$$\begin{aligned}f_0(x_2, y_2) &= \frac{1}{2}\,(R_1 \cos\epsilon + R_3 \cos\epsilon + R_5 \sin\epsilon) \\ f_1(x_2, y_2, \lambda_\odot) &= g_1\, \cos\lambda_\odot + g_2\, \sin\lambda_\odot + g_3\, \cos 2\lambda_\odot + g_4\, \sin 2\lambda_\odot\end{aligned}$$

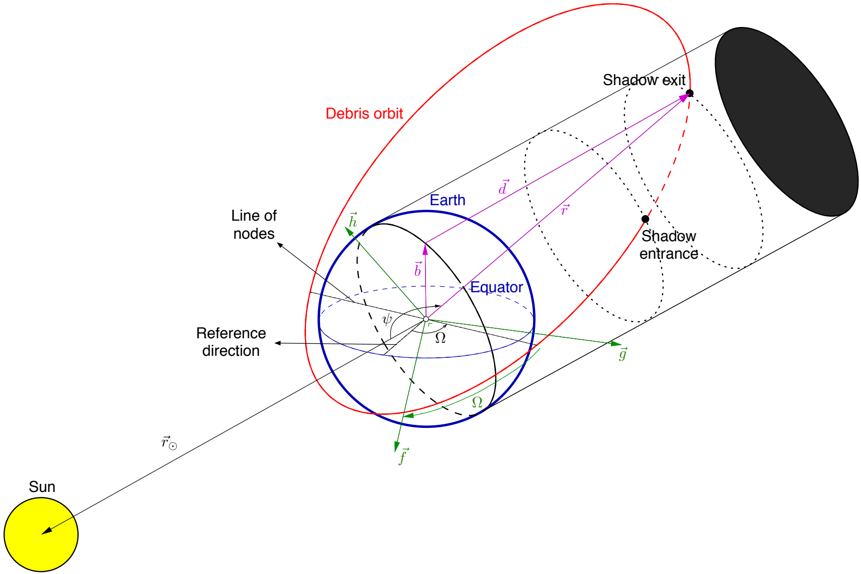
with $g_i = g_i(x_2, y_2)$ and $R_i = R_i(x_2, y_2)$.

The homological equation : $\boxed{\bar{\mathcal{H}}_1 = \mathcal{H}_1 + \{\mathcal{H}_0; \mathcal{W}\} = \mathcal{H}_1 - \frac{\partial \mathcal{H}_0}{\partial \Lambda_\odot}\, \frac{\partial \mathcal{W}}{\partial \lambda_\odot}}$

$$\mathcal{W} = -\kappa^2\,(g_1\, \sin\lambda_\odot - g_2\, \cos\lambda_\odot + \frac{1}{2}\, g_3\, \sin 2\lambda_\odot - \frac{1}{2}\, g_4\, \cos 2\lambda_\odot)$$

$$x_2 = \bar{x}_2 + \frac{\partial \mathcal{W}}{\partial y_2}(\lambda_\odot) \qquad y_2 = \bar{y}_2 - \frac{\partial \mathcal{W}}{\partial x_2}(\lambda_\odot)$$

The Earth umbra



The shadow equation

Simple geometrical problem : cylinder $\bigcap$ ellipse

cylinder : axis in the Sun direction
ellipse : debris orbit

$$s_c(\mathbf{r}) = \frac{\mathbf{r} \cdot \mathbf{r}_\odot}{r_\odot} + \sqrt{r^2 - R_\oplus^2} \quad \begin{array}{ll} < & 0 \text{ inside Earth's shadows} \\ > & 0 \text{ outside Earth's shadows} \\ = & 0 \text{ entry and exit} \end{array}$$

4th degree polynomial in $\tan \frac{E}{2}$ solved by Cardan formula

E_1 entry eccentric anomaly $= E_1(a, e, i, \omega, \Omega, \bar{r}_\odot)$
E_2 exit eccentric anomaly $= E_2(a, e, i, \omega, \Omega, \bar{r}_\odot)$

The toy model with shadow

$$\mathcal{H} = -\frac{\mu^2}{2L^2} + \begin{cases} C_r\, P_r\, \frac{A}{m}\, r\, \bar{r}_\odot\, \cos(\phi) & \text{outside Earth's shadows} \\ 0 & \text{inside Earth's shadows} \end{cases}$$

ϕ the angle between $\mathbf{r}$ and $\mathbf{r}_\odot$, $\quad L = \sqrt{\mu a}$, $\quad \bar{r}_\odot = \frac{r_\odot}{a_\odot}$.

$$\mathcal{H} = -\frac{\mu^2}{2L^2} + \begin{cases} C_r\, P_r\, \frac{A}{m}\, a\,(u\,\xi + v\,\eta) & \text{outside Earth's shadows} \\ 0 & \text{inside Earth's shadows} \end{cases}$$

Debris orbital motion : $u = \cos E - e$ and $v = \sin E\, \sqrt{1 - e^2}$.

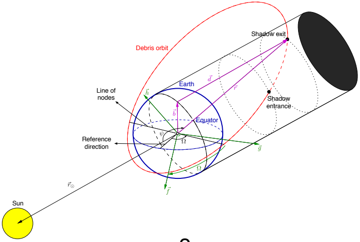
Debris orbit orientation and Sun orbital motion :

$$\begin{aligned} \xi &= \xi_1\, \bar{r}_{\odot,1} + \xi_2\, \bar{r}_{\odot,2} + \xi_3\, \bar{r}_{\odot,3} \\ \eta &= \eta_1\, \bar{r}_{\odot,1} + \eta_2\, \bar{r}_{\odot,2} + \eta_3\, \bar{r}_{\odot,3} \end{aligned}$$

$$\begin{aligned} \xi_1 &= \cos\Omega\ \cos\omega - \sin\Omega\ \cos i\ \sin\omega & \eta_1 &= -\cos\Omega\ \sin\omega - \sin\Omega\ \cos i\ \cos\omega \\ \xi_2 &= \sin\Omega\ \cos\omega + \cos\Omega\ \cos i\ \sin\omega & \eta_2 &= -\sin\Omega\ \sin\omega + \cos\Omega\ \cos i\ \cos\omega \\ \xi_3 &= \sin i\ \sin\omega & \eta_3 &= \sin i\ \cos\omega \end{aligned}$$

Averaging over the short periods : 1 day with shadow

Periods : 1 day (Orbital motion E) and 1 year (Sun $\overline{r}_{\odot,i}$)
Averaging over the fast variable (M the mean anomaly) :



$$
\begin{aligned}
\overline{\mathcal{H}} &= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}\, dM \\
&= -\frac{\mu^2}{2\overline{L}^2} \\
&+ \frac{1}{2\pi}\, C_r\, P_r\, \frac{A}{m}\, \overline{a} \left[\int_0^{M_1} (u\,\xi + v\,\eta)\, dM + \int_{M_2}^{2\pi} (u\,\xi + v\,\eta)\, dM \right]
\end{aligned}
$$

$dM = (1 - e\cos E)\, dE, \quad M1 = E_1 - e\sin E_1, \quad M2 = E_2 - e\sin E_2.$

Aksnes 1976

The averaged Hamiltonian with shadow

$$\begin{aligned}\overline{\mathcal{H}} &= -\frac{\mu^2}{2\overline{L}^2} - \frac{3}{2}\, C_r\, P_r\, \frac{A}{m}\, \frac{\overline{L}^2}{\mu}\, \overline{e}\, \xi + \frac{1}{2\pi} C_r\, P_r\, \frac{A}{m}\, \frac{\overline{L}^2}{\mu}\, [\xi\, \mathcal{A} + \eta\, \mathcal{B}] \\ &= \overline{\mathcal{H}}_0(\overline{L}, \overline{G}, \overline{H}, -, \overline{\varpi}, \overline{\Omega}, \overline{r}_\odot) + \overline{\mathcal{H}}_1(\overline{L}, \overline{G}, \overline{H}, -, \overline{\varpi}, \overline{\Omega}, \overline{r}_\odot) \\ &= \overline{\mathcal{H}}(D = 0) + \overline{\mathcal{H}}_1(D)\end{aligned}$$

$$\begin{aligned}\mathcal{A} &= -2\,(1 + \overline{e}^2)\, \cos\frac{S}{2}\, \sin\frac{D}{2} + \frac{3}{2}\, \overline{e}\, D + \frac{\overline{e}}{2}\, \cos S\, \sin D \\ \mathcal{B} &= \sqrt{1 - \overline{e}^2}\, (-2\, \sin\frac{S}{2}\, \sin\frac{D}{2} + \frac{\overline{e}}{2}\, \sin S\, \sin D)\end{aligned}$$

$$\begin{aligned}S &= E_1 + E_2 \\ D &= E_2 - E_1 = D(\overline{L}, \overline{G}, \overline{H}, -, \overline{\varpi}, \overline{\Omega}, \overline{r}_\odot)\end{aligned}$$

The perturbed problem : with the shadows

$$\mathcal{H} = \mathcal{H}(L, P, Q, \lambda, p, q) \quad \Rightarrow \quad \overline{\mathcal{H}} = \overline{\mathcal{H}}(\overline{L}, \overline{P}, \overline{Q}, -, \overline{p}, \overline{q})$$

At first order :

$$\begin{aligned}
< \dot{P} > = \dot{\overline{P}} &= -\frac{\partial \overline{\mathcal{H}}}{\partial \overline{p}} &= & - < \frac{\partial \mathcal{H}}{\partial p} > \\
< \dot{p} > = \dot{\overline{p}} &= \frac{\partial \overline{\mathcal{H}}}{\partial \overline{P}} &= & < \frac{\partial \mathcal{H}}{\partial P} > \\
< \dot{Q} > = \dot{\overline{Q}} &= -\frac{\partial \overline{\mathcal{H}}}{\partial \overline{q}} &= & - < \frac{\partial \mathcal{H}}{\partial q} > \\
< \dot{q} > = \dot{\overline{q}} &= \frac{\partial \overline{\mathcal{H}}}{\partial \overline{Q}} &= & < \frac{\partial \mathcal{H}}{\partial Q} >
\end{aligned}$$

but not for L or a.

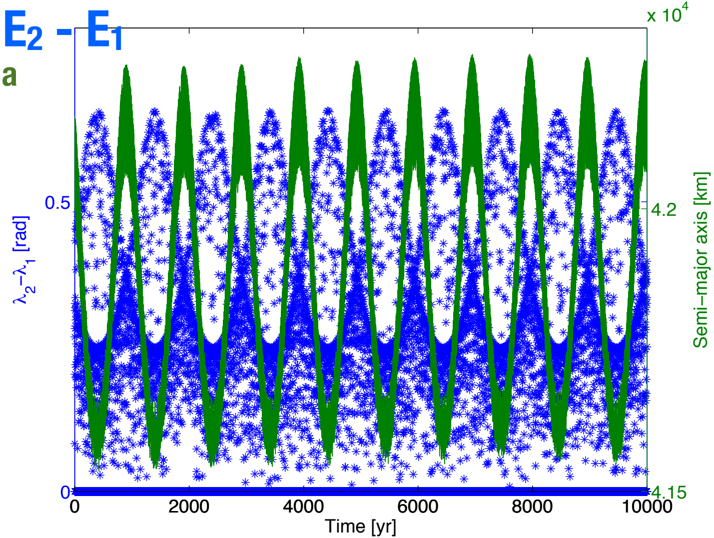
Very long periodic motion of the semi-major axis

$$
\begin{aligned}
< \dot{L} > &= < \frac{\partial \mathcal{H}}{\partial M} >= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial M}\, dM \\
&= \frac{1}{2\pi} \left[\int_0^{E_1} \frac{\partial \mathcal{H}}{\partial M} \, (1 - e \cos E) \, dE + \int_{E_2}^{2\pi} \frac{\partial \mathcal{H}}{\partial M} \, (1 - e \cos E) \, dE \right] \\
&= \frac{1}{\pi} \, C_r \, P_r \, \frac{A}{m} \, \bar{a} \left[\bar{\xi} \, \sin \frac{S}{2} - \bar{\eta} \, \sqrt{1 - \bar{e}^2} \, \cos \frac{S}{2} \right] \, \sin \frac{D}{2}
\end{aligned}
$$

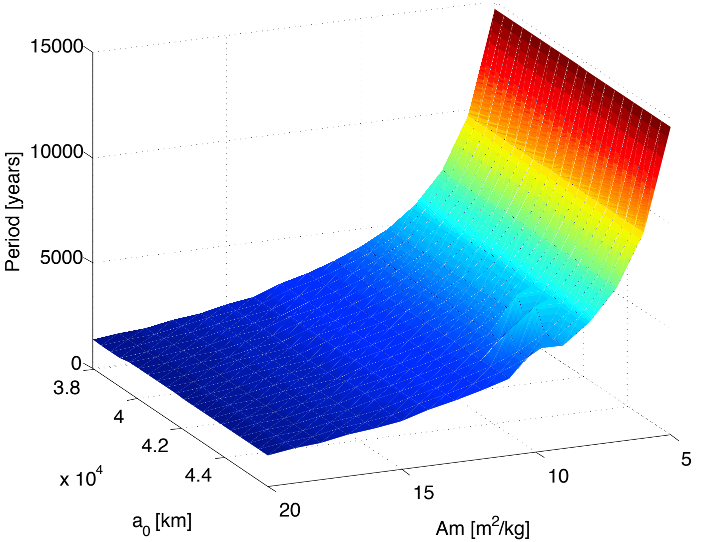
$$
< \dot{a} >= \bar{a}^{\,3/2} \, \frac{2}{\pi \, \sqrt{\mu}} \, C_r \, P_r \, \frac{A}{m} \, \left[\bar{\xi} \sin \frac{S}{2} - \bar{\eta} \sqrt{1 - \bar{e}^2} \, \cos \frac{S}{2} \right] \, \sin \frac{D}{2}
$$

Synchronism between a and $E_2 - E_1$

A/m = 25 m^2/kg - period $\simeq$ 1200 years - $\Delta a \simeq$ 600 km



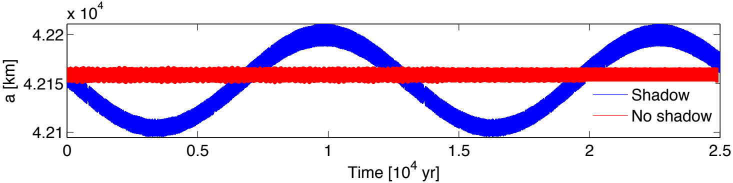
Very long period decreasing with the coefficient A/m



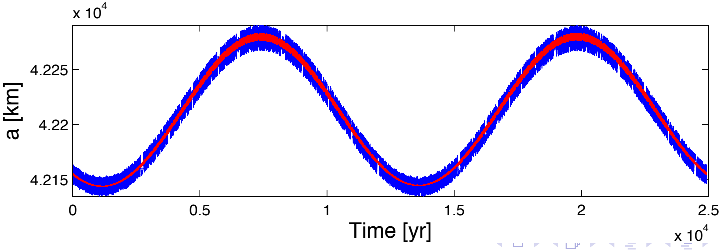
Comparisons

Coefficient A/m = 5 m^2/kg - period $\simeq$ 13 000 years

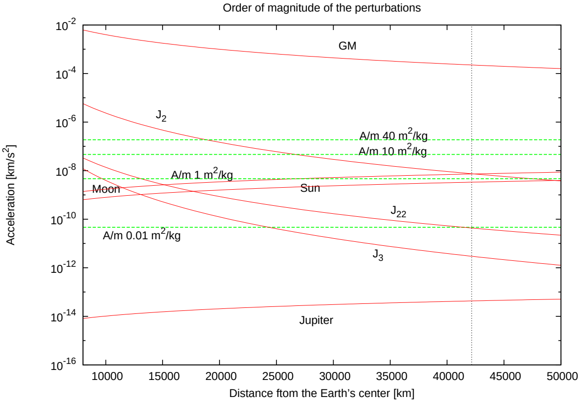
Numerical integration of the simplified system with shadow / without shadow



Symplectic numerical integration with shadow / Simplified system with shadow



Order of magnitude of radiation pressure



J_2

$$\begin{aligned} H_{J_2}(\vec{r}) &= \frac{\mu}{r} J_2 \left(\frac{r_\oplus}{r}\right)^2 P_2\left(\sin\phi_{sat}\right) \\ &= \frac{\mu}{r} J_2 \left(\frac{r_\oplus}{r}\right)^2 \frac{1}{2}\left(3\left(\frac{z}{r}\right)^2 - 1\right) \end{aligned}$$

where ϕ_{sat} represents the latitude of the satellite, and consequently $\sin\phi_{sat} = z/r$.

SRP second order

$$\begin{aligned} H_{SRP}(\vec{r}, \vec{r}_\odot) &= -C_r\, P_r\, \frac{A}{m} a_\odot^2 \, \frac{1}{||\vec{r} - \vec{r}_\odot||} \\ &\simeq -C_r P_r \frac{A}{m} a_\odot^2 \sum_{n=1}^{n=2} \left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi) \end{aligned}$$

Third body : Sun on a circular orbit

$$
\begin{aligned}
H_{3bS}(\vec{r}, \vec{r}_\odot) &= -\mu_\odot \frac{1}{||\vec{r} - \vec{r}_\odot||} + \mu_\odot \frac{\vec{r} \cdot \vec{r}_\odot}{||\vec{r}_\odot||^3} \\
&\simeq -\frac{\mu_\odot}{a_\odot} \sum_{n \geq 0} \left(\frac{r}{a_\odot} \right)^n P_n(\cos\phi) + \mu_\odot \frac{r a_\odot \cos(\phi)}{a_\odot^3} \\
&\simeq -\frac{\mu_\odot}{a_\odot} (1 + \left(\frac{r}{a_\odot} \right)^2 P_2(\cos\phi)),
\end{aligned}
$$

where $\mu_\odot = GM_\odot$ with $M_\odot$ the mass of the Sun.

Third body : Moon on a circular orbit

$$
H_{3bM}(\vec{r}, \vec{r}_☾) = -\frac{\mu_☾}{a_☾} (1 + \sum_{n \geq 2} \left(\frac{r}{a_☾} \right)^n P_n(\cos\phi_M))
$$

where $\mu_☾ = GM_☾$ with $M_☾$ the mass of the Moon, and ϕ_M the angle between the satellite and the Moon

The Sun contributions

$$
\begin{aligned}
& H_{SRP}(\vec{r},\vec{r}_\odot) + H_{3bS}(\vec{r},\vec{r}_\odot) \\
\simeq\ & H_{SRP_1}(\vec{r},\vec{r}_\odot) + H_{SRP_2}(\vec{r},\vec{r}_\odot) + H_{3bS}(\vec{r},\vec{r}_\odot) \\
\simeq\ & C_r P_r \frac{A}{m}\, a_\odot\, r\cos(\phi) \\
+\ & \left[C_r P_r \frac{A}{m} a_\odot - \frac{\mu_\odot}{a_\odot}\right] \left(\frac{r}{a_\odot}\right)^2 P_2(\cos\phi)
\end{aligned}
$$

Averaging over daily period :

$$
\begin{aligned}
\overline{H}(x_1,y_1,x_2,y_2) &= \overline{H}_{kepler} + \overline{H}_{J_2}(x_1,y_1,x_2,y_2) \\
&+ \overline{H}_{SRP_1}(x_1,y_1,x_2,y_2,\vec{r}_\odot) \\
&+ \overline{H}_{SRP_2+3bS}(x_1,y_1,x_2,y_2,\vec{r}_\odot) \\
&+ \overline{H}_{3bM}(x_1,y_1,x_2,y_2,\vec{r}_{☾})
\end{aligned}
$$

Averaging results

$$\overline{H}_{J_2} = C_p\, P + C_q\, Q = \frac{C_p}{2}(x_1^2 + y_1^2) + \frac{C_q}{2}(x_2^2 + y_2^2)$$

$$\overline{H}_{SRP_1} = -\frac{3}{2}\, C_r P_r \,\frac{A}{m}\, a\, e\, \xi$$

$$\overline{H}_{SRP_2+3bS} = -\left[C_r P_r \frac{A}{m} a_\odot - \frac{\mu_\odot}{a_\odot} \right] \frac{3a^2}{4a_\odot^2} w^2$$

$$= -\beta\, \frac{3a^2}{4a_\odot^2} w^2$$

$$\overline{H}_{3bM} = \frac{\mu_☾}{a_☾}\, \frac{3a^2}{4a_☾^2} w_M^2$$

$$w = -\sin q\ \sin i\ \vec{r}_{\odot,1} - \cos q\ \sin i\ \vec{r}_{\odot,2} + \cos i\ \vec{r}_{\odot,3}$$
$$w_M = -\sin q\ \sin i\ \vec{r}_{☾,1} - \cos q\ \sin i\ \vec{r}_{☾,2} + \cos i\ \vec{r}_{☾,3}$$

Short periodic motion : Kepler + J2 + SRP1

$$\begin{aligned}
\dot{x}_1(t) &= -C_2\, y_1 - n_\odot\, k\, r_{\odot,1}\,, \\
\dot{y}_1(t) &= C_2\, x_1 - n_\odot\, k\, r_{\odot,2}\,,
\end{aligned}$$

$$C_2 = \frac{3}{2}\sqrt{\frac{\mu}{a^3}} J_2 \frac{r_\oplus^2}{a^2}$$

$$\begin{aligned}
x_1(t) &= C_x + \frac{k \sin(n_\odot t + \lambda_{\odot,0})}{1 - eta^2}\left[\eta \cos\epsilon + 1\right], \\
y_1(t) &= C_y + \frac{k \cos(n_\odot t + \lambda_{\odot,0})}{1 - \eta^2}\left[\cos\epsilon + \eta\right],
\end{aligned}$$

Long periodic motion

$$\dot{x}_2(t) = C_q\, y_2 - n_\odot k \left[r_{\odot,1}\left(\frac{x_1 x_2}{2L}\right) - r_{\odot,2}\left(\frac{-2x_1 y_2}{2L} + \frac{y_1 x_2}{2L}\right) - r_{\odot,3}\left(\frac{x_1}{\sqrt{L}}\right.\right.$$

$$+ \frac{\partial \bar{H}_{SRP_2+3bS}}{\partial y_2} + \frac{\partial \bar{H}_{3bM}}{\partial y_2}$$

$$\dot{y}_2(t) = -C_q\, x_2 + n_\odot k \left[r_{\odot,1}\left(\frac{-2x_2 y_1}{2L} + \frac{x_1 y_2}{2L}\right) - r_{\odot,2}\left(\frac{y_1 y_2}{2L}\right) - r_{\odot,3}(-\right.$$

$$- \frac{\partial \bar{H}_{SRP_2+3bS}}{\partial x_2} - \frac{\partial \bar{H}_{3bM}}{\partial x_2}.$$

Averaging over the motion of the Sun and of the Moon

$$\dot{x_2}(t) = d_1 \; y_2 + d_3,$$
$$\dot{y_2}(t) = -d_2 \; x_2,$$

$$d_1 = n_\odot \frac{k^2}{4L} \cos\epsilon + \frac{C_q}{2} - \delta - \delta \cos^2\epsilon - \gamma - \gamma \cos^2\epsilon_M,$$
$$d_2 = n_\odot \frac{k^2}{4L} \cos\epsilon + \frac{C_q}{2} - 2\,\delta \cos^2\epsilon - 2\,\gamma \cos^2\epsilon_M,$$
$$d_3 = -n_\odot \frac{k^2}{2\sqrt{L}} \sin\epsilon + 2\,\delta\,\sqrt{L}\,\sin^2\epsilon + 2\,\gamma\,\sqrt{L}\,\sin^2\epsilon_M,$$

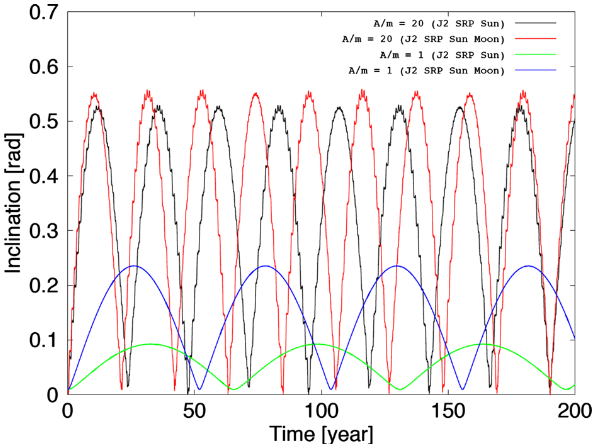
where $\delta = \beta\, \frac{3a^2}{16\,L\,a_\odot^2}$ and $\gamma = -\frac{\mu_☾}{a_☾}\,\frac{3a^2}{16\,L\,a_☾^2}$.

We write the corresponding solution for $x_2(t)$ and $y_2(t)$:

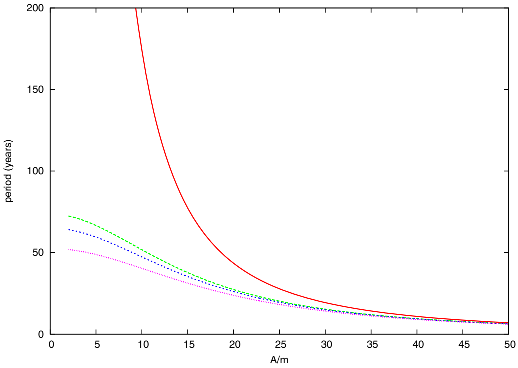
$$x_2(t) = \mathcal{D}\,\sin(\sqrt{d_1 d_2}\;t - \psi),$$
$$y_2(t) = \mathcal{D}\sqrt{\frac{d_2}{d_1}}\,\cos(\sqrt{d_1 d_2}\;t - \psi) - \frac{d_3}{d_1},$$

Eccentricity and inclination motions

Introduction of J_2, Sun and Moon in the description (Casanova)

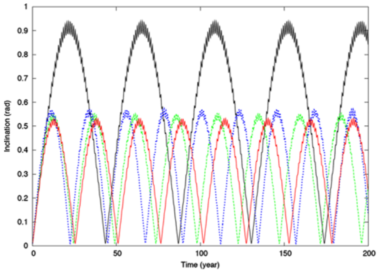


Inclination motion

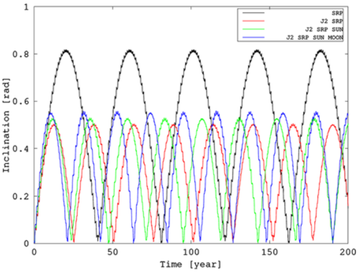


SRP
SRP + J_2
SRP + J_2 + Sun
SRP + J_2 + Sun + Moon

A/M = 20 m^2/kg - period too long for SPR - efficient formulae



(a)



(b)