Integration of $1/\delta_h$

We describe the explicitly computation of the integral of $1/\delta_h$ over

$$\mathbb{T}^2 = \{(\ell', \ell) : -\pi \le \ell' \le \pi, -\pi \le \ell \le \pi\},$$

assuming that (ℓ'_h, ℓ_h) is an internal point of this domain. First, we move the point (ℓ'_h, ℓ_h), corresponding to the local minimum value d_h, into the origin of the reference system by the variable change

$$\sigma_h : \ \mathbb{T}^2 \ni (\ell', \ell) \mapsto (\ell' - \ell'_h, \ell - \ell_h) =: \kappa = (k', k) \in \overline{\mathbb{T}}^2 \tag{1}$$

where we set $\overline{\mathbb{T}}^2 = \sigma_h\left(\mathbb{T}^2\right)$. The matrix

$$\mathcal{T} = \begin{bmatrix} \dfrac{1}{\rho_1} & -\dfrac{\rho_2}{\rho_1\rho_4} \\ & \\ 0 & \dfrac{1}{\rho_4} \end{bmatrix}, \quad \text{with} \quad \rho_1 = |\tau'|, \quad \rho_2 = -\frac{\langle \tau', \tau \rangle}{|\tau'|}, \quad \rho_4 = \frac{\sqrt{\det(\mathcal{A}_{\mathrm{h}})}}{|\tau'|}, \tag{2}$$

is such that

$$\mathcal{T}^t \mathcal{A}_h \mathcal{T} = \mathcal{I}_2 \ . \tag{3}$$

Then, the following coordinate change

$$\mathfrak{R} : \kappa \mapsto \psi = \mathcal{R}\,\kappa\,, \qquad \text{with} \qquad \mathcal{R} = \mathcal{T}^{-1} = \begin{bmatrix} \rho_1 & \rho_2 \\ 0 & \rho_4 \end{bmatrix}, \tag{4}$$

brings $\delta_h^2 \circ \sigma_h^{-1}$ into the form

$$\delta_h^2 \circ \sigma_h^{-1} \circ \mathfrak{R}^{-1}(\psi) = y'^2 + y^2 + d_h^2\,, \qquad \psi = (y', y) \tag{5}$$

and transforms the domain $\overline{\mathbb{T}}^2$ into a parallelogram with two sides parallel to the y axis (see Figure 1).
Using the variable changes (1), (4) and the polar coordinates map $\mathfrak{P}$, with inverse

$$\mathfrak{P}^{-1} : (r, \theta) \mapsto (y_1, y_2) = (r\cos\theta, r\sin\theta)$$

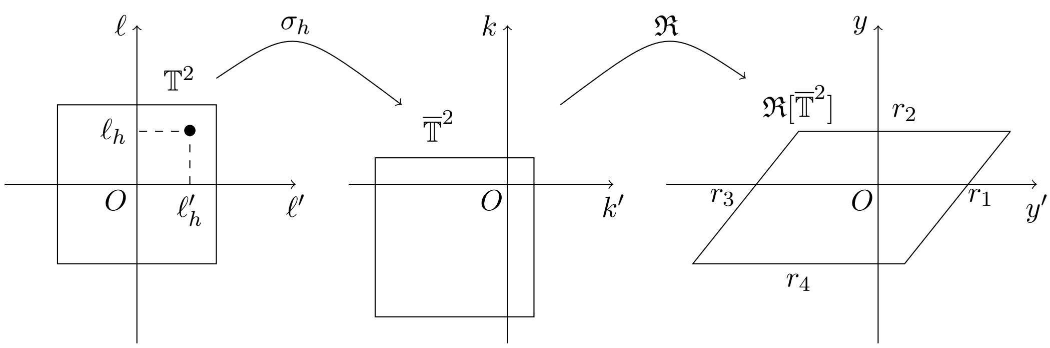

Figure 1: Description of the transformations of the integration domain $\mathbb{T}^2$ with the two coordinate changes (1), (4).

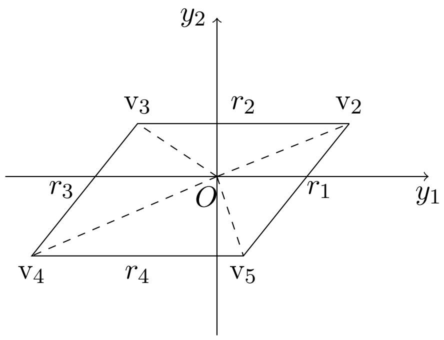

Figure 2: Decomposition of the domain of integration $\mathfrak{R}[\overline{\mathbb{T}}^2]$ to compute the integral of $1/\delta_h$ over $\mathbb{T}^2$ using polar coordinate.

we obtain

$$\int_{\mathbb{T}^2} \frac{1}{\delta_h}\, d\ell'\, d\ell = \frac{1}{\sqrt{\det(\mathcal{A}_h)}} \int_{\mathfrak{R}[\overline{\mathbb{T}}^2]} \frac{1}{\sqrt{y'^2 + y^2 + d_h^2}}\, dy'\, dy = \frac{1}{\sqrt{\det(\mathcal{A}_h)}} \int_{\mathcal{D}_h} \frac{r}{\sqrt{r^2 + d_h^2}}\, dr\, d\theta \tag{6}$$

where $\mathfrak{R}^{-1}\left[\mathfrak{P}^{-1}\left(\mathcal{D}_h\right)\right] = \overline{\mathbb{T}}^2$. Then, we decompose the domain $\mathcal{D}$ into four parts

$$\mathfrak{T} = \bigcup_{j=1}^{4} \left\{(r,\theta) \in \mathbb{R}^2 \ :\ \theta_j \le \theta \le \theta_{j+1} \text{ and } 0 \le r \le r_j(\theta)\right\}$$

where $r_j(\theta)$, with $j = 1 \dots 4$, represent the lines r_j delimiting $\mathfrak{R}[\overline{\mathbb{T}}^2]$ (see Figure 2) and in polar coordinates are given by

$$r_1(\theta) = \frac{\rho_1 \rho_4 (\pi - \ell'_h)}{\rho_4 \cos\theta - \rho_2 \sin\theta}\,, \qquad r_2(\theta) = \frac{\rho_4(\pi - \ell_h)}{\sin\theta}\,,$$

$$r_3(\theta) = \frac{-\rho_1 \rho_4 (\pi + \ell'_h)}{\rho_4 \cos\theta - \rho_2 \sin\theta}\,, \qquad r_4(\theta) = \frac{-\rho_4(\pi + \ell_h)}{\sin\theta}\,.$$

While $\theta_1 = \theta_5 - 2\pi$ and θ_{j+1} are the counter-clockwise angles between the y'-axis and the vertexes v_l seen from the origin of the axes (see Figure 2):

$$0 < \theta_2 < \theta_3 < \pi < \theta_4 < \theta_5 < 2\pi\,.$$

Moreover, the following relations hold for these angles:

$$\tan\theta_2 = \frac{\rho_4(\pi - \ell_h)}{\rho_1(\pi - \ell'_h) + \rho_2(\pi - \ell_h)}\,, \quad \tan\theta_3 = \frac{-\rho_4(\pi - \ell_h)}{\rho_1(\pi + \ell'_h) - \rho_2(\pi - \ell_h)}\,,$$

$$\tan\theta_4 = \frac{\rho_4(\pi + \ell_h)}{\rho_1(\pi + \ell'_h) + \rho_2(\pi + \ell_h)}\,, \quad \tan\theta_5 = \frac{-\rho_4(\pi + \ell_h)}{\rho_1(\pi - \ell'_h) - \rho_2(\pi + \ell_h)}\,.$$

Integrating in the r variable we obtain

$$\int_{\mathbb{T}^2} \frac{1}{\delta_h}\, d\ell'\, d\ell = \frac{1}{\sqrt{\det(\mathcal{A}_h)}} \cdot \left\{ \sum_{j=1}^{4} \int_{\theta_j}^{\theta_{j+1}} \sqrt{d_h^2 + r_j^2(\theta)}\, d\theta - 2\pi d_h \right\}. \tag{7}$$

Note that the integrals in the right hand side of (7) are differentiable functions of the orbital elements.