

# Orbit Determination with the two-body Integrals

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**Abstract** We investigate a method to compute a finite set of preliminary orbits for solar system bodies using the first integrals of the Kepler problem. This method is thought for the applications to the modern sets of astrometric observations, where often the information contained in the observations allows only to compute, by interpolation, two angular positions of the observed body and their time derivatives at a given epoch; we call this set of data *attributable*. Given two attributable of the same body at two different epochs we can use the energy and angular momentum integrals of the two-body problem to write a system of polynomial equations for the topocentric distance and the radial velocity at the two epochs. We define two different algorithms for the computation of the solutions, based on different ways to perform elimination of variables and obtain a univariate polynomial. Moreover we use the redundancy of the data to test the hypothesis that two attributable belong to the same body (*linkage problem*). It is also possible to compute a covariance matrix, describing the uncertainty of the preliminary orbits which results from the observation error statistics. The performance of this method has been investigated by using a large set of simulated observations of the Pan-STARRS project.

## 1 Introduction

With the new observational techniques of the next generation surveys, like Pan-STARRS and LSST,<sup>1</sup> the number of moving objects detected in each night of observations is expected to increase by two orders of magnitude with respect to the current surveys.

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<sup>1</sup> see the web pages <http://pan-starrs.ifa.hawaii.edu>, and <http://www.lsst.org>

To deal with this huge amount of data the interest in the study of orbit determination methods has been renewed, both from the theoretical and the computational point of view. The classical methods of preliminary orbit determination by Laplace [8] and Gauss [4], that have been often revisited in the last two centuries [18], [9], [10], are based on the knowledge of at least three observations of a solar system body in three different nights. Both Laplace's and Gauss' method may produce more than one preliminary orbit for the same object: a detailed analysis of the occurrence of multiple solutions is in [6]. The determination of a preliminary orbit is followed by the *differential corrections* [1], an iterative method to obtain the minimum of a target function, that improves the orbit in the sense of the least squares fit of the residuals: this sequence of operations was already proposed in [4].

The data of the current surveys generally do not provide a single observation for an object in an observing night: in fact the moving objects are distinguished from fixed stars by detecting them a few times in the same night: the sequence of observations usually gives a *too short arc* on the celestial sphere [14], such that the data are not enough to compute an orbit for that body. As the number of detected objects per night is very large, it is difficult to decide whether two sequences of observations made in different nights belong to the same object: this gives rise to the problem of *linkage* of two short arcs of observations.

The information contained in a short arc of observations can be used to define an *attributable* [11], consisting of the angular position and velocity of the body on the celestial sphere at a given time; the topocentric distance and the radial velocity at that time are unknown. Therefore two short arcs of observations belonging to the same object provide us with 8 scalar data, from which we can try to compute an orbit.

In 1977 Taff and Hall proposed to use the angular momentum and the energy integrals to perform orbit determination starting from a data set that corresponds to two attributables of the same observed body (see [19], [20]). They noticed that the problem can be written in an algebraic form, but the total degree is high, thus they suggested to use a Newton-Raphson method to solve the problem. This approach deals with the solutions only locally, and there are alternative possible solutions that can be lost. In this paper we shall start from the same first integrals of the Kepler problem, but we shall exploit the algebraic character of the problem, keeping in this way a global control on the solutions. In particular we shall present two different methods to solve by elimination the polynomial system corresponding to this problem, and to compute all the related preliminary orbits defined by the two attributables.

Since an orbit is defined by 6 scalar data, the available information is redundant, and we can use this to set compatibility conditions for the solutions (see (14)), that should be fulfilled if the two attributables belong to the same solar system object. The unavoidable errors in the observations affect also the computation of the attributables. Given a covariance matrix for the two attributables, expressing their uncertainty, we can use this to compute the value of an *identification norm*, based on the compatibility conditions, to decide if the attributables may be related to the same body (i.e. if the linkage is successful) and to choose among possible alternative solutions.

The plan of the paper is the following: after introducing some notation related to attributables in Section 2, we explain in Section 3 the orbit determination method and derive the bivariate polynomial system whose roots give us the topocentric distances of the observed object at the two epochs. In Section 4 we explain the two algorithms that we propose to search for these roots and the effective computation of the orbits. In Section 5 we deal with the uncertainty of the data, introduce the identification

norm and provide the covariance matrices of the preliminary orbits. Some numerical experiments are presented in the last two sections: in Section 6 we show the results of a test case, illustrating the computation of the preliminary orbits for a numbered asteroid whose orbit is well known, while in Section 7 we investigate the performance of the method for a large database of simulated observations.

## 2 Attributables

Let  $(\rho, \alpha, \delta) \in \mathbb{R}^+ \times [-\pi, \pi) \times (-\pi/2, \pi/2)$  be spherical coordinates for the topocentric position of a solar system body. The angular coordinates  $(\alpha, \delta)$  are defined by a topocentric coordinate system that can be arbitrarily selected. Usually, in the applications,  $\alpha$  is the right ascension and  $\delta$  the declination with respect to an equatorial coordinate system (e.g., J2000).

Given a short arc of observations of a celestial body  $(t_i, \alpha_i, \delta_i)$ , for  $i = 1 \dots m$  with  $m \geq 2$ , it is often possible to compute an attributable,<sup>2</sup> that is a vector

$$\mathcal{A} = (\alpha, \delta, \dot{\alpha}, \dot{\delta}) \in [-\pi, \pi) \times (-\pi/2, \pi/2) \times \mathbb{R}^2,$$

representing the angular position and velocity of the body at a mean time  $\bar{t}$  in the selected coordinates (see [11]). Usually we choose  $\bar{t}$  as the mean  $(\sum_i t_i)/m$ . The attributable is computed by a polynomial fit, typically linear or quadratic, and the observations used in the computation need to be made by the same observatory. If the observations are enough, i.e.  $m \geq 2$  for a linear fit,  $m \geq 3$  for a quadratic one, then we can compute also a covariance matrix  $\Gamma_{\mathcal{A}}$ , representing the uncertainty of the attributable. Note that the topocentric distances  $\rho_i$  at times  $t_i$  are completely unknown. We introduce the heliocentric position and velocity of the body at time  $\bar{t}$

$$\mathbf{r} = \mathbf{q} + \rho \hat{\boldsymbol{\rho}}, \quad \dot{\mathbf{r}} = \dot{\mathbf{q}} + \dot{\rho} \hat{\boldsymbol{\rho}} + \rho(\hat{\boldsymbol{\rho}}_\alpha \dot{\alpha} + \hat{\boldsymbol{\rho}}_\delta \dot{\delta}), \quad (1)$$

with  $\rho, \dot{\rho}$  the topocentric distance and the radial velocity, and with  $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\rho}}_\alpha, \hat{\boldsymbol{\rho}}_\delta$  the observation direction and its partial derivatives with respect to  $\alpha$  and  $\delta$ .

The vectors  $\mathbf{q}, \dot{\mathbf{q}}$  represent the heliocentric position and velocity of the observer on the Earth. The observer position is known as a function of time, but for consistency, if the attributable is computed by a fit to polynomials with low degree, the values  $\mathbf{q}(\bar{t}), \dot{\mathbf{q}}(\bar{t})$  need to be computed by the same interpolation. Therefore we make a quadratic fit with the actual geocentric positions  $\mathbf{q}(t_i) - \mathbf{q}_\oplus(t_i)$  at the times of the individual observations ( $\mathbf{q}_\oplus$  is the heliocentric position of the Earth center) to obtain the interpolating function  $\mathbf{q}_{obs}(t)$ ; then we take  $\mathbf{q}(\bar{t}) = \mathbf{q}_\oplus(\bar{t}) + \mathbf{q}_{obs}(\bar{t})$  and  $\dot{\mathbf{q}}(\bar{t}) = \dot{\mathbf{q}}_\oplus(\bar{t}) + \dot{\mathbf{q}}_{obs}(\bar{t})$ . This method was suggested by Poincaré in [18], and it is important to obtain preliminary orbits of better quality, see [15].

In rectangular coordinates we have

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= (\cos \alpha \cos \delta, \sin \alpha \cos \delta, \sin \delta), \\ \hat{\boldsymbol{\rho}}_\alpha &= (-\sin \alpha \cos \delta, \cos \alpha \cos \delta, 0), \\ \hat{\boldsymbol{\rho}}_\delta &= (-\cos \alpha \sin \delta, -\sin \alpha \sin \delta, \cos \delta). \end{aligned}$$

These vectors form an orthogonal system, in particular

$$|\hat{\boldsymbol{\rho}}| = |\hat{\boldsymbol{\rho}}_\delta| = 1, \quad |\hat{\boldsymbol{\rho}}_\alpha| = \cos \delta, \quad \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}}_\alpha = \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}}_\delta = \hat{\boldsymbol{\rho}}_\alpha \cdot \hat{\boldsymbol{\rho}}_\delta = 0,$$

<sup>2</sup> The name refers to the possibility of attributing the observations of the short arc to an already known orbit.

where the dot indicates the Euclidean scalar product and  $|\cdot|$  the corresponding norm.

For later reference we introduce the orthonormal basis  $\{\hat{\rho}, \hat{\nu}, \hat{\mathbf{n}}\}$  adapted to the apparent path  $\hat{\rho} = \hat{\rho}(t)$  of the observed body on the celestial sphere: we define the unit vector  $\hat{\nu}$  by the relation

$$\frac{d}{dt}\hat{\rho} = \eta \hat{\nu},$$

where  $\eta = \sqrt{\dot{\alpha}^2 \cos^2 \delta + \dot{\delta}^2}$  and is called *proper motion*. Moreover we set  $\hat{\mathbf{n}} = \hat{\rho} \times \hat{\nu}$ .

### 3 Linkage by the two-body integrals

Given two attributables  $\mathcal{A}_1, \mathcal{A}_2$  at different epochs  $\bar{t}_1, \bar{t}_2$ , in the hypothesis that they belong to the same observed body, we write down polynomial equations for the topocentric distance and radial velocity of the body at the two epochs by using the angular momentum and the energy integrals.

#### 3.1 Angular momentum and Energy

For a given attributable  $\mathcal{A}$  the angular momentum vector (per unit mass) can be written as a polynomial function of the radial distance and velocity  $\rho, \dot{\rho}$ :

$$\mathbf{c}(\rho, \dot{\rho}) = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{D}\dot{\rho} + \mathbf{E}\rho^2 + \mathbf{F}\rho + \mathbf{G},$$

where

$$\begin{aligned} \mathbf{D} &= \mathbf{q} \times \hat{\rho}, \\ \mathbf{E} &= \dot{\alpha}\hat{\rho} \times \hat{\rho}_\alpha + \dot{\delta}\hat{\rho} \times \hat{\rho}_\delta = \eta\hat{\mathbf{n}}, \\ \mathbf{F} &= \dot{\alpha}\mathbf{q} \times \hat{\rho}_\alpha + \dot{\delta}\mathbf{q} \times \hat{\rho}_\delta + \hat{\rho} \times \dot{\mathbf{q}}, \\ \mathbf{G} &= \mathbf{q} \times \dot{\mathbf{q}}, \end{aligned}$$

depend only on the attributable  $\mathcal{A}$  and on the motion of the observer  $\mathbf{q}, \dot{\mathbf{q}}$  at the time  $\bar{t}$  of the attributable. For the given  $\mathcal{A}$  we can also write the two-body energy as a function of  $\rho, \dot{\rho}$ , as in [12]

$$2\mathcal{E}(\rho, \dot{\rho}) = \dot{\rho}^2 + c_1\dot{\rho} + c_2\rho^2 + c_3\rho + c_4 - \frac{2k^2}{\sqrt{\rho^2 + c_5\rho + c_0}},$$

where

$$\begin{aligned} c_0 &= |\mathbf{q}|^2, & c_1 &= 2\dot{\mathbf{q}} \cdot \hat{\rho}, & c_2 &= \eta^2, \\ c_3 &= 2(\dot{\alpha}\dot{\mathbf{q}} \cdot \hat{\rho}_\alpha + \dot{\delta}\dot{\mathbf{q}} \cdot \hat{\rho}_\delta), & c_4 &= |\dot{\mathbf{q}}|^2, & c_5 &= 2\mathbf{q} \cdot \hat{\rho}, \end{aligned}$$

depend only on  $\mathcal{A}, \mathbf{q}, \dot{\mathbf{q}}$ .

#### 3.2 Equating the integrals

Now we take two attributables  $\mathcal{A}_1 = (\alpha_1, \delta_1, \dot{\alpha}_1, \dot{\delta}_1)$ ,  $\mathcal{A}_2 = (\alpha_2, \delta_2, \dot{\alpha}_2, \dot{\delta}_2)$  at epochs  $\bar{t}_1, \bar{t}_2$ ; we shall use the notation of Section 3.1, with index 1 or 2 referring to the epoch. If  $\mathcal{A}_1, \mathcal{A}_2$  correspond to the same physical object, then the angular momentum vectors at the two epochs must coincide:

$$\mathbf{D}_1\dot{\rho}_1 - \mathbf{D}_2\dot{\rho}_2 = \mathbf{J}(\rho_1, \rho_2), \quad (2)$$

where

$$\mathbf{J}(\rho_1, \rho_2) = \mathbf{E}_2 \rho_2^2 - \mathbf{E}_1 \rho_1^2 + \mathbf{F}_2 \rho_2 - \mathbf{F}_1 \rho_1 + \mathbf{G}_2 - \mathbf{G}_1 .$$

Relation (2) is a system of three equations in the four unknowns  $\rho_1, \dot{\rho}_1, \rho_2, \dot{\rho}_2$ , with constraints

$$\rho_1 > 0, \quad \rho_2 > 0 .$$

By scalar multiplication of (2) with  $\mathbf{D}_1 \times \mathbf{D}_2$  we eliminate the variables  $\dot{\rho}_1, \dot{\rho}_2$  and obtain the equation

$$\mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{J}(\rho_1, \rho_2) = 0 . \quad (3)$$

The left hand side in (3) is a quadratic form in the variables  $\rho_1, \rho_2$ ; we write it as

$$q(\rho_1, \rho_2) \stackrel{\text{def}}{=} q_{20} \rho_1^2 + q_{10} \rho_1 + q_{02} \rho_2^2 + q_{01} \rho_2 + q_{00} , \quad (4)$$

with

$$\begin{aligned} q_{20} &= -\mathbf{E}_1 \cdot \mathbf{D}_1 \times \mathbf{D}_2, & q_{02} &= \mathbf{E}_2 \cdot \mathbf{D}_1 \times \mathbf{D}_2, \\ q_{10} &= -\mathbf{F}_1 \cdot \mathbf{D}_1 \times \mathbf{D}_2, & q_{01} &= \mathbf{F}_2 \cdot \mathbf{D}_1 \times \mathbf{D}_2, \\ q_{00} &= (\mathbf{G}_2 - \mathbf{G}_1) \cdot \mathbf{D}_1 \times \mathbf{D}_2 . \end{aligned}$$

Equation (4) defines a conic section in the  $(\rho_1, \rho_2)$  plane, with symmetry axes parallel to the coordinate axes. Since the directions of  $\mathbf{E}_1, \mathbf{E}_2$  correspond to  $\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2$ , for a time span  $\bar{t}_2 - \bar{t}_1$  small enough the angle between these two directions is small and the coefficients  $q_{20}, q_{02}$  have opposite signs, thus in this case (4) defines a hyperbola.

We can compute the radial velocities  $\dot{\rho}_1, \dot{\rho}_2$  by vector multiplication of (2) with  $\mathbf{D}_1$  and  $\mathbf{D}_2$ , projecting on the direction of  $\mathbf{D}_1 \times \mathbf{D}_2$ :

$$\dot{\rho}_1(\rho_1, \rho_2) = \frac{(\mathbf{J} \times \mathbf{D}_2) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{|\mathbf{D}_1 \times \mathbf{D}_2|^2}, \quad \dot{\rho}_2(\rho_1, \rho_2) = \frac{(\mathbf{J} \times \mathbf{D}_1) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{|\mathbf{D}_1 \times \mathbf{D}_2|^2} . \quad (5)$$

For the given  $\mathcal{A}_1, \mathcal{A}_2$  we can also equate the corresponding two-body energies  $\mathcal{E}_1, \mathcal{E}_2$ . We use the expressions of  $\dot{\rho}_1(\rho_1, \rho_2), \dot{\rho}_2(\rho_1, \rho_2)$  above and substitute them into  $\mathcal{E}_1 = \mathcal{E}_2$ ,

$$\mathcal{F}_1(\rho_1, \rho_2) - \frac{2k^2}{\sqrt{\mathcal{G}_1(\rho_1)}} = \mathcal{F}_2(\rho_1, \rho_2) - \frac{2k^2}{\sqrt{\mathcal{G}_2(\rho_2)}} ,$$

for some polynomial functions  $\mathcal{F}_1(\rho_1, \rho_2), \mathcal{F}_2(\rho_1, \rho_2), \mathcal{G}_1(\rho_1), \mathcal{G}_2(\rho_2)$  with degrees  $\deg(\mathcal{F}_1) = \deg(\mathcal{F}_2) = 4$  and  $\deg(\mathcal{G}_1) = \deg(\mathcal{G}_2) = 2$ . By squaring twice we obtain the polynomial equation

$$p(\rho_1, \rho_2) \stackrel{\text{def}}{=} \left[ (\mathcal{F}_1 - \mathcal{F}_2)^2 \mathcal{G}_1 \mathcal{G}_2 - 4k^4 (\mathcal{G}_1 + \mathcal{G}_2) \right]^2 - 64k^8 \mathcal{G}_1 \mathcal{G}_2 = 0 \quad (6)$$

with total degree 24. Some spurious solutions may have been added as a result of squaring expressions with unknown sign.

Note that, in the geocentric approximation for the observations,  $\mathbf{G}_i$  ( $i = 1, 2$ ) is the angular momentum of the Earth at epochs  $\bar{t}_1, \bar{t}_2$ , thus  $\mathbf{G}_1 = \mathbf{G}_2$  and  $q_{00} = 0$ . In this case  $\rho_1 = \rho_2 = 0$  is a solution of the system  $q(\rho_1, \rho_2) = p(\rho_1, \rho_2) = 0$ , thus we always have a spurious solution corresponding to the Earth center. This spurious solution also appears in the geocentric version of the method of Laplace for a preliminary orbit from 3 observations. With topocentric observations the zero solution may be replaced by one with both  $\rho_1$  and  $\rho_2$  very small.

### 3.3 Degenerate cases

The quadratic form (4) degenerates into a linear function when

$$\mathbf{E}_1 \cdot \mathbf{D}_1 \times \mathbf{D}_2 = \mathbf{E}_2 \cdot \mathbf{D}_1 \times \mathbf{D}_2 = 0 .$$

A simple computation shows that

$$\begin{aligned} \mathbf{E}_1 \cdot \mathbf{D}_1 \times \mathbf{D}_2 &= \eta_1 (\hat{\mathbf{n}}_1 \cdot \mathbf{q}_1) (\hat{\rho}_1 \times \hat{\rho}_2 \cdot \mathbf{q}_2), \\ \mathbf{E}_2 \cdot \mathbf{D}_1 \times \mathbf{D}_2 &= \eta_2 (\hat{\mathbf{n}}_2 \cdot \mathbf{q}_2) (\hat{\rho}_1 \times \hat{\rho}_2 \cdot \mathbf{q}_1), \end{aligned}$$

thus, assuming that the proper motions  $\eta_1, \eta_2$  do not vanish and setting  $\mathbf{n}_{12} = \hat{\rho}_1 \times \hat{\rho}_2$ , the degeneration occurs when either  $\mathbf{n}_{12}$  vanishes (C0) or at least one the following relations holds:

$$\hat{\mathbf{n}}_1 \cdot \mathbf{q}_1 = \hat{\mathbf{n}}_2 \cdot \mathbf{q}_2 = 0, \quad (\text{C1})$$

$$\mathbf{n}_{12} \cdot \mathbf{q}_1 = \mathbf{n}_{12} \cdot \mathbf{q}_2 = 0, \quad (\text{C2})$$

$$\hat{\mathbf{n}}_1 \cdot \mathbf{q}_1 = \mathbf{n}_{12} \cdot \mathbf{q}_1 = 0, \quad (\text{C3})$$

$$\hat{\mathbf{n}}_2 \cdot \mathbf{q}_2 = \mathbf{n}_{12} \cdot \mathbf{q}_2 = 0. \quad (\text{C4})$$

The interpretation of these conditions is the following: (C0) means that  $\hat{\rho}_1, \hat{\rho}_2$  point to either exactly the same or exactly the opposite direction in the sky; (C1) means that both sets  $\{\hat{\rho}_1, \hat{\mathbf{v}}_1, \mathbf{q}_1\}$  and  $\{\hat{\rho}_2, \hat{\mathbf{v}}_2, \mathbf{q}_2\}$  are constituted of coplanar vectors; (C2) says that  $\hat{\rho}_1, \hat{\rho}_2, \mathbf{q}_1, \mathbf{q}_2$  are coplanar. Let us discuss condition (C3):  $\hat{\mathbf{n}}_1 \cdot \mathbf{q}_1 = 0$  means that  $\mathbf{q}_1, \hat{\rho}_1, \hat{\mathbf{v}}_1$  are coplanar and  $\mathbf{n}_{12} \cdot \mathbf{q}_1 = 0$  means that  $\hat{\rho}_1, \hat{\rho}_2, \mathbf{q}_1$  are coplanar as well. If  $\mathbf{D}_1 \neq 0$  we obtain that the four vectors  $\mathbf{q}_1, \hat{\rho}_1, \hat{\mathbf{v}}_1, \hat{\rho}_2$  all lie in the same plane. In particular (C3) implies that  $\hat{\rho}_2$  belongs to the great circle defined by the intersection of the plane generated by  $\hat{\rho}_1, \hat{\mathbf{v}}_1$  with the celestial sphere. This degeneration condition can be compared with the failure condition of the classical orbit determination methods with three observations by Gauss and Laplace [17], due to vanishing of the curvature in the apparent path of the observed body on the celestial sphere. The discussion of condition (C4) is similar to the previous one and corresponds to the coplanarity of  $\mathbf{q}_2, \hat{\rho}_2, \hat{\mathbf{v}}_2, \hat{\rho}_1$ .

## 4 Computation of the solutions

In this section we introduce two different methods to search for the solutions of the semi-algebraic problem

$$\begin{cases} p(\rho_1, \rho_2) = 0 \\ q(\rho_1, \rho_2) = 0 \end{cases}, \quad \rho_1, \rho_2 > 0 \quad (7)$$

for the polynomials  $p, q$  introduced in (6), (4) respectively. Moreover we explain the full procedure for the computation of the preliminary orbits and introduce compatibility conditions to decide whether the attributables used to define the problem are related to the same solar system body.

#### 4.1 Computation of the resultant via DFT

The first method consists in writing the resultant (see [3]) of  $p$  and  $q$  with respect to one variable, say  $\rho_1$ . In this way we find a univariate polynomial in the  $\rho_2$  variable whose real positive roots are the only possible  $\rho_2$ -components of a solution of (7). By grouping the monomials with the same power of  $\rho_1$  we can write

$$p(\rho_1, \rho_2) = \sum_{j=0}^{20} a_j(\rho_2) \rho_1^j, \quad \text{where} \quad (8)$$

$$\deg(a_j) = \begin{cases} 20 & \text{for } j = 0 \dots 4 \\ 24 - (j + 1) & \text{for } j = 2k - 1 \text{ with } k \geq 3 \\ 24 - j & \text{for } j = 2k \text{ with } k \geq 3 \end{cases}$$

and

$$q(\rho_1, \rho_2) = b_2 \rho_1^2 + b_1 \rho_1 + b_0(\rho_2) \quad (9)$$

for some univariate polynomial coefficients  $a_i, b_j$  depending on  $\rho_2$  (actually  $b_1, b_2$  are constant). We consider the resultant  $Res(\rho_2)$  of  $p, q$  with respect to  $\rho_1$ : it is generically a degree 48 polynomial defined as the determinant of the Sylvester matrix

$$\mathbf{S}(\rho_2) = \begin{pmatrix} a_{20} & 0 & b_2 & 0 & \dots & \dots & 0 \\ a_{19} & a_{20} & b_1 & b_2 & 0 & \dots & 0 \\ \vdots & \vdots & b_0 & b_1 & b_2 & \dots & \vdots \\ \vdots & \vdots & 0 & b_0 & b_1 & \dots & \vdots \\ a_0 & a_1 & \vdots & \vdots & \vdots & b_0 & b_1 \\ 0 & a_0 & 0 & 0 & 0 & 0 & b_0 \end{pmatrix}. \quad (10)$$

The positive real roots of  $Res(\rho_2)$  are the only possible values of  $\rho_2$  for a solution  $(\rho_1, \rho_2)$  of (7). We could use the resultant method to eliminate the variable  $\rho_2$  by a different grouping of the terms of  $p, q$ :

$$p(\rho_1, \rho_2) = \sum_{j=0}^{20} a'_j(\rho_1) \rho_2^j, \quad q(\rho_1, \rho_2) = b'_2 \rho_2^2 + b'_1 \rho_2 + b'_0(\rho_1),$$

where the degrees of  $a'_j$  are described by the same rules as for  $a_j$ .

Apart from non-real and non-positive solutions, we shall see that there are additional different ways to discard some pairs of solutions of (7), thus we expect that the number of acceptable ones is not large. We use a scheme similar to [5] to compute the coefficients of the resultant  $Res(\rho_2)$ :

- 1) evaluate  $a_i(\rho_2), b_j(\rho_2)$  at the 64-th roots of unit  $\omega_k = e^{2\pi i \frac{k}{64}}, k = 0 \dots 63$ , by a DFT (Discrete Fourier Transform) algorithm;
- 2) compute the determinant of the 64 Sylvester matrices; by relation

$$\det(\mathbf{S}(\rho_2)|_{\rho_2=\omega_k}) = (\det \mathbf{S}(\rho_2))|_{\rho_2=\omega_k}$$

we have the values of  $Res(\rho_2)$  at the 64-th roots of unit;

- 3) apply an IDFT (Inverse Discrete Fourier Transform) algorithm to obtain the coefficients of  $Res(\rho_2)$  from its evaluations.

The complete set of complex roots of  $Res(\rho_2)$  are computed using the algorithm described in [2], based on simultaneous iterations. Let  $\rho_2(k)$ ,  $k = 1 \dots n \leq 48$ , be the subset of the real and positive roots of  $Res(\rho_2)$ . Then for each  $k$  we perform the sequence of operations below:

- 4) solve the equation  $q(\rho_1, \rho_2(k)) = 0$  and compute the two possible values for  $\rho_1(k, 1)$ ,  $\rho_1(k, 2)$ , discarding negative solutions. Then compute  $p(\rho_1(k, 1), \rho_2(k))$ ,  $p(\rho_1(k, 2), \rho_2(k))$  and select  $\rho_1(k)$  from the pair that gives the smaller absolute value;
- 5) discard spurious solutions, resulting from the squaring used to reduce the energy equality to the polynomial equation (6);
- 6) compute the corresponding values of  $\dot{\rho}_1(k)$ ,  $\dot{\rho}_2(k)$  by (5) and obtain a pair of orbits defined by the sets  $(\alpha_i, \delta_i, \dot{\alpha}_i, \dot{\delta}_i, \rho_i, \dot{\rho}_i)$  of *attributable elements*,<sup>3</sup> for  $i = 1, 2$ ;
- 7) change from attributable elements to Cartesian heliocentric coordinates by relation  $\mathbf{r}_i = \rho_i(k)\hat{\rho}_i + \mathbf{q}_i$  for  $i = 1, 2$ , and by the corresponding formula for  $\dot{\mathbf{r}}_i$ . Note that the observer position  $\mathbf{q}_i$  is not the actual  $\mathbf{q}(t_i)$ , but is obtained by interpolation as proposed by Poincaré (see Section 2). Then a standard coordinate change allows us to obtain the related pairs of orbital elements: we shall use Keplerian elements  $(a, e, I, \Omega, \omega, \ell)$ , where  $\ell$  is the mean anomaly<sup>4</sup>. The epochs of the orbits are  $\tilde{t}_1(k)$ ,  $\tilde{t}_2(k)$ , corrected by aberration due to the finite velocity of the light  $c$ :  $\tilde{t}_i(k) = \bar{t}_i - \rho_i(k)/c$  for  $i = 1, 2$ .

We have implemented this algorithm in FORTRAN 90 using *quadruple precision* for part of these computations, in particular the ones related to DFT and IDFT. This feature appeared necessary to obtain reliable results starting from our first numerical experiments.

#### 4.2 Normal form of the problem

Another method to compute the solutions of (7) is based on a coordinate change to variables  $(\xi_1, \xi_2)$ , that allows to perform easily the elimination of either  $\xi_1$  or  $\xi_2$ . Let us set

$$p(\rho_1, \rho_2) = \sum_{i,j=0}^{20} p_{i,j} \rho_1^i \rho_2^j.$$

First we consider the affine transformation to intermediate variables  $(\zeta_1, \zeta_2)$

$$\mathcal{T} : \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \rightarrow \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^{-1} \rho_1 - \tau_1 \\ \sigma_2^{-1} \rho_2 - \tau_2 \end{pmatrix},$$

where, to eliminate the linear terms in (4), we set

$$\sigma_1 \tau_1 = -\frac{q_{1,0}}{2q_{2,0}} \stackrel{\text{def}}{=} \alpha, \quad \sigma_2 \tau_2 = -\frac{q_{0,1}}{2q_{0,2}} \stackrel{\text{def}}{=} \beta,$$

<sup>3</sup> The attributable elements are the same as spherical polar coordinates with their time derivatives: the coordinates are just reordered in such a way that the first four form the attributable, hence the name.

<sup>4</sup> Any other set of orbital elements in which the first four are defined by the two-body energy and angular momentum can be used, e.g. cometary elements  $(p_d, e, I, \Omega, \omega, t_p)$  where  $p_d$  is the perihelion distance and  $t_p$  is the time of perihelion passage: this set would allow to handle also parabolic and hyperbolic orbits.



so that

$$q \circ \mathcal{T}^{-1}(\zeta_1, \zeta_2) = q_{2,0} \sigma_1^2 \left[ \zeta_1^2 + \frac{q_{0,2} \sigma_2^2}{q_{2,0} \sigma_1^2} \zeta_2^2 + \frac{\kappa}{q_{2,0} \sigma_1^2} \right], \quad \text{with } \kappa = q_{0,0} - \frac{q_{1,0}^2}{4q_{2,0}} - \frac{q_{0,1}^2}{4q_{0,2}}.$$

If we set, for an arbitrary  $\sigma_2 \in \mathbb{R}$ ,

$$\sigma_1 = \gamma \sigma_2, \quad \gamma = \sqrt{-\frac{q_{0,2}}{q_{2,0}}}$$

we obtain

$$q \circ \mathcal{T}^{-1}(\zeta_1, \zeta_2) = q_{2,0} \sigma_1^2 \left[ \zeta_1^2 - \zeta_2^2 - 2c_\star \right], \quad \text{with } c_\star = -\frac{\kappa}{2 q_{2,0} \sigma_1^2}.$$

We already observed that, for  $\bar{t}_2 - \bar{t}_1$  small enough,  $q_{02}$  and  $q_{20}$  have opposite signs, hence in this case the variable change  $\mathcal{T}$  is real. However, in general, we have to consider  $\mathcal{T}$  as a transformation of the complex domain  $\mathbb{C}^2$ . We also have

$$p \circ \mathcal{T}^{-1}(\zeta_1, \zeta_2) = \sum_{i,j=0}^{20} \tilde{p}_{i,j} \zeta_1^i \zeta_2^j,$$

where

$$\tilde{p}_{i,j} = \sigma_2^{i+j} \gamma^i \sum_{h=i}^{20} \sum_{k=j}^{20} p_{h,k} \binom{h}{i} \binom{k}{j} \alpha^{h-i} \beta^{k-j}, \quad (11)$$

and  $\alpha, \beta, \gamma$  depend only on the coefficients of  $q(\rho_1, \rho_2)$ .

Now we apply a rotation of angle  $\pi/4$  to pass to the  $(\xi_1, \xi_2)$  variables:

$$\mathcal{R} : \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}.$$

We have

$$\begin{aligned} q \circ \mathcal{T}^{-1} \circ \mathcal{R}^{-1}(\xi_1, \xi_2) &= 2 q_{2,0} \sigma_1^2 [\xi_1 \xi_2 - c_\star], \\ p \circ \mathcal{T}^{-1} \circ \mathcal{R}^{-1}(\xi_1, \xi_2) &= \sum_{i,j=0}^{24} p_{i,j}^* \xi_1^i \xi_2^j, \end{aligned}$$

where

$$p_{i,j}^* = \begin{cases} \tilde{p}_{i+j,i}^* & \text{if } i+j \leq 24 \\ 0 & \text{if } i+j > 24 \end{cases} \quad (12)$$

and

$$\begin{aligned} \bar{p}_{m,n}^* &= \sum_{h+k=m} \tilde{p}_{h,k} \sum_{i+j=n} \binom{h}{i} \binom{k}{j} \frac{(-1)^j}{2^{(h+k)/2}} = \sum_{h=0}^m \tilde{p}_{h,m-h} \sum_{i=0}^n \binom{h}{i} \binom{m-h}{n-i} \frac{(-1)^{n-i}}{2^{m/2}} = \\ &= \sum_{h=\max\{m-20,0\}}^{\min\{m,20\}} \tilde{p}_{h,m-h} \sum_{i=0}^n \binom{h}{i} \binom{m-h}{n-i} \frac{(-1)^{n-i}}{2^{m/2}}. \end{aligned}$$

The last equality is obtained taking into account that  $\tilde{p}_{h,m-h} = 0$  for  $h > 20$  or  $m - h > 20$ . Using the relation  $\xi_1 \xi_2 = c_\star$  we can consider in place of  $p \circ T^{-1} \circ \mathcal{R}^{-1}(\xi_1, \xi_2)$  the polynomial

$$\begin{aligned} p^\star(\xi_1, \xi_2) &= \sum_{\substack{h, k=0 \\ h > k}}^{24} p_{h,k}^\star c_\star^k \xi_1^{h-k} + \sum_{h=0}^{24} p_{h,h}^\star c_\star^h + \sum_{\substack{h, k=0 \\ h < k}}^{24} p_{h,k}^\star c_\star^h \xi_2^{k-h} = \\ &= \sum_{j=1}^{24} \left( \sum_{\substack{h, k=0 \\ h-k=j}}^{24} p_{h,k}^\star c_\star^{h-j} \right) \xi_1^j + \sum_{j=1}^{24} \left( \sum_{\substack{h, k=0 \\ k-h=j}}^{24} p_{h,k}^\star c_\star^{k-j} \right) \xi_2^j + \sum_{h=0}^{24} p_{h,h}^\star c_\star^h = \\ &= A_{24} \xi_1^{24} + \dots + A_1 \xi_1 + A_0(\xi_2), \end{aligned}$$

with

$$A_j = \sum_{\substack{h, k=0 \\ h-k=j}}^{24} p_{h,k}^\star c_\star^{h-j} = \sum_{h=j}^{24} p_{h,h-j}^\star c_\star^{h-j}, \quad j = 1 \dots 24,$$

$$A_0(\xi_2) = B_{24} \xi_2^{24} + \dots + B_1 \xi_2 + B_0,$$

$$B_j = \sum_{\substack{h, k=0 \\ k-h=j}}^{24} p_{h,k}^\star c_\star^{k-j} = \sum_{k=j}^{24} p_{k-j,k}^\star c_\star^{k-j}, \quad j = 0 \dots 24.$$

We consider the algebraic problem in normal form

$$\begin{cases} p^\star(\xi_1, \xi_2) = 0 \\ \xi_1 \xi_2 - c_\star = 0 \end{cases}. \quad (13)$$

In this case we have to consider all the solutions of (13), not only the ones with real and positive components.

If  $c_\star = 0$ , then the solutions  $(\xi_1, \xi_2)$  of (13) are of the form  $(\xi_1(k), 0)$  or  $(0, \xi_2(k))$ , where  $\xi_1(k), \xi_2(k)$ ,  $k = 1 \dots 24$  are the roots of  $A_{24} \xi_1^{24} + \dots + A_1 \xi_1 + B_0$  and  $B_{24} \xi_2^{24} + \dots + B_1 \xi_2 + B_0$  respectively.

If  $c_\star \neq 0$ , using the relation  $\xi_1 \xi_2 = c_\star$  we can eliminate one variable, say  $\xi_1$ , from  $p^\star$ . Thus we obtain the univariate polynomial

$$\mathfrak{p}(\xi_2) = \sum_{k=0}^{48} \mathfrak{p}_k \xi_2^k, \quad \text{with } \mathfrak{p}_k = \begin{cases} A_{24-k} c_\star^{24-k}, & 0 \leq k \leq 23 \\ B_{k-24}, & 24 \leq k \leq 48 \end{cases}.$$

We compute all the complex roots  $\xi_2(k)$ ,  $k = 1 \dots 48$  of  $\mathfrak{p}(\xi_2)$  by the algorithm in [2]; then for each  $k$  we define the other component of the solutions by

$$\xi_1(k) = \frac{c_\star}{\xi_2(k)}.$$

Given all the complex solutions of (13) we compute the corresponding points in the  $(\rho_1, \rho_2)$  plane by

$$(\rho_1(k), \rho_2(k)) = T^{-1} \circ \mathcal{R}^{-1}(\xi_1(k), \xi_2(k)), \quad k = 1 \dots 48,$$

discarding the ones with non-real or non-positive components. At this point the preliminary solutions can be computed following the same steps 5), 6), 7) of the algorithm explained in Subsection 4.1.

From a few experiments performed this method seems to require more than quadruple precision, thus the advantage in the simple elimination of the variable  $\xi_1$  must be balanced with the introduction of heavier computations.

### 4.3 Compatibility conditions

The knowledge of the angular momentum vector and of the energy at a given time allows us to compute the Keplerian elements

$$a, e, I, \Omega .$$

In fact the semimajor axis  $a$  and the eccentricity  $e$  can be computed from the energy and the size of the angular momentum through the relations

$$\mathcal{E} = -\frac{k^2}{2a}, \quad \|\mathbf{c}\| = k\sqrt{a(1-e^2)};$$

the longitude of the node  $\Omega$  and the inclination  $I$  are obtained from the direction of the angular momentum

$$\hat{\mathbf{c}} = (\sin \Omega \sin I, -\cos \Omega \sin I, \cos I) .$$

The two attributables  $\mathcal{A}_1, \mathcal{A}_2$  at epochs  $\bar{t}_1, \bar{t}_2$  give 8 scalar data, thus the problem is over-determined. From a non-spurious pair  $(\bar{\rho}_1, \bar{\rho}_2)$ , solution of (7), we obtain the same values of  $a, e, I, \Omega$  at both times  $\bar{t}_i, i = 1, 2$ , but we must check that the orbit is indeed the same, that is check the compatibility conditions

$$\omega_1 = \omega_2, \quad \ell_1 = \ell_2 + n(\bar{t}_1 - \bar{t}_2), \quad (14)$$

where  $\omega_1, \omega_2$  and  $\ell_1, \ell_2$  are the arguments of perihelion and the mean anomalies of the body at times  $\bar{t}_1, \bar{t}_2$  and  $n = ka^{-3/2}$  is the mean motion, which is also the same for the two orbits;  $k$  is Gauss' constant, corresponding to  $\sqrt{GM_\odot}$  in a suitable system of units. The first of conditions (14) corresponds to the use of the fifth integral of the Kepler problem, related to Lenz-Laplace's integral vector

$$\mathbf{L} = \frac{1}{k^2} \dot{\mathbf{r}} \times \mathbf{c} - \frac{\mathbf{r}}{|\mathbf{r}|} .$$

Indeed the compatibility conditions (14) can not be exactly satisfied, due to both the errors in the observations and to the planetary perturbations. Actually the latter are important only when the observed body undergoes a close approach to some planet in the interval between  $\bar{t}_1$  and  $\bar{t}_2$ . Thus we may be able to discard some solutions, for which the compatibility conditions are largely violated. Nevertheless, we need a criterion to assess whether smaller discrepancies from the exact conditions (14) are due to the measurement uncertainty or rather due to the fact that the two attributables do not belong to the same physical object. This will be introduced in the next section.

## 5 Covariance of the solutions

Given a pair of attributables  $\mathbf{A} = (\mathcal{A}_1, \mathcal{A}_2)$  with covariance matrices  $\Gamma_{\mathcal{A}_1}, \Gamma_{\mathcal{A}_2}$ , we call  $\mathbf{R} = (\rho_1, \dot{\rho}_1, \rho_2, \dot{\rho}_2)$  one of the solutions of the equation  $\Phi(\mathbf{R}; \mathbf{A}) = \mathbf{0}$ , with

$$\Phi(\mathbf{R}; \mathbf{A}) = \begin{pmatrix} \mathbf{D}_1 \dot{\rho}_1 - \mathbf{D}_2 \dot{\rho}_2 - \mathbf{J}(\rho_1, \rho_2) \\ \mathcal{E}_1(\rho_1, \dot{\rho}_1) - \mathcal{E}_2(\rho_2, \dot{\rho}_2) \end{pmatrix} . \quad (15)$$

We can repeat what follows for each solution of  $\Phi(\mathbf{R}; \mathbf{A}) = \mathbf{0}$ .

Let  $\mathbf{R} = \mathbf{R}(\mathbf{A}) = (\mathcal{R}_1(\mathbf{A}), \mathcal{R}_2(\mathbf{A}))$ , where  $\mathcal{R}_i(\mathbf{A}) = (\rho_i(\mathbf{A}), \dot{\rho}_i(\mathbf{A}))$  for  $i = 1, 2$ . If both the elements  $(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A}))$ ,  $(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A}))$  give negative two-body energy orbits, then we can compute the corresponding Keplerian elements at times

$$\tilde{t}_i = \tilde{t}_i(\mathbf{A}) = \bar{t}_i - \frac{\rho_i(\mathbf{A})}{c}, \quad i = 1, 2$$

through the transformation

$$(\alpha, \delta, \dot{\alpha}, \dot{\delta}, \rho, \dot{\rho}) = (\mathcal{A}, \mathcal{R}) \mapsto \mathcal{E}_{Kep}(\mathcal{A}, \mathcal{R}) = (a, e, I, \Omega, \omega, \ell).$$

We have, for example, a smooth function

$$a_i = a_i(\mathbf{A}) = a(\mathcal{A}_i, \mathcal{R}_i(\mathbf{A})), \quad i = 1, 2$$

and similar functional relations for  $e, I, \Omega, \omega, \ell$ . We use the vector differences

$$\Delta_{1,2} = (\Delta\omega, \Delta\ell),$$

where  $\Delta\omega$  is the difference of the two angles  $\omega_1$  and  $\omega_2$ ,  $\Delta\ell$  is the difference of the two angles  $\ell_1$  and  $\ell_2 + n(\tilde{t}_2 - \tilde{t}_1)$  and  $n = ka_1^{-3/2} = ka_2^{-3/2}$  is the mean motion of both orbits. Here we compute the difference of two angles in such a way that it is a smooth function near a vanishing point; for example we define  $\Delta\omega = [\omega_1 - \omega_2 + \pi(\bmod 2\pi)] - \pi$ . With this caution, the vector  $\Delta_{1,2} = \Delta_{1,2}(\mathbf{A})$  represents the discrepancy in perihelion argument and mean anomaly of the two orbits, comparing the anomalies at the same time  $\tilde{t}_1$ . We introduce the map

$$\begin{aligned} \Psi : \left( [-\pi, \pi] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^2 \right)^2 &\longrightarrow [-\pi, \pi] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R} \times S^1 \times S^1 \\ (\mathcal{A}_1, \mathcal{A}_2) = \mathbf{A} &\mapsto \Psi(\mathbf{A}) = (\mathcal{A}_1, \mathcal{R}_1, \Delta_{1,2}), \end{aligned}$$

giving the orbit  $(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A}))$  in attributable elements at time  $\tilde{t}_1$  (the epoch of the first attributable corrected by aberration), together with the difference  $\Delta_{1,2}(\mathbf{A})$  in the angular elements, which are not constrained by the angular momentum and the energy integrals. By the covariance propagation rule we have

$$\Gamma_{\Psi(\mathbf{A})} = \frac{\partial \Psi}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[ \frac{\partial \Psi}{\partial \mathbf{A}} \right]^T, \quad (16)$$

where

$$\frac{\partial \Psi}{\partial \mathbf{A}} = \begin{bmatrix} I & 0 \\ \frac{\partial \mathcal{R}_1}{\partial \mathcal{A}_1} & \frac{\partial \mathcal{R}_1}{\partial \mathcal{A}_2} \\ \frac{\partial \Delta_{1,2}}{\partial \mathcal{A}_1} & \frac{\partial \Delta_{1,2}}{\partial \mathcal{A}_2} \end{bmatrix} \quad \text{and} \quad \Gamma_{\mathbf{A}} = \begin{bmatrix} \Gamma_{\mathcal{A}_1} & 0 \\ 0 & \Gamma_{\mathcal{A}_2} \end{bmatrix},$$

so that the covariance of  $\Psi(\mathbf{A})$  is given by the  $8 \times 8$  matrix

$$\Gamma_{\Psi(\mathbf{A})} = \begin{bmatrix} \Gamma_{\mathcal{A}_1} & \Gamma_{\mathcal{A}_1, \mathcal{R}_1} & \Gamma_{\mathcal{A}_1, \Delta_{1,2}} \\ \Gamma_{\mathcal{R}_1, \mathcal{A}_1} & \Gamma_{\mathcal{R}_1} & \Gamma_{\mathcal{R}_1, \Delta_{1,2}} \\ \Gamma_{\Delta_{1,2}, \mathcal{A}_1} & \Gamma_{\Delta_{1,2}, \mathcal{R}_1} & \Gamma_{\Delta_{1,2}} \end{bmatrix},$$

where

$$\Gamma_{\mathcal{A}_1, \mathcal{R}_1} = \Gamma_{\mathcal{A}_1} \left[ \frac{\partial \mathcal{R}_1}{\partial \mathcal{A}_1} \right]^T, \quad \Gamma_{\mathcal{A}_1, \Delta_{1,2}} = \Gamma_{\mathcal{A}_1} \left[ \frac{\partial \Delta_{1,2}}{\partial \mathcal{A}_1} \right]^T,$$

$$\Gamma_{\mathcal{R}_1, \Delta_{1,2}} = \frac{\partial \mathcal{R}_1}{\partial \mathcal{A}_1} \Gamma_{\mathcal{A}_1} \left[ \frac{\partial \Delta_{1,2}}{\partial \mathcal{A}_1} \right]^T + \frac{\partial \mathcal{R}_1}{\partial \mathcal{A}_2} \Gamma_{\mathcal{A}_2} \left[ \frac{\partial \Delta_{1,2}}{\partial \mathcal{A}_2} \right]^T,$$

$$\Gamma_{\mathcal{R}_1, \mathcal{A}_1} = \Gamma_{\mathcal{A}_1, \mathcal{R}_1}^T, \quad \Gamma_{\Delta_{1,2}, \mathcal{A}_1} = \Gamma_{\mathcal{A}_1, \Delta_{1,2}}^T, \quad \Gamma_{\Delta_{1,2}, \mathcal{R}_1} = \Gamma_{\mathcal{R}_1, \Delta_{1,2}}^T,$$

and

$$\Gamma_{\mathcal{A}_1} = \frac{\partial \mathcal{A}_1}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[ \frac{\partial \mathcal{A}_1}{\partial \mathbf{A}} \right]^T, \quad \Gamma_{\mathcal{R}_1} = \frac{\partial \mathcal{R}_1}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[ \frac{\partial \mathcal{R}_1}{\partial \mathbf{A}} \right]^T, \quad \Gamma_{\Delta_{1,2}} = \frac{\partial \Delta_{1,2}}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[ \frac{\partial \Delta_{1,2}}{\partial \mathbf{A}} \right]^T.$$

The matrices  $\frac{\partial \mathcal{R}_i}{\partial \mathcal{A}_j}$ ,  $i, j = 1, 2$ , can be computed from the relation

$$\frac{\partial \mathbf{R}}{\partial \mathbf{A}}(\mathbf{A}) = - \left[ \frac{\partial \Phi}{\partial \mathbf{R}}(\mathbf{R}(\mathbf{A}), \mathbf{A}) \right]^{-1} \frac{\partial \Phi}{\partial \mathbf{A}}(\mathbf{R}(\mathbf{A}), \mathbf{A}).$$

We also have

$$\begin{aligned} \frac{\partial \Delta \omega}{\partial \mathcal{A}_1} &= \frac{\partial \omega}{\partial \mathcal{A}}(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A})) + \frac{\partial \omega}{\partial \mathcal{R}}(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A})) \frac{\partial \mathcal{R}_1(\mathbf{A})}{\partial \mathcal{A}_1} - \frac{\partial \omega}{\partial \mathcal{R}}(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A})) \frac{\partial \mathcal{R}_2(\mathbf{A})}{\partial \mathcal{A}_1}, \\ \frac{\partial \Delta \omega}{\partial \mathcal{A}_2} &= \frac{\partial \omega}{\partial \mathcal{R}}(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A})) \frac{\partial \mathcal{R}_1(\mathbf{A})}{\partial \mathcal{A}_2} - \frac{\partial \omega}{\partial \mathcal{A}}(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A})) - \frac{\partial \omega}{\partial \mathcal{R}}(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A})) \frac{\partial \mathcal{R}_2(\mathbf{A})}{\partial \mathcal{A}_2} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \Delta \ell}{\partial \mathcal{A}_1} &= \frac{\partial \ell}{\partial \mathcal{A}}(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A})) + \frac{\partial \ell}{\partial \mathcal{R}}(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A})) \frac{\partial \mathcal{R}_1(\mathbf{A})}{\partial \mathcal{A}_1} - \frac{\partial \ell}{\partial \mathcal{R}}(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A})) \frac{\partial \mathcal{R}_2(\mathbf{A})}{\partial \mathcal{A}_1} - \\ &\quad - \frac{3}{2} \frac{n_1(\mathbf{A})}{a_1(\mathbf{A})} \left[ \frac{\partial a}{\partial \mathcal{A}}(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A})) + \frac{\partial a}{\partial \mathcal{R}}(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A})) \frac{\partial \mathcal{R}_1(\mathbf{A})}{\partial \mathcal{A}_1} \right] [t_2(\mathbf{A}) - t_1(\mathbf{A})] + \\ &\quad + \frac{n_1(\mathbf{A})}{c} \left[ \frac{\partial \rho_1}{\partial \mathcal{A}_1}(\mathbf{A}) - \frac{\partial \rho_2}{\partial \mathcal{A}_1}(\mathbf{A}) \right], \\ \frac{\partial \Delta \ell}{\partial \mathcal{A}_2} &= \frac{\partial \ell}{\partial \mathcal{R}}(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A})) \frac{\partial \mathcal{R}_1(\mathbf{A})}{\partial \mathcal{A}_2} - \frac{\partial \ell}{\partial \mathcal{A}}(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A})) - \frac{\partial \ell}{\partial \mathcal{R}}(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A})) \frac{\partial \mathcal{R}_2(\mathbf{A})}{\partial \mathcal{A}_2} - \\ &\quad - \frac{3}{2} \frac{n_1(\mathbf{A})}{a_1(\mathbf{A})} \left[ \frac{\partial a}{\partial \mathcal{A}}(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A})) + \frac{\partial a}{\partial \mathcal{R}}(\mathcal{A}_2, \mathcal{R}_2(\mathbf{A})) \frac{\partial \mathcal{R}_2(\mathbf{A})}{\partial \mathcal{A}_2} \right] [t_2(\mathbf{A}) - t_1(\mathbf{A})] + \\ &\quad + \frac{n_1(\mathbf{A})}{c} \left[ \frac{\partial \rho_1}{\partial \mathcal{A}_2}(\mathbf{A}) - \frac{\partial \rho_2}{\partial \mathcal{A}_2}(\mathbf{A}) \right]. \end{aligned}$$

### 5.1 Identification of attributables

One important step is to decide if trying to link  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  has produced at least one reliable orbit, so that we can state the two sets of observations defining the  $\mathcal{A}_i$ ,  $i = 1, 2$  may belong to one and the same solar system body. Neglecting the unavoidable errors in the observations and the approximations made both with the interpolation to compute  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  and with the use of a two-body model, if the observations belong to the same solar system body, then  $\Delta_{1,2}(\mathbf{A}) = \mathbf{0}$ . We need to check whether the failure of this condition is within the acceptable range of values which is statistically expected to be generated by the errors in the available observations.

The marginal covariance matrix of the compatibility conditions is

$$\Gamma_{\Delta_{1,2}} = \frac{\partial \Delta_{1,2}}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[ \frac{\partial \Delta_{1,2}}{\partial \mathbf{A}} \right]^T.$$

The inverse matrix  $C^{\Delta_{1,2}} = \Gamma_{\Delta_{1,2}}^{-1}$  defines a norm  $\|\cdot\|_*$  in the  $(\Delta\omega, \Delta\ell)$  plane, allowing to test an identification between the attributables  $\mathcal{A}_1, \mathcal{A}_2$ : the test is

$$\|\Delta_{1,2}\|_*^2 = \Delta_{1,2} C^{\Delta_{1,2}} \Delta_{1,2}^T \leq \chi_{max}^2,$$

where  $\chi_{max}$  is a control parameter. The value of the control could be selected on the basis of  $\chi^2$  tables, if we could assume that the observations errors are Gaussian and their standard deviations, mean values and correlations were known. Since this hypothesis is not satisfied in practice, the control value  $\chi_{max}$  needs to be selected on the basis of large scale tests, see Section 7.

## 5.2 Uncertainty of the orbits

The methods explained in Section 4 also allow to assign an uncertainty to the preliminary orbits that we compute. A solution  $(\mathcal{A}_1, \mathcal{R}_1(\mathbf{A}))$ , in attributable elements, has the marginal covariance matrix

$$\begin{bmatrix} \Gamma_{\mathcal{A}_1} & \Gamma_{\mathcal{A}_1, \mathcal{R}_1} \\ \Gamma_{\mathcal{R}_1, \mathcal{A}_1} & \Gamma_{\mathcal{R}_1} \end{bmatrix}.$$

The preliminary orbits obtained by the other available algorithms do not produce a covariance matrix: this is usually computed in the differential correction step of the orbit determination procedure. With the algorithm of [13] a covariance matrix may be defined, but it is not positive definite. The advantage of having a covariance matrix already from the preliminary orbit step could be important in two ways. First, the covariance matrix describes a confidence ellipsoid where a two-body orbit, compatible with the observations and their errors, can be found. The size of this ellipsoid can provide useful hints on the difficulty of the differential corrections procedure. Second, even if the differential corrections are divergent, the covariance matrix of the preliminary orbit can be used to compute a prediction with confidence region, allowing for a planned recovery, for assessment of impact risk, and so on.

## 6 A test case

We show a test of the linkage procedure using the attributables

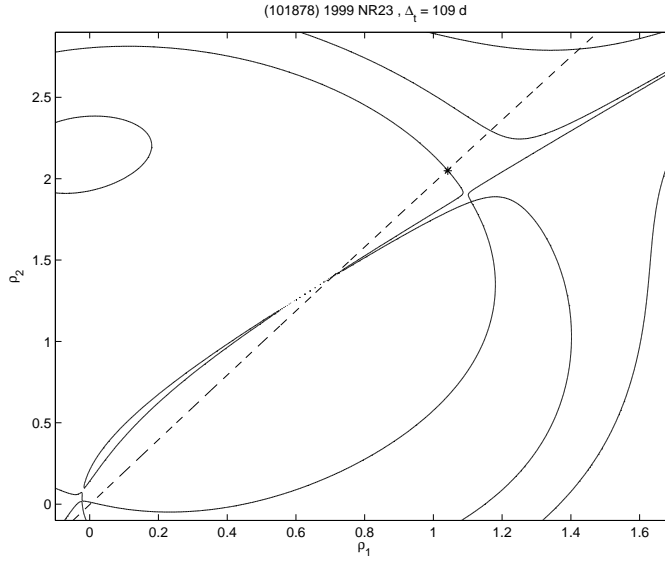
$$\begin{aligned} \mathcal{A}_1 &= (0.2872656, 0.1106342, -0.00375115, -0.00167695), \\ \mathcal{A}_2 &= (0.2820817, 0.1086542, 0.00514465, 0.00215975) \end{aligned}$$

of the asteroid (101878) 1999 NR<sub>23</sub> at epochs  $\bar{t}_1 = 54000$ ,  $\bar{t}_2 = 54109$  respectively (time in MJD). The values of the components of  $\mathcal{A}_1, \mathcal{A}_2$  are in radians/radians per day. They have been computed from two groups of observations, separated by more than 100 days, made from two different observatories: Mauna Kea (568) and Mt. Lemmon Survey (G96). From the known nominal orbit of this asteroid we obtain the values

$$\rho_1 = 1.0419, \quad \rho_2 = 2.0485$$

of the topocentric distance (in AU) at the two mean epochs of the observations.

In Figure 1 we show the intersections between the curves defined by  $p(\rho_1, \rho_2)$  and  $q(\rho_1, \rho_2)$ . By solving the corresponding problem (7) with the method described in



**Fig. 1** Intersections of the curves  $p = 0, q = 0$  (solid and dashed, respectively) in the plane  $\rho_1, \rho_2$  for the asteroid (101878) 1999 NR<sub>23</sub>: the asterisk corresponds to the true orbit.

	$\rho_1$	$\rho_2$
1	0.0059	0.0097
2	0.7130	1.4100
3	0.7045	1.3933
4	1.0409	2.0517
5	1.1659	2.2952
6	1.4246	2.7968

**Table 1** Solutions of the system (7) for (101878) 1999 NR<sub>23</sub>.

Subsection 4.1 we find the 6 positive pairs of solutions  $(\rho_1, \rho_2)$  displayed in Table 1.

After removing the solutions with very small components ( $\rho_1$  or  $\rho_2 < 0.02$  AU), duplicates and spurious solutions we are left with the values labeled 2 and 4 in Table 1. For

	1		2	
$\ \Delta_{1,2}\ _*$	487.65806		0.19505	
$a$	6.87384	6.87384	2.25828	2.25828
$e$	0.81798	0.81798	0.19787	0.19787
$I$	0.51733	0.51733	0.59995	0.59995
$\Omega$	156.55521	156.55521	156.42531	156.42531
$\omega$	144.68146	321.78289	144.39580	145.26330
$\ell$	4.66178	355.27766	47.75173	78.65378
$t$ (MJD)	53999.8205	54109.1368	53999.8186	54109.1331

**Table 2** Keplerian elements (angles in degrees) corresponding to the pairs  $(\rho_1, \rho_2)$  labeled with 2 and 4 in Table 1. The value of  $\|\Delta_{1,2}\|_*$  is shown for the two solutions.

these we succeed in computing Keplerian orbits, that we list in Table 2. The values of  $a, e, I, \Omega$  are the same for each pair: this is due to the fact that each pair of orbits shares the same angular momentum and the same energy. The value of the identification norm  $\|\Delta_{1,2}\|_*$  strongly suggests to select the second solution. The results appear pretty good, in fact the differences with the true solution are of the order of  $3 \times 10^{-5}$  AU and the errors in the Keplerian elements are comparable with the planetary perturbations; this is intrinsically bound to the use of a two-body approximation.

## 7 Numerical experiments with simulated observations

We have tested our identification method with the DFT algorithm, explained in Subsection 4.1, using simulated observations of objects in a solar system model. The data have been given to us by R. Jedicke and L. Dennau from the Institute of Astronomy, University of Hawaii, and the data quality resemble the one which should be achieved by the Pan-STARRS telescope when it will be fully operative. The RMS of the observations vary from 0.01 to 0.02 *arcsec*, that is rather optimistic for the current surveys. The current astrometric data quality of the Pan-STARRS 1 telescope is such that the RMS of the residuals for well determined asteroid orbits is between 0.11 and 0.13 *arcsec*. Better results should be achieved when the astrometric reduction of asteroid detections will be performed with respect to a catalogue generated by the Pan-STARRS survey itself.

These observations cover 31 observing nights, in three consecutive lunations and are grouped into *tracklets*. Each tracklet is composed by observations presumably belonging to the same object and covering a short arc: some of them are false (e.g. joint observations of different objects). From each tracklet we can compute an attributable. We have first applied to the database of tracklets the identification procedures defined in [13], [7]. Then we have tested our method on the leftover database, for which the previous procedure has failed. These remaining observations corresponds to 19441 objects, and 24590 tracklets, but only 4132 objects have at least two tracklets, that is a necessary requirement for the application of our method. The hyperbolic orbits have been removed from the solar system model: in fact our current method does not search for them, but it could be easily modified to include their orbit determination. To reduce the computational complexity, we need to define a filter to select the pairs of attributables which we try to link. In Section 7.1 we describe the two filters we have used in processing the simulated data.

### 7.1 Filtering pairs of attributables

#### 7.1.1 First filter: guessing the second angular position.

For a given pair of attributables at epochs  $\bar{t}_1, \bar{t}_2$ , consider for  $i = 1, 2$  the corresponding proper motions  $\eta_i$  and the mobile bases  $\{\hat{\rho}_i, \hat{\nu}_i, \hat{\mathbf{n}}_i\}$ , defined in Section 2. A first simple way to discard pairs of attributables is to constrain the time span  $\delta t = \bar{t}_2 - \bar{t}_1$ : we impose  $\delta t_{min} \leq \delta t \leq \delta t_{max}$  for suitable positive constants  $\delta t_{min}, \delta t_{max}$ . In our experiment we have used  $\delta t_{min} = 0.5$  days, that practically means we have tried to link attributables obtained in different nights. Then we form the orthogonal matrices  $V_1 = [\hat{\rho}_1 | \hat{\nu}_1 | \hat{\mathbf{n}}_1]$  and  $V_2 = [\hat{\rho}_2 | \hat{\nu}_2 | \hat{\mathbf{n}}_2]$ : these are rotation matrices to the mobile bases  $\{\hat{\rho}_i, \hat{\nu}_i, \hat{\mathbf{n}}_i\}$ ,



$i = 1, 2$ . Let  $R_{\phi, \hat{\mathbf{e}}}$  denote the rotation of an angle  $\phi$  around the unit vector  $\hat{\mathbf{e}}$ . Then  $R_{\eta_1 \delta t \hat{\mathbf{n}}_1} = V_1 R_{\eta_1 \delta t \hat{\mathbf{z}}} V_1^T$  ( $\hat{\mathbf{z}}$  is the third unit vector of the reference frame defining our rectangular coordinates) is the parallel transport matrix along the geodesic on the unit sphere defined by  $\mathcal{A}_1$  to time  $\bar{t}_2$ ; hence  $\hat{\boldsymbol{\rho}}_{12} = R_{\eta_1 \delta t \hat{\mathbf{n}}_1} \hat{\boldsymbol{\rho}}_1$  is the predicted observation direction at time  $\bar{t}_2$ , assuming the trajectory is a great circle and the proper motion is constant. By exchanging the order of the two attributables we can compute  $R_{-\eta_2 \delta t \hat{\mathbf{n}}_2} = V_2 R_{-\eta_2 \delta t \hat{\mathbf{z}}} V_2^T$  and  $\hat{\boldsymbol{\rho}}_{21} = R_{-\eta_2 \delta t \hat{\mathbf{n}}_2} \hat{\boldsymbol{\rho}}_2$ , that is the prediction at time  $\bar{t}_1$ . We use the metric

$$d(\hat{\boldsymbol{\rho}}_1, \hat{\boldsymbol{\rho}}_2) = \min\{\widehat{\hat{\boldsymbol{\rho}}_{12}, \hat{\boldsymbol{\rho}}_2}, \widehat{\hat{\boldsymbol{\rho}}_{21}, \hat{\boldsymbol{\rho}}_1}\},$$

that is the minimum between the two angular differences, discarding pairs of attributables that give rise to a large value of this metric.

### 7.1.2 Second filter: symmetric LLS fit.

Given the two attributables  $\mathcal{A}_1, \mathcal{A}_2$  at times  $\bar{t}_1, \bar{t}_2$  we perform a quadratic approximation of the apparent motion on the celestial sphere  $S^2$  by using a Linear Least Squares (LLS) fit. The apparent motion is given by the functions  $\alpha(t), \delta(t)$ . We approximate  $\alpha(t), \delta(t)$  with second degree polynomials whose coefficients are derived from a least squares fit. We denote the approximating quadratic functions as

$$\alpha(t) = \alpha_q + \dot{\alpha}_q(t - \bar{t}) + \frac{1}{2} \ddot{\alpha}_q(t - \bar{t})^2, \quad \delta(t) = \delta_q + \dot{\delta}_q(t - \bar{t}) + \frac{1}{2} \ddot{\delta}_q(t - \bar{t})^2, \quad (17)$$

where  $\bar{t} = \frac{1}{2}(\bar{t}_1 + \bar{t}_2)$  is the mean of the times of the attributables. The corresponding time derivatives are

$$\dot{\alpha}(t) = \dot{\alpha}_q + \ddot{\alpha}_q(t - \bar{t}), \quad \dot{\delta}(t) = \dot{\delta}_q + \ddot{\delta}_q(t - \bar{t}).$$

We want to determine the 6 quantities  $\alpha_q, \dot{\alpha}_q, \ddot{\alpha}_q, \delta_q, \dot{\delta}_q, \ddot{\delta}_q$  using the data coming from the attributables. The vector of residuals is

$$\boldsymbol{\xi} = (\mathcal{A}_1 - \mathcal{A}(\bar{t}_1), \mathcal{A}_2 - \mathcal{A}(\bar{t}_2))^T,$$

with  $\mathcal{A}(t) = (\alpha(t), \delta(t), \dot{\alpha}(t), \dot{\delta}(t))$ .

Given the covariance matrices  $\Gamma_{\mathcal{A}_1}, \Gamma_{\mathcal{A}_2}$  associated to the attributables we use them to *weight* the residuals in the definition of the target function:

$$Q(\boldsymbol{\xi}) = \frac{1}{8} \boldsymbol{\xi} \cdot W \boldsymbol{\xi}, \quad \text{where } W^{-1} = \begin{pmatrix} \Gamma_{\mathcal{A}_1} & 0 \\ 0 & \Gamma_{\mathcal{A}_2} \end{pmatrix}.$$

We introduce the notation

$$\mathbf{x} = (\alpha_q, \dot{\alpha}_q, \ddot{\alpha}_q, \delta_q, \dot{\delta}_q, \ddot{\delta}_q)^T, \quad \boldsymbol{\lambda} = (\mathcal{A}_1, \mathcal{A}_2)^T.$$

The value of  $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$  that minimizes the target function is obtained by solving the normal equation

$$C \mathbf{x} = -B^t W \boldsymbol{\lambda}, \quad \text{where } B = \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}, \quad C = B^t W B,$$

and the matrix  $B$  has the form

$$B = - \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad \text{with} \quad B_i = \begin{pmatrix} 1 & (t_i - \bar{t}) & \frac{1}{2}(t_i - \bar{t})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & (t_i - \bar{t}) & \frac{1}{2}(t_i - \bar{t})^2 \\ 0 & 1 & (t_i - \bar{t}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (t_i - \bar{t}) \end{pmatrix},$$

for  $i = 1, 2$ . Once the value of  $\mathbf{x}$  is given, we compute separately the residuals for the first and the second attributable  $\boldsymbol{\xi}_1 = (\mathcal{A}_1 - \mathcal{A}(\bar{t}_1))^T$ ,  $\boldsymbol{\xi}_2 = (\mathcal{A}_2 - \mathcal{A}(\bar{t}_2))^T$  and use the norm  $\sqrt{Q(\boldsymbol{\xi})}$  to decide which are the pairs to discard. We also discard couples of attributables giving rise to a large value of the quantity

$$\kappa_q \eta_q^2 = \frac{1}{\eta_q} \left[ (\ddot{\delta}_q \dot{\alpha}_q - \ddot{\alpha}_q \dot{\delta}_q) \cos \delta_q + \dot{\alpha}_q (\eta_q^2 + \dot{\delta}_q^2) \sin \delta_q \right],$$

where  $\eta_q = \sqrt{\dot{\delta}_q^2 + \dot{\alpha}_q^2 \cos^2 \delta_q}$  and  $\kappa_q$  is the geodesic curvature (see [16], Chapter 9).

## 7.2 Results

The accuracy of the linkage method can be measured by the number of true identifications over the total number of identifications found. The total number is 3625 and the true ones (that may be related to the same object if it has more than 2 tracklets) are 2908, i.e. 80.2% of the total. We could eliminate almost half of the 717 false identifications by lowering from 0.15 to 0.0625 *arcsec* the control on the RMS for acceptable orbits: but this would make us lose 95 true identifications.

In Table 3 we show the efficiency of the linkage procedure, that is we write the number of objects for which at least a pair of tracklets has been correctly linked, giving the details for the MB (Main Belt) and the NEO (Near Earth Object) class. As expected, the efficiency appears greater if there are three tracklets that can be pairwise linked. We stress that we have tested our method with data for which the

with 2 tracklets in 2 nights	Total	Found		Lost	
all	1074	951	88.5%	123	11.5%
MB	1038	936	90.2%	102	9.8%
NEO	19	9	47.4%	10	52.6%
with 3 tracklets in 3 nights	Total	Found		Lost	
All	214	205	95.8%	9	4.2%
MB	197	194	98.5%	3	1.5%
NEO	3	2	66.7%	1	33.3%

**Table 3** Efficiency of the identification procedure.

other available methods in [13], [7] could not perform the linkage.

## 8 Conclusions and future work

We have investigated an orbit determination method that is based on the integrals of the Kepler problem and is suitable to be used with modern data sets of observations.

The linkage procedure itself allows us to write covariance matrices for the preliminary orbits found. An interesting feature that comes out from our numerical experiments is that this method appears to work also when the time span between the two attributables is large, hence it can be used in cases where other linkage methods fail. The efficiency and the performance of the algorithm explained in Subsection 4.1 have been studied with a large scale test. The number of alternative solutions of the problem deserves a deeper investigation, however we expect that the acceptable ones should often be much less than 48, the maximum number of solutions of the polynomial system (7). Moreover the performance of the second algorithm to solve (7), described in Subsection 4.2, has not been tested yet: we would like to perform further experiments to decide if it allows to decrease the computation time.

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## References

1. Bate, R. R., Mueller, D. D. and White, J. E.: 1971, *Fundamentals of Astrodynamics*, Dover publications.
2. Bini, D. A.: 1997, *Numerical computation of polynomial zeros by means of Aberth method*, Numer. Algorithms, **13**, no. 3-4, 179–200.
3. Cox, D. A., Little, J. B., O’Shea, D.: 1996, *Ideals, Varieties and Algorithms*, Springer.
4. Gauss, C. F.: 1809, *Theoria motus corporum coelestium in sectionibus conicis solem ambientium*, reprinted by Dover publications (1963).
5. Gronchi, G. F.: 2002, *On the stationary points of the squared distance between two ellipses with a common focus*, SIAM Journ. Sci. Comp. **24/1**, 61–80.
6. Gronchi, G. F.: 2009, *Multiple solutions in preliminary orbit determination from three observations*, CMDA **103** n.4, 301–326.
7. Kubica, J., Denneau, L., Grav, T., Heasley, J., Jedicke, R., Masiero, J., Milani, A., Moore, A., Tholen and D., Wainscoat, R. J.: 2007, *Efficient intra- and inter-night linking of asteroid detections using kd-trees*, Icarus **189**, 151–168.
8. Laplace, P. S.: 1780, Mém. Acad. R. Sci. Paris, in *Laplace’s collected works* **10**, 93–146.
9. Leuschner, A.O.: 1912, *On the Laplacean orbit methods*, Proceedings of the ICM, 209–217.
10. Merton, G.: 1925, *A modification of Gauss’s Method for the Determination of Orbits*, MNRAS **85**, 693–731
11. Milani, A., Sansaturio, M. E., Chesley, S. R.: 2001, *The Asteroid Identification Problem IV: Attributions*, Icarus **151**, 150–159.
12. Milani, A., Gronchi, G. F., de’Michieli Vitturi, M., Knežević, Z.: 2004, *Orbit Determination with Very Short Arcs I. Admissible Regions*, CMDA **90**, 59–87.
13. Milani, A., Gronchi, G. F., Knežević, Z., Sansaturio, M. E., Arratia, O.: 2005, *Orbit Determination with Very Short Arcs II. Identifications*, Icarus **79**, 350–374.
14. Milani, A., Gronchi, G. F., Knežević, Z. : 2007, *New Definition of Discovery for Solar System Objects*, Earth Moon Planets, **100**, 83–116.
15. Milani, A., Gronchi, G. F., Farnocchia, D., Knežević, Z., Jedicke, R., Denneau, L. & Pierfederici, F.: 2008, *Topocentric Orbit Determination: algorithms for the next generation surveys*, Icarus **195**, 474–492.
16. Milani, A., Gronchi, G. F.: 2009, *Theory of Orbit Determination*, Cambridge University Press
17. Plummer, H. C.: 1918, *An introductory treatise on Dynamical Astronomy*, Cambridge University press, reprinted by Dover publications, New York (1960).

18. Poincaré, H.: 1906, *Sur la détermination des orbites par la méthode de Laplace*, Bulletin astronomique **23**, 161–187.
19. Taff, L. G., Hall, D. L.: 1977, *The use of angles and angular rates. I - Initial orbit determination*, CMDA **16**, 481–488
20. Taff, L. G., Hall, D. L.: 1980, *The use of angles and angular rates. II - Multiple Observation Initial orbit determination*, CMDA **21**, 281–290