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Light-time computations for the BepiColombo radioscience experiment

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Abstract The radioscience experiment is one of the on board experiment of the Mercury ESA mission BepiColombo that will be launched in 2014. The goals of the experiment are to determine the gravity field of Mercury and its rotation state, to determine the orbit of Mercury, to constrain the possible theories of gravitation (for example by determining the post-Newtonian (PN) parameters), to provide the spacecraft position for geodesy experiments and to contribute to planetary ephemerides improvement. This is possible thanks to a new technology which allows to reach great accuracies in the observables range and range rate; it is well known that a similar level of accuracy requires studying a suitable model taking into account numerous relativistic effects. In this paper we deal with the modelling of the space-time coordinate transformations needed for the light-time computations and the numerical methods adopted to avoid rounding-off errors in such computations.

Keywords Mercury · Interplanetary tracking · Light-time · Relativistic effects · Numerical methods

1 Introduction

BepiColombo is an European Space Agency mission to be launched in 2014, with the goal of an in-depth exploration of the planet Mercury; it has been identified as one of the most challenging long-term planetary projects. Only two NASA missions had Mercury as target in the past, the Mariner 10, which flew by three times in 1974-5 and

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Messenger, which carried out its flybys on January and October 2008, September 2009 before it starts its year-long orbiter phase in March 2011.

The BepiColombo mission is composed by two spacecraft to be put in orbit around Mercury. The radioscience experiment is one of the on board experiments, which would coordinate a gravimetry, a rotation and a relativity experiment, using a very accurate range and range rate tracking. These measurements will be performed by a full 5-way link [Iess and Boscagli 2001] to the Mercury orbiter; by exploiting the frequency dependence of the refraction index, the differences between the Doppler measurements (done in Ka and X band) and the delay give information on the plasma content along the radiowave path. In this way most of the measurements errors introduced can be removed, improving of about two orders of magnitude with respect to the past technologies. The accuracies that can be achieved are 10 cm in range and 3×10^{-4} cm/s in range rate.

How we compute these observables? For example, a first approximation of the range could be given by the formula

$$r = |\mathbf{r}| = |(\mathbf{x}_{\rm sat} + \mathbf{x}_{\rm M}) - (\mathbf{x}_{\rm EM} + \mathbf{x}_{\rm E} + \mathbf{x}_{\rm ant})| \ , \tag{1}$$

which models a very simple geometrical situation (see Figure 1). The vector $\mathbf{x}_{\rm sat}$ is the mercurycentric position of the orbiter, the vector $\mathbf{x}_{\rm M}$ is the position of the center of mass of Mercury (M) in a reference system with origin at the Solar System Barycenter (SSB), the vector $\mathbf{x}_{\rm EM}$ is the position of the Earth-Moon center of mass in the same reference system, $\mathbf{x}_{\rm E}$ is the vector from the Earth-Moon Barycenter (EMB) to the center of mass of the Earth (E), the vector $\mathbf{x}_{\rm ant}$ is the position of the reference point of the ground antenna with respect to the center of mass of the Earth.

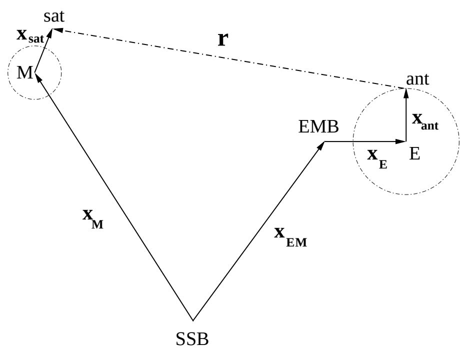

Fig. 1 Geometric sketch of the vectors involved in the computation of the range. SSB is the Solar System Barycenter, M is the center of Mercury, EMB is the Earth-Moon Barycenter, E is the center of the Earth.

Using (1) means to model the space as flat arena (r is an Euclidean distance) and the time as absolute parameter. This is obviously not possible because it is clear that, beyond some threshold of accuracy, these quantities have to be formulated within the framework of Einstein's theory of gravity (general relativity theory, GRT). Moreover we have to take into account the different times at which the events have to be computed: the transmission of the signal at the transmit time (t_t), the signal at the Mercury orbiter at the time of bounce (t_b) and the reception of the signal at the receive time (t_r).

Formula (1) could be a good starting point to construct a correct relativistic formulation; with the word "correct" we do not mean all the possible relativistic effects, but the effects measurables by the experiment. This paper deals with the corrections to apply to this formula to obtain a consistent relativistic model for the computations of the observables and the practical implementation of such computations.

In Section 2 we discuss the relativistic four-dimensional reference systems used and the transformations adopted to make the sums in (1) consistent; according to [Soffel *et al.* 2003], with "reference system" we mean a purely mathematical construction, while a "reference frame" is a some physical realization of a reference system. The relativistic contribution to the time delay due to the Sun's gravitational field, the Shapiro effect, is described in Section 3. Section 4 describes the theoretical procedure to compute the light-time (range) and the Doppler shift (range rate). In Section 5 we discuss the practical implementation of the algorithms showing how we eliminate rounding-off problems.

The equations of motion for the planets Mercury and Earth, including all the relativistic effects (and potential violations of GRT) required to the accuracy of the BepiColombo radioscience experiment have already been discussed in [Milani *et al.* 2009], thus this paper concentrates on the computation of the observables.

2 Space-time reference frames and transformations

The five vectors involved in formula (1) have to be computed at their own time, the epoch of different events: e.g., $\mathbf{x}_{\mathrm{ant}}$, $\mathbf{x}_{\mathrm{EM}}$ and $\mathbf{x}_{\mathrm{E}}$ are computed at both the antenna *transmit time* t_t and the *receive time* t_r of the signal. $\mathbf{x}_{\mathrm{M}}$ and $\mathbf{x}_{\mathrm{sat}}$ are computed at the *bounce time* t_b (when the signal has arrived to the orbiter and is sent back, with correction for the delay of the transponder). To be able to perform the vector sums and differences, these vectors have to be converted to a common space-time reference system, the only possible choice being some realization of the BCRS (Barycentric Celestial Referece System). We adopt for now a realization of the BCRS that we call SSB (Solar System Barycentric) reference frame and in which the time is TDB (Barycentric Dynamic Time); other possible choices, such as a TCB (Barycentric Celestial Time), only can differ by linear scaling. The TDB choice of the SSB timescale entails also the appropriate linear scaling of space-coordinates and planetary masses as described for instance in [Klioner 2008] or [Klioner *et al.* 2009].

The vectors $\mathbf{x}_{\mathrm{M}}$, $\mathbf{x}_{\mathrm{E}}$, and $\mathbf{x}_{\mathrm{EM}}$ are already in SSB as provided by numerical integration and external ephemerides; thus the vectors $\mathbf{x}_{\mathrm{ant}}$ and $\mathbf{x}_{\mathrm{sat}}$ have to be converted to SSB from the Geocentric and Mercurycentric systems, respectively. Of course the conversion of reference systems implies also the conversion of the time coordinate. There are three different time coordinates to be considered. The currently published planetary ephemerides are provided in TDB. The observations are based on averages of clock and frequency measurements on the Earth surface: this defines to another time coordinate called TT (Terrestrial Time). Thus for each observation the times of transmission t_t and receiving t_r need to be converted from TT to TDB to find the corresponding positions of the planets, e.g., the Earth and the Moon, by combining information from the precomputed ephemerides and the output of the numerical integration for Mercury and the Earth-Moon barycenter. This time conversion step is necessary for the accurate processing of each set of interplanetary tracking data; the main term in the difference TT-TDB is periodic, with period 1 year and amplitude $\simeq 1.6 \times 10^{-3}$ s, while there is essentially no linear trend, as a result of a suitable definition of the TDB.

The equation of motion of a Mercurycentric orbiter can be approximated, to the required level of accuracy, by a Newtonian equation provided the independent variable is the proper time of Mercury. Thus, for the BepiColombo radioscience experiment, it is necessary to define a new time coordinate TDM (Mercury Dynamic Time) containing terms of 1-PN order depending mostly upon the distance from the Sun and velocity of Mercury [Milani *et al.* 2009].

From now on we shall call the quantites related to the SSB frame “TDB-compatible”, the quantites related to the Geocentric frame “TT-compatible”, and the quantites related to the Mercurycentric frame “TDM-compatible”, in accordance with the paper [Klioner *et al.* 2009], and label them DB, DT and DM, respectively.

The differential equation giving the local time T as a function of the SSB time t , which we are currently assuming to be TDB, is the following:

$$\frac{dT}{dt} = 1 - \frac{1}{c^2}\left[U + \frac{v^2}{2} - L\right] , \tag{2}$$

where U is the gravitational potential (the list of contributing bodies depends upon the accuracy required: in our implementation we use Sun, Mercury to Neptune, Moon) at the planet center and v is the SSB velocity of the same planet. The constant term L is used to perform the conventional rescalings motivated by removal of secular terms, e.g., for the Earth we use L_C.

The space-time transformations we have to perform involve essentially the position of the antenna and the position of the orbiter. The Geocentric coordinates of the antenna should be transformed into TDB-compatible coordinates; the transformation is expressed by the formula

$$\mathbf{x}_{\rm ant}^{DB} = \mathbf{x}_{\rm ant}^{DT}\left(1 - \frac{U}{c^2} - L_C\right) - \frac{1}{2}\left(\frac{\mathbf{v}_{\rm E}^{DB}\cdot\mathbf{x}_{\rm ant}^{DT}}{c^2}\right)\mathbf{v}_{\rm E}^{DB} ,$$

where U is the gravitational potential at the geocenter (excluding the Earth mass), $L_C = 1.48082686741 \times 10^{-8}$ is a scaling factor given as definition, supposed to be a good approximation for removing secular terms from the trasformation and $\mathbf{v}_{\rm E}^{TDB}$ is the barycentric velocity of the Earth. The next formula contains the effect on the velocities of the time coordinate change, which should be consistently used together with the coordinate change:

$$\mathbf{v}_{\rm ant}^{DB} = \left[\mathbf{v}_{\rm ant}^{DT}\left(1 - \frac{U}{c^2} - L_C\right) - \frac{1}{2}\left(\frac{\mathbf{v}_{\rm E}^{DB}\cdot\mathbf{v}_{\rm ant}^{DT}}{c^2}\right)\mathbf{v}_{\rm E}^{DB}\right]\cdot\left[\frac{dT}{dt}\right] .$$

Note that the previous formula contains the factor dT/dt (expressed by eq. (2)) that deals with time transformation: T is the local time for Earth, that is TT, and t is the corresponding TDB time.

The Mercurycentric coordinates of the orbiter should be transformed into TDB-compatible coordinates through the formula

$$\mathbf{x}_{\rm sat}^{DB} = \mathbf{x}_{\rm sat}^{DM}\left(1 - \frac{U}{c^2} - L_{CM}\right) - \frac{1}{2}\left(\frac{\mathbf{v}_{\rm M}^{DB}\cdot\mathbf{x}_{\rm sat}^{DM}}{c^2}\right)\mathbf{v}_{\rm M}^{DB} ,$$

where U is the gravitational potential at the center of mass of Mercury (excluding the Mercury mass) and L_{CM} could be used to remove the secular term in the time transformation (thus defining a TM scale, implying a rescaling of the mass of Mercury). We believe this is not necessary: the secular drift of TDM with respect to other time scales is significant [Milani *et al.* 2009][Figure 5], but a simple iterative scheme is very efficient in providing the inverse time transformation. Thus we set $L_{CM} = 0$, assuming the reference frame is TDM-compatible. As for the antenna we have a formula expressing the velocity transformation that contains the derivative of time T for Mercury, that is TDM, with rispect to time t, that is TDB:

$$\mathbf{v}_{\rm sat}^{DB} = \left[\mathbf{v}_{\rm sat}^{DM}\left(1 - \frac{U}{c^2} - L_{CM}\right) - \frac{1}{2}\left(\frac{\mathbf{v}_{\rm M}^{TD}\cdot\mathbf{v}_{\rm sat}^{TD}}{c^2}\right)\mathbf{v}_{\rm M}^{DB}\right]\cdot\left[\frac{dT}{dt}\right] \ .$$

In all the formulas for these coordinate changes we have neglected the terms of the SSB acceleration of the planet center [Damour *et al.* 1994], because they contain beside $1/c^2$ the additional small parameter (distance from planet center)/(planet distance to the Sun), which is of the order of 10^{-4} even for a Mercury orbiter.

To assess the relevance of the relativistic corrections of this section to the accuracy of the BepiColombo radioscience experiment, we have computed the observables range and range rate with and without these corrections. As shown in Figure 2, the differences are significant, at a signal-to-noise ratio $S/N \simeq 1$ for range, much more for range rate, with an especially strong signature from the orbital velocity of the mercurycentric orbit (with $S/N > 50$).

3 Shapiro effect

The correct modelling of space-time transformations is not sufficient to have a precise computation of the signal delay: we have to take into account the general relativistic contribution to the time delay due to the spacetime curvature under the effect of the Sun's gravitational field, the *Shapiro effect* [Shapiro 1964]. The Shapiro time delay $\varDelta t$ at the 1-PN level is [Will 1993, Moyer 2003]

$$\varDelta t = \frac{(1+\gamma)\,\mu_0}{c^3}\,\ln\left(\frac{r_t + r_r + r}{r_t + r_r - r}\right) \ , \quad S(\gamma) = c\,\varDelta t \ ;$$

$r_t = |\mathbf{r}_{\rm t}|$ and $r_r = |\mathbf{r}_{\rm r}|$ are the heliocentric distances of the transmitter and the receiver at the corresponding time instants of photon transmission and reception, μ_0 is the mass of the Sun and $r = |\mathbf{r}_r - \mathbf{r}_t|$. The planetary terms, similar to the solar one, can also be included but they are smaller than the accuracy needed for our measurements. Parameter γ is the only post-Newtonian parameter used for the light-time effect and, in fact, it could be best constraint during superior conjuction [Milani *et al.* 2002]. The total amount of the Shapiro effect in range is shown in Figure 3.

The question arises whether the very high signal to noise in the range requires other terms in the solar gravity influence, due to either (i) motion of the source, or (ii) higher-order corrections when the radio waves are passing near the Sun, at just a few solar radii (and thus the denominator in the log-function of the Shapiro formula is small). The corrections (i) are of the post-Newtonian order 1.5, that is containing a factor $1/c^3$, but it has been shown in [Milani *et al.* 2009] that they are too small to affect our accuracy. The corrections (ii) are of order 2, that is $1/c^4$, but they can be

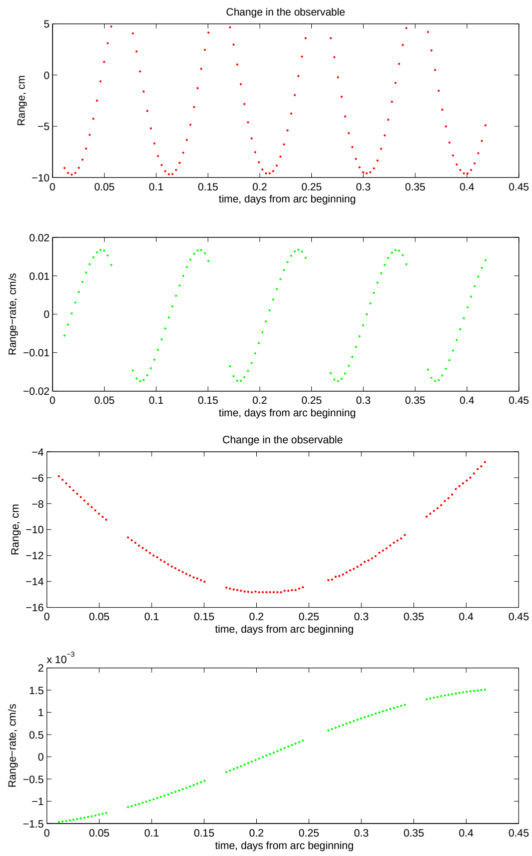

Fig. 2 The difference in the observables range and range rate for one pass of Mercury above the horizon for a ground station, by using an hybrid model in which the position and velocity of the orbiter have not transformed to TDB-compatible quantities and a correct model in which all quantities are TDB-compatible. Interruptions of the signal are due to spacecraft passage behind Mercury as seen for the Earth station. Top: for an hybrid model with the satellite position and velocity not transformed to TDB-compatible. Bottom: for an hybrid model with the position and velocity of the antenna not transformed to TDB-compatible.

actually larger for an experiment involving Mercury. The relevant correction is most easily obtained by adding $1/c^4$ terms in the Shapiro formula, due to the bending of the light path:

$$S(\gamma) = \frac{(1+\gamma)\,\mu_0}{c^2}\,\ln\left(\frac{r_t + r_r + r + \frac{(1+\gamma)\,\mu_0}{c^2}}{r_t + r_r - r + \frac{(1+\gamma)\,\mu_0}{c^2}}\right)\,.$$

This formulation has been proposed by [Moyer 2003] and, recently, it has been justified in the small impact parameter regime by much more theoretically rooted derivations by [Klioner and Zschocke 2007], [Teyssandier & Le Poncin-Lafitte 2008] and [Ashby and Bertotti 2008]. Figure 4 shows that the order 2 correction is relevant for our experiment, especially when there is a superior conjunction with a small impact parameter of the radio wave path passing near the Sun. Note that the $1/c^4$ correction

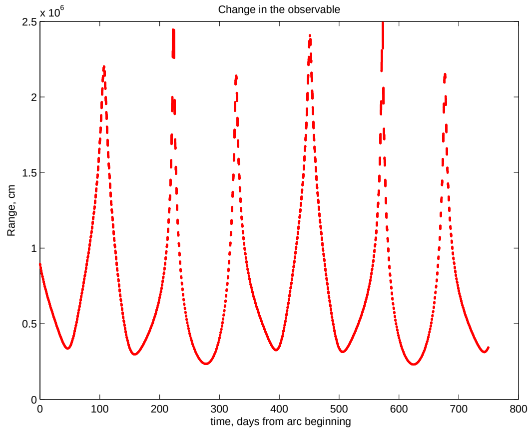

Fig. 3 Total amount of the Shapiro effect in range over 2-year simulation. The sharp peaks correspond to superior conjunctions, when Mercury is "behind the Sun" as seen from Earth, with values as large as 24 km for radiowaves passing at 3 solar radii from the center of the Sun. Interruptions of the signal are due to spacecraft visibility from the Earth station (in this simulation we assume just one station).

($\sim$ 10 cm) in the Shapiro formula effectively corresponds to $\sim 3 \times 10^{-5}$ correction in the value of the post-Newtonian parameter γ.

The Shapiro correction for the computation of the range rate is:

$$\dot{S} = \frac{2(1+\gamma)\mu_0}{c^2}\left[\frac{-r\,(\dot{r}_t+\dot{r}_r)+\dot{r}\,\left(r_t+r_r+\frac{(1+\gamma)\mu_0}{c^2}\right)}{(r_t+r_r+\frac{(1+\gamma)\mu_0}{c^2})^2-r^2}\right] .$$

This formula is almost never found in the literature and has not been much used in the processing of the past radioscience experiments, such as [Bertotti *et al.* 2003], because the observable range rate is typically computed as difference of ranges divided by time; however, for reasons explained in Section 5, this formula is now necessary.

4 Light-time iterations

Since radar measurements are usually referred to the receive time t_r the observables are seen as functions of this time, and the computation sequence works backward in time: starting from t_r, the bounce time t_b is computed iteratively, and, using this information the transmit time t_t is computed.

The vectors $\mathbf{x}_{\mathrm{M}}^{DB}$ and $\mathbf{x}_{\mathrm{EM}}^{DB}$ are obtained integrating the post-Newtonian equations of motions. The vectors $\mathbf{x}_{\mathrm{sat}}^{DM}$ are obtained by integrating the orbit in the mercurycentric TDM-compatible frame. The vector $\mathbf{x}_{\mathrm{ant}}^{DT}$ is obtained from a standard IERS model of Earth rotation, given accurate station coordinates, and $\mathbf{x}_{\mathrm{E}}^{DT}$ from lunar ephemerides [Milani and Gronchi 2009].

In the following subsections we shall describe the procedure to compute the range observable (Section 4.1) and the range rate (Section 4.2).

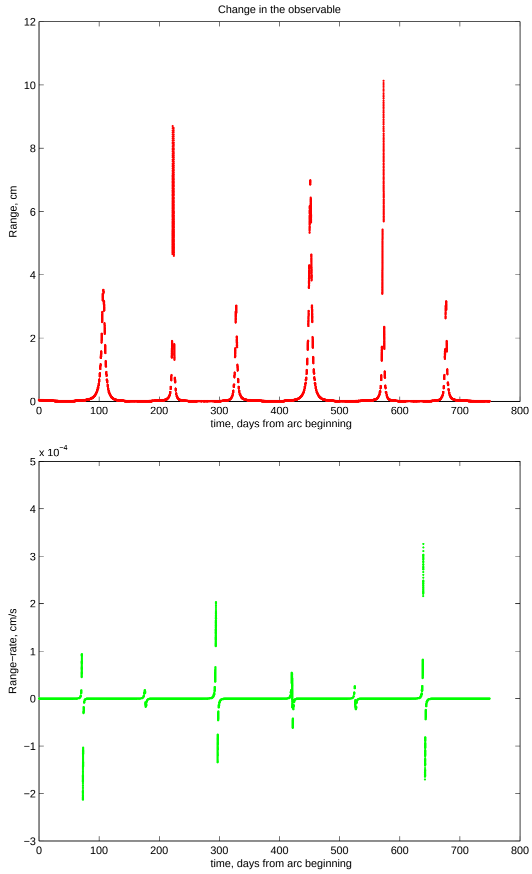

Fig. 4 Differences in range (top) and range rate (bottom) by using an order 1 and an order 2 post-Newtonian formulation ($\gamma = 1$); the correction is relevant for BepiColombo, at least when a superior conjunction results in a small impact parameter b. E.g., in this figure we have plotted data assumed to be available down to $b \simeq 3R_0$. For larger values of b the effect decreases as $1/b^2$.

4.1 Range

Once the five vectors are available at the appropriate times and in a consistent SSB system, there are two different light-times, the up-leg $\Delta t_{up} = t_b - t_t$ for the signal from the antenna to the orbiter, and the down-leg $\Delta t_{down} = t_r - t_b$ for the return signal. They are defined implicitly by the distances up-leg and down-leg

$$\begin{aligned} &\mathbf{r}_{do}(t_r) = \mathbf{x}_{\mathrm{sat}}(t_b(t_r)) + \mathbf{x}_{\mathrm{M}}(t_b(t_r)) - \mathbf{x}_{\mathrm{EM}}(t_r) - \mathbf{x}_{\mathrm{E}}(t_r) - \mathbf{x}_{\mathrm{ant}}(t_r) \ , \\ &r_{do}(t_r) = |\mathbf{r}_{do}(t_r)| \ , \qquad c(t_r - t_b) = r_{do}(t_r) + S_{do}(\gamma) \ , \end{aligned} \tag{3}$$

$$\begin{aligned} &\mathbf{r}_{up}(t_r) = \mathbf{x}_{\mathrm{sat}}(t_b(t_r)) + \mathbf{x}_{\mathrm{M}}(t_b(t_r)) - \mathbf{x}_{\mathrm{EM}}(t_t(t_r)) - \mathbf{x}_{\mathrm{E}}(t_t(t_r)) - \mathbf{x}_{\mathrm{ant}}(t_t(t_r)) \ , \\ &r_{up}(t_r) = |\mathbf{r}_{up}(t_r)| \ , \qquad c(t_b - t_t) = r_{up}(t_r) + S_{up}(\gamma) \ , \end{aligned} \tag{4}$$

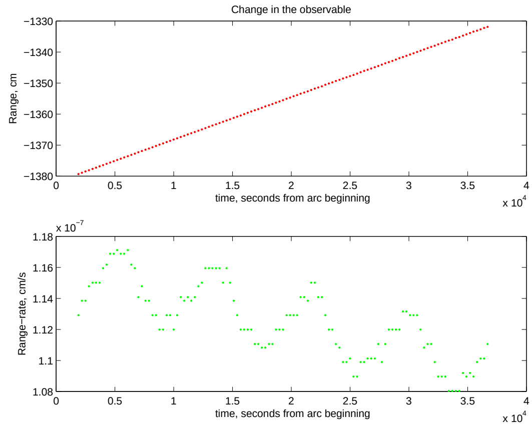

Fig. 5 The difference in the observables range and range rate using a light-time in TT and a light time in TDB: the difference in range is very high, more than 13 meters in one day, while the difference in range rate is less than the accuracy of the experiment.

respectively, with somewhat different Shapiro effects. Then $t_r - t_b$ and $t_b - t_t$ are the two portions of the light-time, in the time attached to the SSB, that is TDB; this provides the computation of t_t. Then these times are to be converted back in the time system applicable at the receiving station, where the time measurement is performed, which is TT (or some other form of local time, such as the standard UTC). t_r is already available in the local time scale, from the original measurement, while t_t needs to be converted back from TDB to TT. The difference between these two TT times is Δt_{tot}, from which we can conventionally define $r(t_r) = c\,\Delta t_{tot}/2$. Note that the difference Δt_{tot} in TT is significantly different from $t_r - t_t$ in TDB, by an amount of the order of 10^{-7} s, while the sensitivity of the BepiColombo radioscience experiment is of the order of 10^{-9} s, thus these conversions change the computed observable in a significant way, see Figure 5.

The practical method for solving $t_b(t_r)$ and $t_t(t_r)$ in Eqs. (3) and (4) is as follows. Since the measurement is labeled with the receive time t_r, the iterative procedure needs to start from eq. (3) by computing the states $\mathbf{x}_{\mathrm{EM}}$, $\mathbf{x}_{\mathrm{E}}$ and $\mathbf{x}_{\mathrm{ant}}$ at epoch t_r, then selecting a rough guess t_b^0 for the bounce time (e.g., $t_b^0 = t_r$). Then the states $\mathbf{x}_{\mathrm{sat}}$ and $\mathbf{x}_{\mathrm{M}}$ are computed at t_b^0 and a successive guess t_b^1 is given by (3). This is repeated computing t_b^2, and so on until convergence, that is, until $t_b^k - t_b^{k-1}$ is smaller than the required accuracy. This fixed point iteration to solve the implicit equation for t_b is convergent because the motion of the satellite and of Mercury, in the time $t_r - t_b$, is a small fraction of the total difference vector. After accepting the last value of t_b we start with the states $\mathbf{x}_{\mathrm{sat}}$ and $\mathbf{x}_{\mathrm{M}}$ at t_b and with a rough guess t_t^0 for the transmit time (e.g., $t_t^0 = t_b$). Then $\mathbf{x}_{\mathrm{EM}}$, $\mathbf{x}_{\mathrm{E}}$ and $\mathbf{x}_{\mathrm{ant}}$ are computed at epoch t_t^0 and t_t^1 is given by eq. (4), and the same procedure is iterated to convergence, that is to achieve a small enough $t_t^k - t_t^{k-1}$. This double iterative procedure to compute range is consistent with what has been used for a long time in planetary radar [Yeomans *et al.* 1992]. We conventionally define $r = (r_{do} + S_{do} + r_{up} + S_{up})/2$.

4.2 Range rate

After the two iterations providing at convergence t_b and t_t are complete, we can proceed to compute the range rate. We rewrite the expression for the Euclidean range (down-leg and up-leg) as a scalar product:

$$
\begin{aligned}
r_{do}^2(t_r) &= [\mathbf{x}_{\mathrm{Ms}}(t_b) - \mathbf{x}_{\mathrm{Ea}}(t_r)] \cdot [\mathbf{x}_{\mathrm{Ms}}(t_b) - \mathbf{x}_{\mathrm{Ea}}(t_r)] \ , \\
r_{up}^2(t_r) &= [\mathbf{x}_{\mathrm{Ms}}(t_b) - \mathbf{x}_{\mathrm{Ea}}(t_t)] \cdot [\mathbf{x}_{\mathrm{Ms}}(t_b) - \mathbf{x}_{\mathrm{Ea}}(t_t)]
\end{aligned}
$$

where $\mathbf{x}_{\mathrm{Ms}} = \mathbf{x}_{\mathrm{M}} + \mathbf{x}_{\mathrm{sat}}$ and $\mathbf{x}_{\mathrm{Ea}} = \mathbf{x}_{\mathrm{EM}} + \mathbf{x}_{\mathrm{E}} + \mathbf{x}_{\mathrm{ant}}$. The light-time equation contains also the Shapiro terms, thus the range rate observable contains also additive terms $\dot{S}_{do}$ and $\dot{S}_{up}$, with significant effects (a few cm/s during superior conjuctions). Since the equations giving t_b and t_t are still (3) and (4), in computing the time derivatives, we need to take into account that $t_b = t_b(t_r)$ and $t_t = t_t(t_r)$, with non-unit derivatives.

Computing the derivative with respect to the receive time t_r, and using the dot notation to stand for d/dt_r, we obtain:

$$
\dot{r}_{do}(t_r) = \hat{\mathbf{r}}_{do} \left[\dot{\mathbf{x}}_{\mathrm{Ms}}(t_b) \left(1 - \frac{\dot{r}_{do}(t_r) + \dot{S}_{do}}{c} \right) - \dot{\mathbf{x}}_{\mathrm{Ea}}(t_r) \right] \ , \tag{5}
$$

$$
\begin{aligned}
\dot{r}_{up}(t_r) = \hat{\mathbf{r}}_{up} \Bigg[& \dot{\mathbf{x}}_{\mathrm{Ms}}(t_b) \left(1 - \frac{\dot{r}_{do}(t_r) + \dot{S}_{do}}{c} \right) - \\
& \dot{\mathbf{x}}_{\mathrm{Ea}}(t_t) \left(1 - \frac{\dot{r}_{do}(t_r) + \dot{S}_{do}}{c} - \frac{\dot{r}_{up}(t_r) + \dot{S}_{up}}{c} \right) \Bigg]
\end{aligned} \tag{6}
$$

where

$$
\hat{\mathbf{r}}_{do} = \frac{1}{r_{do}(t_r)} \left(\mathbf{x}_{\mathrm{Ms}}(t_b) - \mathbf{x}_{\mathrm{Ea}}(t_r) \right)
$$

and

$$
\hat{\mathbf{r}}_{up} = \frac{1}{r_{up}(t_r)} \left(\mathbf{x}_{\mathrm{Ms}}(t_b) - \mathbf{x}_{\mathrm{Ea}}(t_t) \right) .
$$

However, the contribution of the time derivatives of the Shapiro effect to the dt_b/dt_r and dt_t/dt_r corrective factors is small, of the order of 10^{-10}, which is marginally significant for the BepiColombo radioscience experiment. We conventionally define $\dot{r} = c(1 - \dot{t}_t)/2 = (\dot{r}_{do} + \dot{S}_{do} + \dot{r}_{up} + \dot{S}_{up})/2$. These equations are compatible with [Yeomans *et al.* 1992], taking into account that they use a single iteration.

Since the time derivatives of the Shapiro effects contain $\dot{r}_t$ and $\dot{r}_r$ the equations (5) and (6) are implicit, thus we can again use a fixed point iteration. It is also possible to use a very good approximation which solves explicitly for $\dot{r}_{do}$ and then for $\dot{r}_{up}$, neglecting the very small contribution of Shapiro terms:

$$
\dot{r}_{do} = \hat{\mathbf{r}}_{do} \cdot \left[\dot{\mathbf{x}}_{\mathrm{Ms}}(t_b) \left(1 - \frac{\dot{S}_{do}}{c} \right) - \dot{\mathbf{x}}_{\mathrm{Ea}}(t_r) \right] \left[1 + \frac{\dot{\mathbf{x}}_{\mathrm{Ms}}(t_b) \cdot \hat{\mathbf{r}}_{do}}{c} \right]^{-1}
$$

where the right hand side is weakly dependent upon $\dot{r}_{do}$ only through $\dot{S}_{do}$, thus a moderately accurate approximation could be used in the computation of $\dot{S}_{do}$, followed by a single iteration. For the other leg

$$\dot{r}_{up}(t_r) = \hat{\mathbf{r}}_{up} \cdot \left[\dot{\mathbf{x}}_{\mathrm{Ms}}(t_b) \left(1 - \frac{\dot{r}_{do}(t_r) + \dot{S}_{do}}{c} \right) - \dot{\mathbf{x}}_{\mathrm{Ea}}(t_t) \left(1 - \frac{\dot{r}_{do}(t_r) + \dot{S}_{do}}{c} - \frac{\dot{S}_{up}}{c} \right) \right]$$
$$\left[1 - \frac{\dot{\mathbf{x}}_{\mathrm{Ea}}(t_t) \cdot \hat{\mathbf{r}}_{up}}{c} \right]^{-1} .$$

All the above computations are in SSB with TDB; however, the frequency measurements, at both t_t and t_r, are done on Earth, that is with a time which is TT. This introduces a change in the measured frequencies at both ends, and because this change is not the same (the Earth having moved by about 3×10^{-4} of its orbit) there is a correction needed to be performed. The quantity we are measuring is essentially the derivative of t_t with respect to t_r, but this in two different time systems: for readability, we use T for TT, t for TDB

$$\frac{dT_t}{dT_r} = \frac{dT_t}{dt_t} \frac{dt_t}{dt_r} \frac{dt_r}{dT_r} ,$$

where the derivatives of the time coordinate changes are the same as the right hand sides of the differential equation giving T as a function of t in the first factor and the inverse of the same for the last factor. However, the accuracy required is such that the main term with the mass of the Sun μ_0 and the position of the Sun $\mathbf{x}_0$ is enough:

$$\frac{dT_t}{dT_r} = \left[1 - \frac{\mu_0}{|\mathbf{x}_E(t_t) - \mathbf{x}_0(t_t)|\, c^2} - \frac{|\dot{\mathbf{x}}_{\mathrm{E}}(t_t)|^2}{2\, c^2} \right] \frac{dt_t}{dt_r}$$
$$\left[1 - \frac{\mu_0}{|\mathbf{x}_{\mathrm{E}}(t_r) - \mathbf{x}_0(t_r)|\, c^2} - \frac{|\dot{\mathbf{x}}_{\mathrm{E}}(t_r)|^2}{2\, c^2} \right]^{-1} . \tag{7}$$

Note that we do not need the L_C constant term discussed above because it cancels in the first and last terms in the right hand sides of Eq. (7). The correction in the above formula is required for consistency, but in fact the correction has an order of magnitude of 10^{-7} cm/s and is negligible for the sensitivity of the BepiColombo radioscience experiment (Figure 5).

5 Numerical problems and solutions

The computation of the observables, as presented in the previous section, is already complex, but still the list of subtle technicalities is not complete.

A problem well known in radioscience is that for top accuracy the range rate measurement cannot be the instantaneous value $\dot{r}(t_r) = (\dot{r}_{do}(t_r) + \dot{S}_{do} + \dot{r}_{up}(t_r) + \dot{S}_{up})/2$. In fact, the measurement is not instantaneous: an accurate measure of a Doppler effect requires to fit the difference of phase between carrier waves, the one generated at the station and the one returned from space, accumulated over some *integration time* Δ, typically between 10 and 1000 s. Thus the observable is really a difference of ranges

$$\dot{r}_{\Delta}(t_r) = \frac{r(t_b + \Delta/2) - r(t_b - \Delta/2)}{\Delta} \tag{8}$$

or, equivalently, an averaged value of range rate over the integration interval

$$\dot{r}_{\Delta}(t_r) = \frac{1}{\Delta} \int_{t_b - \Delta/2}^{t_b + \Delta/2} \dot{r}(s)\, ds . \tag{9}$$

In order to understand the computational difficulty we need to take also into account the orders of magnitude. As said in the introduction, for state of the art tracking systems, such as those using a multi-frequency link in the X and Ka bands, the accuracy of the range measurements can be $\simeq 10$ cm and the one of range-rate 3×10^{-4} cm/s (over an integration time of 1 000 s). Let us take an integration time $\Delta = 30$ s, which is adequate for measuring the gravity field of Mercury; in fact if the orbital period is $\simeq 8\,000$ s, the harmonics of order $m = 26$ have periods as short as $\simeq 150$ s.

The accuracy over 30 s of the range rate measurement can be, by Gaussian statistics, $\simeq 3 \times 10^{-4}\ \sqrt{1\,000/30} \simeq 17 \times 10^{-4}$ cm/s, and the required accuracy in the computation of the difference $r(t_b + \Delta/2) - r(t_b - \Delta/2)$ is $\simeq 0.05$ cm. The distances can be as large as $\simeq 2 \times 10^{13}$ cm, thus the relative accuracy in the difference needs to be 2.5×10^{-15}. This implies that *rounding off* is a problem with current computers, with relative rounding off error of $\varepsilon = 2^{-52} = 2.2 \times 10^{-16}$; extended precision is supported in software, but it has many limitations. The practical consequences are that the computer program processing the tracking observables, at this level of precision and over interplanetary distances, needs to be a mixture of ordinary and extended precision variables. Any imperfection may result in "banding", that is residuals showing a discrete set of values, implying that some information corresponding to the real accuracy of the measurements has been lost in the digital processing.

As an alternative, the use of a quadrature formula for the integral in eq. (9) can provide a numerically more stable result, because the S/N of the range rate measurement is $\ll 1/\varepsilon$. Figure 5 shows that a very small model change, generating a range rate signal ≤ 2 micron/s over one pass, can be computed smoothly by using a 7 nodes Gauss quadrature formula.

6 Conclusions

By combining the results of the previous paper [Milani *et al.* 2009] and of this one, we have completed the task of showing that it is possible to build a consistent relativistic model of the dynamics and of the observations for a Mercury orbiter tracked from the Earth, at a level of accuracy and self-consistency compatible with the very demanding requirements of the BepiColombo radioscience experiment.

In particular, in this paper we have given the algorithm definitions for the computation of the observables range and range rate, including the reference system effects and the Shapiro effect. We have shown which computation can be performed explicitly and which ones need to be obtained from an iterative procedure. We have also shown how to push these computations, when implemented in a realistic computer with rounding-off, to the needed accuracy level, even without the cumbersome usage of quadruple precision. The list of "relativistic corrections", assuming we can distinguish their effects separately, is long, and we have shown that many subtle effects are relevant to the required accuracy. However, in the end what is required is just to be fully consistent with a post-Newtonian formulation to some order, to be adjusted when necessary. In fact, the Moyer's correction to the Shapiro effect in range is the only second order effect we have found to be necessary in our model.

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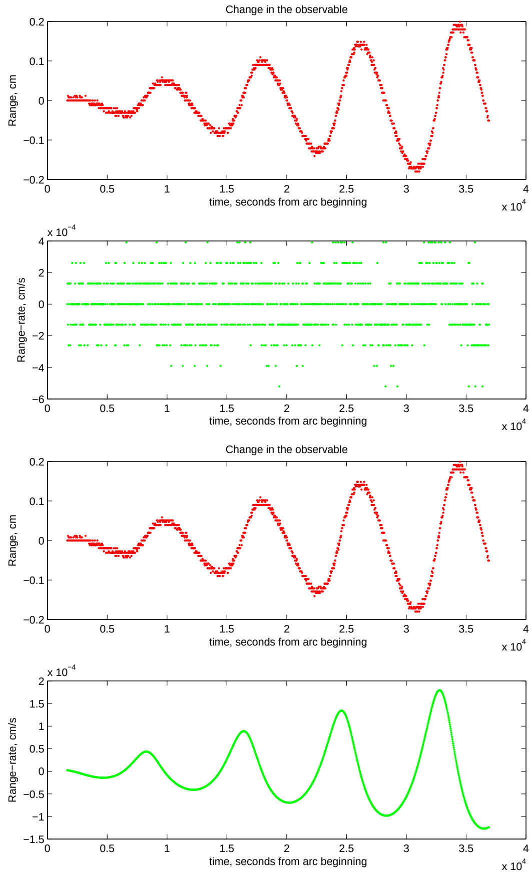

Fig. 6 Range and range rate differences due to a change by 10^{-11} of the C_{22} harmonic coefficient. Top: the range rate computed as range difference divided by the integration time of 30 s, eq. (8), is obscured by the rounding off. Bottom: the range rate computed as an integral, eq. (9), is smooth; the difference is marginally significant with respect to the measurement accuracy.

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