APPROXIMATE SYMMETRIES OF INTERPLANETARY ORBIT DETERMINATION. THE BEPICOLOMBO CASE

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Abstract

The Radio science experiment of the BepiColombo mission to the planet Mercury shall allow to solve for the gravity field of the planet up to degree and order 25, for an improved orbit of Mercury and a very accurate verification of Einstein's theory of gravitation, for measuring the rotation state of the planet, for mapping the topography (in combination with a laser altimeter). To actually obtain all the above results from the range and range-rate data we will have to solve a complex orbit determination problem which is not automatically well posed and numerically stable. The theory of symmetries and rank deficiencies provides the conceptual tools to find the critical steps of the data processing procedure and to overcome the difficulties, providing results adequate for the achievement of the mission science goals.

BepiColombo is the next cornerstone mission of planetary exploration of the European Space Agency [Balogh et al. 2000]. It shall launch two spacecraft in orbit around Mercury; one of these , the Mercury Planetary Orbiter (MPO), will investigate mostly the solid planet surface and interior. Information on the interior structure can be obtained only by a geophysical investigation based upon gravimetry, magnetometry and measurement of the planet's rotation state. Another of the mission goals is to test Einstein's theory of gravitation by measuring very accurately the orbit of Mercury around the Sun and the propagation of radio waves in the gravity field of the Sun.

The main tool to achieve the goals above is the Mercury Orbiter Radio science Experiment (MORE). The on-board accelerometer (ISA), the highest resolution imager and the laser altimeter (BELA) instruments shall also contribute to the above goals. Thanks to a sophisticated multi-frequency radio wave link, MORE will provide extreme accuracy *range* (errors $\simeq 10\ cm$) and *range-rate* (errors $\simeq 1.5\ \mu/s$) between the BepiColombo MPO and 1-2 ground stations.

1 The BepiColombo orbit determination

One passage of Mercury above the horizon of a ground station lasts $\simeq 8$ hours (more in summer, less in winter). Thus the range and range-rate data are naturally split into *arcs*, $\simeq 1$ per day; the MPO should orbit around Mercury for at least one year, thus there are hundreds of arcs. For each arc we need to solve for the *local parameters*, relevant only for the data of the arc. With all the data of all arcs we need to solve for the *global parameters*, including the science goals, in a least square fit.

Science goals include the *harmonic coefficients* of Mercury's gravitational potential up to degree and order 25, the *Post-Newtonian Parameters* (PPN) controlling the relativistic orbit and a set of coefficients defining the *rotation state* of the planet. The following is a list of parameters to be solved in the fit, including the science goals as well as calibration and intermediate parameters [Milani et al. 2001, Milani et al. 2002]

Local parameters:

- 6 MPO initial conditions for the arc,
- 3 calibration coefficients for the accelerometer,
- 6 corrections to heliocentric Mercury $(*)$

for a total of $\simeq 15 \cdot 365$ parameters.

Global parameters:

- $26^2 - 3$ harmonics of Mercury's potential,
- Mercury's dynamical Love number k_2,
- 12 initial conditions for Mercury and Earth $(*)$,
- γ, β, η and other PPN, mass and J_2 of the Sun $(*)$,
- ≥ 2 par. describing Mercury's rotation $(**)$

for a total of $\simeq 700$ parameters. (The stars mark some of the alternative choices discussed later). Overall there shall be $m \simeq 1,000,000$ range and range-rate data points, thus equations, in $N \simeq 6,000$ unknown. The main problem is not the size of the least square fit, which is quite manageable even without special computing facilities: state of the art satellite geodesy solution for the gravity field of the Earth are significantly larger. The problem to assess the performance of MORE is whether the orbit determination problem is well posed and numerically stable.

2 Rank deficiency and symmetries

We shall discuss the MPO orbit determination problem in the context of the classical least squares principle, which can be modified to include the appropriate weighing of the residuals to take into account the observational accuracy and also the correlations. If the problem is summarized in the classical *normal equation* relating the corrections ΔX of the parameters vector X to the normalized residuals vector Ξ

$$B^T\, B\, \Delta X = -B^T\, \Xi$$

where $B = \partial \Xi / \partial X$, then the *normal matrix* $C = B^T\, B$ is numerically singular, the *covariance matrix* $\Gamma = C^{-1}$ cannot be reliably computed and anyway the confidence ellipsoid (with matrix C) is huge. If the differential corrections are iterated, the procedure diverges. If the iterations are stopped, the solution obtained is disastrously wrong.

This disaster is due to a well known, although not fully understood, phenomenon called *rank deficiency.* Rank deficiency is *exact* if C is singular, *approximate* if C is very badly conditioned (ratio of largest to smallest eigenvalue comparable to the inverse of machine accuracy).

Would this imply the total failure of 3 BepiColombo experiments (MORE, ISA, BELA)? No, the experts in orbit determination know *recipes* to remove rank deficiency, but this is kraftmanship, not science. Thus we have set as a goal for our BepiColombo related research to build a rigorous theory of rank deficiency and of the ways to cope with it. The first result of such a theory is not new, although it is not found in textbooks:

Theorem: If there is an effective one-parameter group of transformations which are exact symmetries of all the observations then the $N \times N$ normal matrix C has rank $N-1$. If there is an effective dimension d Lie group of exact symmetries then C has rank $N-d$.

Effective means that only the unit element of the group acts as the identity transformation. *Exact symmetry* means that after applying the transformation the residuals are exactly the same. *Approximate symmetry* means that a value ϵ of the symmetry parameter changes the residuals by $\mathcal{O}(\epsilon^2)$. The converse of the above theorem is not always true: if the rank of C is $N-d$ there could be approximate symmetries only (unless some other hypothesis is applicable); if the rank of C is everywhere $N-1$ there is a local one-parameter of exact symmetries.

There are classical examples of symmetries and rank deficiency in the n-body problem, that is in the orbit determinations for the planets of our Solar System: if the observations are only range and/or range-rate between planets (e.g., radar), the 3-dimensional group $SO(3)$ of rotations is an exact and effective group of symmetries. If the observations are angles only, the changes of scale by λ in length and μ in mass are exact symmetries for $\lambda^3 = \mu$.

A similar case of symmetry applies to the BepiColombo case. The initial conditions for the Earth, for Mercury and the mass of the Sun cannot be adjusted at once, by using only range/range-rate data between the Earth and the Mercury system. If there were only Earth and Mercury there would be an exact symmetry. Due to the weak coupling with the other planets, there is an approximate symmetry and the normal matrix C is very badly conditioned. As we have shown in [Milani et al. 2002], 4 constraints, which essentially reduce the 13 parameters listed above to only 9, are needed to stabilize the problem, which can then be solved in stable and accurate way. The PPN can be solved simultaneously, and the resulting accuracy shall allow to improve very significantly our knowledge of the theory of gravitation.

The above example shows the method we use: we first find an exact symmetry, by analyzing geometrically a simplified problem, then identify the approximate symmetry resulting from adding more realistic details and use it to understand an approximate rank deficiency. Once a rank deficiency is identified, it can be remove it by either some *constraints* imposed upon the parameters of the problem or by *descoping*, that is removing some parameters from the list of those to be solved.

One exact symmetry is well known in the limit case for distance $\to +\infty$, which occurs in the now fashionable problem of orbit determination for extrasolar planets. If the orbit of a satellite is rotated around the fixed direction from the Earth to the central body (assumed to be spherical), the residuals in range and range-rate are exactly the same.

For extrasolar planets (or dim companion stars), observed by their range-rate perturbations on the central star, since the distance is practically infinite and the direction of observations hardly changes, this symmetry is exact. The best solution, in this case, is descoping: the longitude of the node Ω, using as reference plane the *sky plane* orthogonal to the direction to the Earth, is removed from the parameters to be fit and remains unknown[1].

In the Mercury orbiter case the symmetry is approximate, since the direction from the Earth to Mercury changes, but over time scales much longer than the period of the mercury-centric orbit [Bonanno and Milani 2002]. In this case descoping is not appropriate, we use constraints to the orbit arising from long arc solutions, for which the symmetry is broken. However, long arc solutions are less accurate because of the difficulty of modeling non-gravitational perturbations (the accelerometer needs to be calibrated, and this is very difficult during the time span between two observed arcs). E.g., in a simulation we have assumed an a priori knowledge of the MPO orbit within 3 meters in position, the final results is a thin confidence ellipse for the spacecraft position with a long axis of $\simeq 3$ m and shorter axes of a few cm [Milani et al. 2001].

3 Symmetries and degeneracy for a planetary orbiter

The case of a planetary orbiter is somewhat more complicated than the ones described in the classical literature on orbit determination, thus some comparatively simple symmetries have never been discussed in published papers.

The ranging symmetry

If the corrections to Mercury's orbit are considered as *local parameters* (later fit in an orbit determination for the planets), then the *translations* in position and in velocity along the plane orthogonal to the Earth-Mercury direction are approximate symmetries. Thus already in the solution for the local parameters the 15×15 normal matrix is very badly conditioned.

This can be solved by descoping. If the arc is short, ranging to the MPO only provides an estimated correction to the range to Mercury's *center of mass* (CoM). The range-rate to MPO allows to correct the range-rate to Mercury CoM. Thus the maximum number of local parameters is 11, not 15. However, this is a quick solution, adopted in [Milani et al. 2001] to perform a simulation of the MORE experiment with a shorter development time for the software. A more rigorous, and presumably somewhat more accurate solution would be obtained by removing the corrections to the orbit of Mercury from the local parameters, solving simultaneously for the initial

[1] Another symmetry of this case does not allow to solve independently for the mass ratio and for the inclination of the orbit with respect sky plane.

conditions of the planetary orbits and for the PPN as global parameters in a single least squares fit. This choice requires a more complicated software, with the advantage of a simpler theory and more numerically stable results, thus it is more appropriate if time and resources allow it, as it will be the case before the real data are available.

The photogravitational symmetry

This is a symmetry involving parameters of a different nature, thus it is not easy to find, although given the basic idea the equations end up being simple. The three accelerometer axes are orthogonal and oriented radially (to Mercury's CoM), in the orbit plane and out of plane. Let there be a constant miscalibration f along the out of plane axis; or, let there be a constant radiation pressure acceleration in the out of plane direction and no accelerometer, which gives the same.

If the spacecraft has a circular orbit of radius r lying on a plane displaced, with respect to the CoM of the planet, by a distance h in the direction opposite to the one to the Sun, then the three vectors of the gravitational monopole attraction $\vec{G}$, of the radiation pressure acceleration $\vec{F}$ and of the centrifugal acceleration $\omega^2\vec{R}$ can have a zero sum if the following equations are satisfied

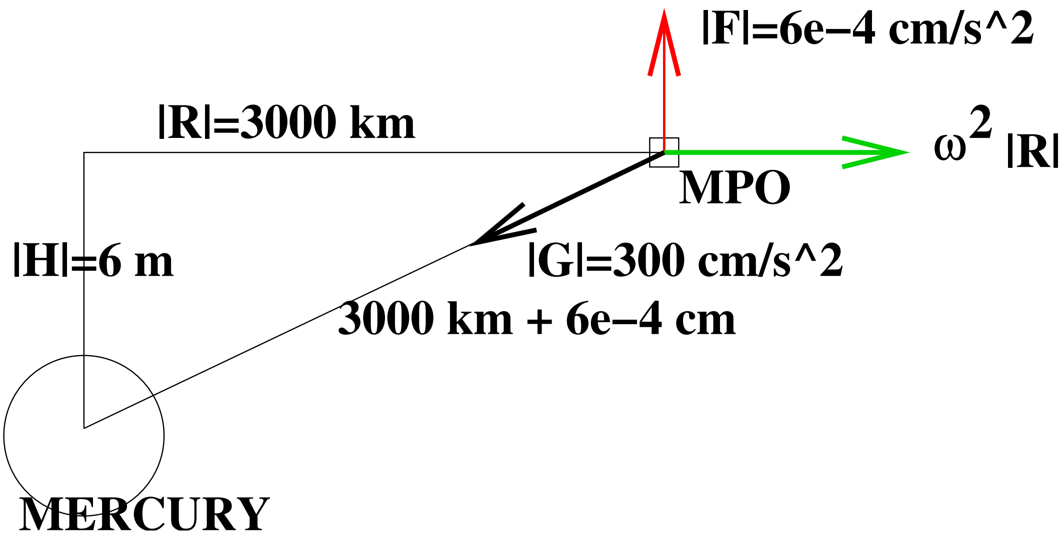

Figure 1: The photogravitational symmetry for a simplified version of the MPO orbit determination.

$$\vec{0} \;=\; \omega^2\,\vec{R}+\vec{G}+\vec{F}$$

$$
\begin{aligned}
\vec{G} &= -\frac{GM\ (\vec{R}+\vec{H})}{(r^2+h^2)^{3/2}}\ ;\ \omega^2\ r = \frac{GM\ r}{(r^2+h^2)^{3/2}} \\
f &= \frac{GM\ h}{(r^2+h^2)^{3/2}} \Longrightarrow \frac{h}{r} = \frac{f}{\omega^2\ r} \\
\frac{M'}{M} &= \left(1+\frac{h^2}{r^2}\right)^{-3/2} \Longrightarrow M' = M\ (1-6\times 10^{-12})
\end{aligned}
$$

The last equation shows that the symmetry can be exact if the parameters f, h, M are changed simultaneously in a way satisfying the equations above: that is, the position of the CoM of Mercury, the accelerometer calibration (or radiation pressure coefficient) and the mass of Mercury cannot be estimated at once. If the mass of the planet is estimated otherwise, an approximate symmetry remains because the change in the estimated mass M is very small, for the order of magnitude values of the other parameters corresponding to the MPO case shown in Figure 1. If the miscalibration f is ignored, the position of Mercury CoM is estimated incorrectly, by several meters, which is relevant for the relativity experiment.

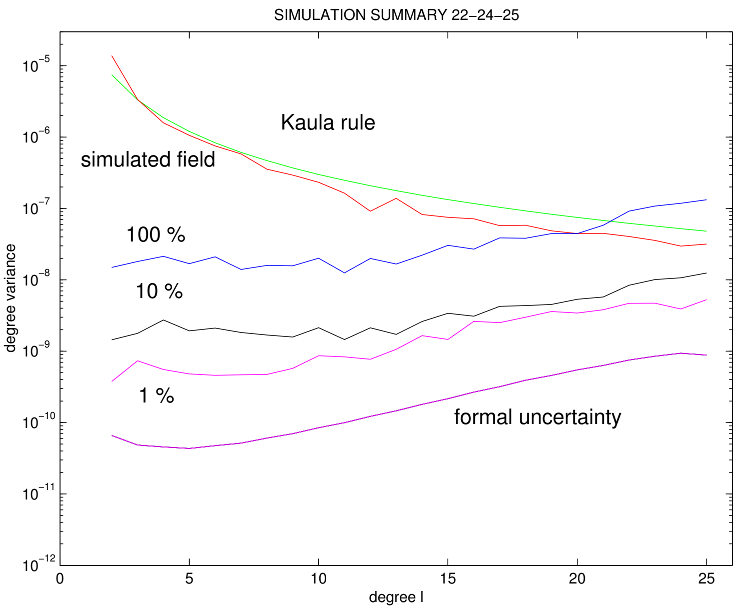

Figure 2: Results of the solution for the gravity field of Mercury. The two upper lines show the (simulated) signal as a function of the harmonic degree, the four below describe the accuracy of the results as discussed in the text.

The results of the simulations (gravimetry)

If the accelerometer miscalibrations are too large, by some form of photogravitational symmetry (which has not yet been analyzed in detail) the gravity field of Mercury is too much in error, as shown by the line corresponding to 100% accelerometer calibration error; also the orbit of the MPO is too poorly determined (e.g., to allow for estimating topography with BELA). If the accelerometer is calibrated a priori (by thermal measurement/control) at the 1% level, the results meet the scientific requirements of MORE. Note that anyway the true error is larger than the formal accuracy (bottom line).

4 The rotation experiment and the missing symmetry

According to science historians, Isaac Newton refused to publish his solution of the 2-body problem until when he had the proof that a spherically symmetric body generates, outside the body, the same gravity field as a point mass placed in the CoM. Unfortunately, this also proves that the *inverse gravimetry problem* is ill-posed, even a perfect knowledge of the gravity field outside a body does not allow to solve for the internal mass distribution.

A question often arising in discussions on the science goals of planetary exploration missions is the following. Can a planetary mission *constrain the internal structure* without landing on the planet? Newton's classical result, and its modern versions, show that even a perfect knowledge of the gravity field outside the surface of the planet does not constrain the concentration of the mass towards the center, thus does not allow to constrain the size and density of the core. E.g., the 6 coefficients of the moment of inertia tensor are linearly related to the 5 harmonic coefficients of degree 2: if the latter are measured with remote gravimetry, one free parameter remains[2].

A solution of the problem could be to directly observe the rotation state of the planet, e.g., by using the high resolution camera (as it has been proposed for BepiColombo at Mercury), or by using the radar images of the surface (as proposed for Cassini at Titan). From a suitably defined obliquity of the rotation axis it is possible to estimate the absolute value of the principal moment of inertia, thus scaling correctly the moment of inertia tensor. It is also possible to measure the libration in longitude resulting from the coupling of the permanent equatorial ellipticity of Mercury with the Sun's tidal pull,

[2]That is, there is a rank deficiency of 1 corresponding to a symmetry moving all the mass of the planet closer to the CoM.

and from this to detect the presence of a decoupling (liquid layer) between core and mantle (see Peale, these proceedings).

However, this *rotation experiment* proposed for BepiColombo imposes very tough constraints on the thermo-mechanical design of the MPO, because of the need to measure spacecraft to surface directions in an absolute reference frame. Thus it would be desirable to measure the rotation from the gravity field above the surface. In principle, the time-dependent gravity field generated by a rotating planet, measured in an inertial frame, depends upon the rotation state. Thus by tracking for long enough and accurately enough a satellite it could be possible to measure the planet rotation without looking at the planet! We have tested this hypothesis by a numerical experiment of MPO orbit determination, adding to the best case shown in Figure 2 (1% accelerometer calibration) just two parameters (obliquity, amplitude of libration in longitude).

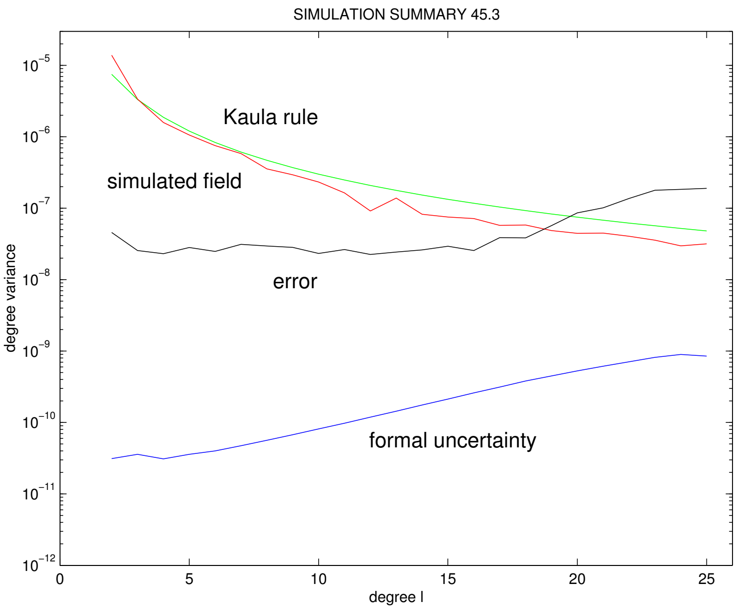

Figure 3: Results of the simultaneous solution for the gravity field and the rotation of Mercury. The error is larger than the signal for the harmonic coefficients of degree ≥ 20.

The results of this combined *gravimetry-rotation experiment* have been disastrous (see Figure 3). Just by adding two rotation parameters to be solved the error in the gravity field increases by a factor $\simeq 100$ with respect to the comparable case of Figure 2. The rotation parameters are essentially undetermined: the iterative correction procedure is divergent, with changes in the rotation parameters as large as the expected value in each iteration. Note that the formal uncertainty of the libration in longitude is very small

(1/6000 of the expected value) and totally meaningless as prediction of the actual accuracy: this should be a strong warning against presenting such formal results as assessment of mission performance.

What we may call *Newton's principle*: "no mass distribution from gravimetry alone", is preserved in the hypothetical BepiColombo gravimetry-rotation experiment by some *hidden symmetry*! An unknown combination of changes in initial conditions, in harmonic coefficients, in calibrations and in Mercury's CoM reproduces a different rotation of Mercury. Thus we plan to go ahead with a separate *rotation experiment* using direct observations by imaging of the rotation of the planet surface.

The open problem is to identify such symmetry: does it involve only the harmonic coefficients and the parameters appearing in the rotation model? In this case it would be a straightforward generalization of Newton's principle to the time dependent, rotating planet case. Or does it result from a combination of local parameters (including initial conditions, accelerometer calibrations, corrections to Mercury's position and velocity) with global ones, thus it is a symmetry involving parameters of a very different physical nature, like the photogravitational symmetry? Is there a way to cope with this rank deficiency, either by constraints or by descoping? Could a future mission with an even more advanced technology, e.g., with a much better a priori accelerometer calibration, prevail on Newton's principle and measure the properties of some planetary core with the Radio Science data only? We hope that these questions will be answered in the future.

References

[Balogh et al. 2000] A. Balogh, et al., BepiColombo: An interdisciplinary cornerstone mission to the planet Mercury, ESA-SCI(2000)1;

[Bonanno and Milani 2002] Bonanno, C., Milani, A. 2002, *Symmetries and rank deficiency in the orbit determination around another planet*, Cel. Mech. Dyn. Ast., **83**, 17–33.

[Milani et al. 2001] Milani, A., Rossi, A., Vockrouhlicky, D., Villani, D., Bonanno, C. 2001, *Gravity field and rotation state of Mercury from the BepiColombo Radio Science Experiments*, Planetary and Space Science, **49**, 1579–1596.

[Milani et al. 2002] Milani, A, Vokrouhlický, D., Villani, D., Bonanno, C. & Rossi, A. 2002 *Testing general relativity with the BepiColombo radio science experiment*, Physical Review D, 66.