Chapter 3

Perturbation theory and the averaging principle

3.1 Integrable systems and action-angle variables

We say that a system of Ordinary Differential Equations (ODEs) is *integrable* if its solutions can be expressed by analytic formulas up to inversions (by the implicit function theorem) or quadratures; we call the system *non-integrable* if this is not possible.

It is easy to show that if we consider a system of $2n$ differential equations (where n is an integer) and we have $2n$ first integrals, then we can integrate the equations and solve the problem. If the system is canonical, in several cases we only need to know n first integrals satisfying some properties to integrate the system. We have in fact the following:

THEOREM 10. *(Liouville-Arnold) Let $\mathcal{M}$ be a symplectic manifold $2n$-dimensional and let $F_i : \mathcal{M} \to \mathbb{R}, i = 1 \dots n$ be functions, defined on $\mathcal{M}$, that are* in involution, *that is the Poisson brackets $\{F_i, F_j\}$ vanish for each $0 \leq i < j \leq n$. We consider the level set of the F_i:*

$$\mathcal{M}_f = \{x : F_i(x) = f_i, i = 1 \dots, n\}.$$

Let us assume that the F_i are independent on $\mathcal{M}_f$ (the 1-forms dF_i are linearly independent in each point of $\mathcal{M}_f$). Then we can conclude that $\mathcal{M}_f$ is a smooth manifold, invariant with respect to the phase flow with Hamilton function $H = F_1$.

If $\mathcal{M}_f$ is a compact connected manifold, it is diffeomorphic to the n-dimensional torus

$$\mathbb{T}^n = \{(\phi_1, \dots, \phi_n) \ mod \ 2\pi\},$$

and

1. *the phase flow with Hamilton function H determines on $\mathcal{M}_f$ a quasi-periodic motion, that is, using angular variables $\phi = (\phi_1, \dots, \phi_n)$, we have*

$$\dot{\phi} = \omega; \qquad \omega = \omega(f);$$

2. *the canonical equations with Hamilton function H can be integrated.*

If we are in the hypotheses of Theorem 10 we can select symplectic coordinates (I,ϕ) such that the first integrals F depend only on the coordinates I (*action* variables), while ϕ (*angle* variables) are angular coordinates on the torus $\mathbb{T}^n$.

3.2 Perturbation methods

Most of the problems in Dynamics, included the three body problem (see [64]), are not integrable, anyway several of them can be represented by differential equations that are a small perturbation of an integrable problem. Let $\mathcal{M}$ be a smooth manifold, let $F,G:\mathcal{M}\longrightarrow T\mathcal{M}$ be smooth maps from the manifold $\mathcal{M}$ to its tangent bundle $T\mathcal{M}$ and let ε be a natural small parameter; we can consider the system

$$\dot{X}=F(X)+\varepsilon G(X) \qquad (\textit{perturbed problem}), \tag{3.1}$$

obtained by perturbing the integrable system

$$\dot{X}=F(X) \qquad (\textit{unperturbed problem}).$$

Even if the system (3.1) is non integrable, we can study it in a qualitative sense, that is we can study some properties of the solutions or we can try to approximate them. Perturbation theory consists of the methods that allow to approximate the solutions of a perturbed problem, like (3.1), starting from the solutions of the unperturbed one.

3.3 The averaging principle

The averaging principle is a powerful tool to study the qualitative behavior of the solutions of Ordinary Differential Equations. It consists in solving averaged equations, obtained by an integral average of the original equations over some angular variables; then we consider the solutions of the averaged equations as representative of the solutions of the original equations for a long time span (of the order of $1/\varepsilon$). A review of the classical results on averaging methods in perturbation theory can be found in [2].

Let us assume that $\mathcal{M}=\Omega\times\mathbb{T}^n$ where Ω is an open set in $\mathbb{R}^k$ (with k integer). We consider the *unperturbed equations*

$$\begin{cases}\dot{\phi}=\omega(I)\\ \dot{I}=0\end{cases}$$

and the corresponding *perturbed equations*

$$\begin{cases}\dot{\phi}=\omega(I)+\varepsilon f(I,\phi,\varepsilon)\\ \dot{I}=\qquad\quad\ \varepsilon g(I,\phi,\varepsilon)\end{cases} \tag{3.2}$$

where $f(I,\phi,\varepsilon),g(I,\phi,\varepsilon)\in C^1(\Omega\times\mathbb{T}^n\times[0,1])$ are 2π-periodic function in the variable $\phi\in\mathbb{T}^n$.

We define as *averaged equations*

$$\dot{J} = \varepsilon G(J)$$

where

$$G(J) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} g(J,\phi,0)\, d\phi_1 \ldots d\phi_n$$

is the *spatial average* over the torus $\mathbb{T}^n$.

Let us consider a time interval $[0,T]$, with $1 \ll T \ll 1/\varepsilon$, that is T is big with respect to 1, and small if compared to $1/\varepsilon$. The increment in the I variable after time T has elapsed is approximated by

$$\Delta I = \varepsilon T \left[\frac{1}{T} \int_0^T g(I,\phi(t),0)\, dt \right] + o(\varepsilon T) \tag{3.3}$$

where $I(t), \phi(t)$ are the solutions of the unperturbed problem, that is $I(t) = I(0)$ is constant (we have called it I for brevity) and $\phi(t) = \phi(0) + \omega(I)t$.

We define the *time average* of the function g as

$$g^*(\phi(0)) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T g(I,\phi(t),0)\, dt$$

(it obviously depends on the initial phase $\phi(0)$) and we notice that the expression between the square parentheses in (3.3) is representative of the time average of the function g as the time T is large. If we introduce the *slow time* $\tau = \varepsilon t$, we can write

$$\frac{\Delta I}{\Delta \tau} \approx g^*(\phi(0))$$

and, if we consider the limit for $\varepsilon T \to 0$ and we use a prime for the derivation with respect to the slow time, we obtain

$$I' = g^*(\phi(0)). \tag{3.4}$$

From relation (3.4) we can see that the passage to the averaged equations corresponds to the substitution of the time average with the spatial average.

3.3.1 Averaging in 1-frequency systems

We assume that the phase space D is the direct product of an open set $\Omega \subseteq \mathbb{R}^n$ and the circle S^1; then we can take into account the average of the perturbed equations (3.2) over the angular variable $\phi \in [0,2\pi[$. In this case the averaged equations are

$$\dot{J} = \varepsilon G(J); \qquad G(J) = \frac{1}{2\pi} \int_0^{2\pi} g(J,\phi,0)\, d\phi\, . \tag{3.5}$$

THEOREM 11. Let $I_0 \in \Omega \subseteq \mathbb{R}^n$ and let us call $I(t)$ the solution of equations (3.2) and $J(t)$ the solution of the averaged equations (3.5) starting from the same initial conditions $I(0) = J(0) = I_0$. If $\varepsilon > 0$ is small enough, if $\omega(I) \neq 0$ for all $I \in \Omega$ and if $J(t)$ remains in a compact domain $K \subset \Omega$ for $t \in [0, T/\varepsilon]$, then there exists a constant C, *independent from* ε, such that

$$|I(t) - J(t)| \leq \varepsilon C$$

for all $t \in [0, T/\varepsilon]$.

In other words, provided that ε is small enough, the difference between the solution of the perturbed equations and the solution of the averaged equations, starting from the same initial conditions, will remain small during a time span T/ε.

Proof. This proof is based on the one given in [1]. We divide it into 5 steps:

Step 1 We shall see that if ε is small enough, it is possible to define a variable change

$$\Psi_\varepsilon : (I, \phi) \longrightarrow (P, \phi)$$

where

$$P = P(I, \phi, \varepsilon) = I + \varepsilon h(I, \phi) \tag{3.6}$$

and $h(I, \phi)$ is 2π-periodic function in the variable ϕ. We want to select $h(I, \phi)$ in order to eliminate the dependence on ϕ from the differential equation for P, up to the first order in ε included. Let us differentiate equation (3.6) with respect to time: we obtain

$$\begin{aligned}\dot{P} &= \dot{I} + \varepsilon \left[\frac{\partial h}{\partial I}(I,\phi)\dot{I} + \frac{\partial h}{\partial \phi}(I,\phi)\dot{\phi}\right] = \\ &= \varepsilon g(I,\phi,\varepsilon)\left[1 + \varepsilon \frac{\partial h}{\partial I}(I,\phi)\right] + \varepsilon\left[\omega(I) + \varepsilon f(I,\phi,\varepsilon)\right]\frac{\partial h}{\partial \phi}(I,\phi) = \\ &= \varepsilon\left[g(I,\phi,\varepsilon) + \omega(I)\frac{\partial h}{\partial \phi}(I,\phi)\right] + \varepsilon^2 r(I,\phi,\varepsilon)\end{aligned} \tag{3.7}$$

where

$$r(I,\phi,\varepsilon) = \left[g(I,\phi,\varepsilon)\frac{\partial h}{\partial I}(I,\phi) + f(I,\phi,\varepsilon)\frac{\partial h}{\partial \phi}(I,\phi)\right] .$$

We search for a function $h(I, \phi)$ such that

$$\left[g(I,\phi,\varepsilon) + \omega(I)\frac{\partial h}{\partial \phi}(I,\phi)\right] = 0 \; :$$

let us set

$$h(I,\phi) = -\frac{1}{\omega(I)}\int_0^\phi \tilde{g}(I,\psi)\, d\psi \tag{3.8}$$

with

$$\tilde{g}(I,\phi) = g(I,\phi,0) - G(I).$$

so that $h(I, \phi)$ is 2π-periodic in ϕ ($\tilde{g}$ has zero average), and the following relation holds:

$$g(I,\phi,\varepsilon) + \omega\frac{\partial h(I,\phi)}{\partial \phi} = g(I,\phi,\varepsilon) - g(I,\phi,0) + G(I) \, . \tag{3.9}$$

Using (3.9) we can write equation (3.7) as follows

$$\dot{P} = \varepsilon\,[G(I) + g(I,\phi,\varepsilon) - g(I,\phi,0)] + \varepsilon^2\, r(I,\phi,\varepsilon)\ . \tag{3.10}$$

Let $K \subset \Omega$ be a *compact, convex* set containing I_0 and such that $J(t) \in K$ for each $t \in [0, T/\varepsilon]$; for each function $u = u(I,\phi,\varepsilon) \in C^1(\Omega \times S^1 \times [0,1])$ we set

$$\begin{aligned} |u|_0 &= \max_{(I,\phi,\varepsilon)\in K\times S^1\times[0,1]} |u(I,\phi,\varepsilon)|\ ; \\ |u|_1 &= |u|_0 + \max_{(I,\phi,\varepsilon)\in K\times S^1\times[0,1]} \left| \frac{\partial}{\partial(I,\phi,\varepsilon)} u(I,\phi,\varepsilon) \right|\ . \end{aligned}$$

Remark 4. When we consider functions $v = v(I,\phi)$, not depending on ε we shall use the notation $|v|_0 := |\tilde{v}|_0$ and $|v|_1 := |\tilde{v}|_1$ where $\tilde{v}(I,\phi,\varepsilon) := v(I,\phi)$.

We select a constant c_1 such that

$$|f|_1 \le c_1; \qquad |g|_1 \le c_1; \qquad |\omega^{-1}|_1 \le c_1;$$

$(\omega(I,\phi) = \omega(I))$.

Step 2 Let us show that the application $\Psi_\varepsilon : (I,\phi) \longrightarrow (P,\phi)$ is a diffeomorphism for ε small enough. From Definition 3.8 we have $h \in C^1(\Omega \times S^1)$ and $|\varepsilon h|_1 < 1$ for ε small enough. First we prove that Ψ_ε is one-to-one: if $\Psi_\varepsilon(I_1,\phi_1) = \Psi_\varepsilon(I_2,\phi_2)$ then $\phi_1 = \phi_2 := \phi$ and

$$I_1 + \varepsilon h(I_1,\phi) = I_2 + \varepsilon h(I_2,\phi).$$

This means

$$\varepsilon h(I_1,\phi) - \varepsilon h(I_2,\phi) = I_2 - I_1\ ,$$

that is in contradiction with the relation $|\varepsilon h|_1 < 1$, that gives an upper bound to the Lipschitz constant of the function εh (note that the convexity of the domain K is needed at this step to use Lagrange's theorem and obtain a Lipschitz constant smaller than 1).

From $|\varepsilon h|_1 < 1$ also follows that Ψ_ε is a local diffeomorphism, in fact

$$\left|\frac{\partial P(I,\phi,\varepsilon)}{\partial I}\right| = \left|\frac{\partial}{\partial I}\,(I + \varepsilon h(I,\phi))\right| = \left|1 + \frac{\partial}{\partial I}\,(\varepsilon h(I,\phi))\right| > \left|1 - \left|\frac{\partial}{\partial I}\,(\varepsilon h(I,\phi))\right|\right| > |1 - |\varepsilon h|_1| > 0$$

for each $(I,\phi,\varepsilon) \in K \times S^1 \times [0,\bar{\varepsilon}]$ with $\bar{\varepsilon} > 0$ small enough, and we are in the hypotheses of the *local inversion theorem*; we can conclude that Ψ_ε is a *diffeomorphism* for ε small enough.

Step 3 Writing the differential equation (3.10) in the (P,ϕ) variables, we obtain

$$\dot{P} = \varepsilon\,[G(P) + R(P,\phi,\varepsilon)]\ . \tag{3.11}$$

We shall show that $R(P,\phi,\varepsilon)$ is an infinitesimal of the first order in ε, so that the differential equation for P, up to terms of the second order in ε, is given by

$$\dot{P} = \varepsilon G(P)\ ;$$

we shall do the computations using the variables (I,ϕ). As

$$G(P) = G(I+\varepsilon h(I,\phi)) = G(I) + \varepsilon h(I,\phi)\frac{\partial}{\partial I}G(\xi(P,I))$$

for $\xi(P,I)$ on the line joining I and P, we have

$$R(P(I,\phi,\varepsilon),\phi,\varepsilon) = -\varepsilon h(I,\phi)\frac{\partial}{\partial I}G(\xi(P,I)) + g(I,\phi,\varepsilon) - g(I,\phi,0) + \varepsilon r(I,\phi,\varepsilon) =$$
$$= -\varepsilon h(I,\phi)\frac{\partial}{\partial I}G(\xi(P,I)) + g(I,\phi,\varepsilon) - g(I,\phi,0) + \varepsilon\left[g(I,\phi,\varepsilon)\frac{\partial h}{\partial I}(I,\phi) + f(I,\phi,\varepsilon)\frac{\partial h}{\partial \phi}(I,\phi)\right] .$$

We use the following bounds for each $(I,\phi,\varepsilon) \in K \times S^1 \times [0,1]$:

$$|h(I,\phi)| \le |h|_0 \le 2\pi\left|\omega^{-1}\right|_0|\tilde{g}|_0 \le 4\pi\left|\omega^{-1}\right|_0|g|_0 \; ;$$
$$\left|\frac{\partial G}{\partial I}(\xi(P,I))\right| \le |g|_1 \; ;$$
$$|g(I,\phi,\varepsilon) - g(I,\phi,0)| \le \varepsilon|g|_1 \; ;$$
$$|r(I,\phi,\varepsilon)| \le |g|_0\left|\frac{\partial h}{\partial I}(I,\phi)\right| + |f|_0\left|\frac{\partial h}{\partial \phi}(I,\phi)\right| \; ;$$
$$\left|\frac{\partial h}{\partial I}(I,\phi)\right| \le 4\pi\left\{|\omega|_1\left|\omega^{-2}\right|_0|g|_0 + \left|\omega^{-1}\right|_0|g|_1\right\} \; ;$$
$$\left|\frac{\partial h}{\partial \phi}(I,\phi)\right| \le 2\pi\left|\omega^{-1}\right|_0|g|_1 \; ;$$

and we conclude that the differential equations (3.11) differ from the averaged equations (3.5) only for terms of the second order in ε; in particular there exists a constant c_2, independent on I,ϕ,ε and such that

$$|R(P(I,\phi,\varepsilon),\phi,\varepsilon)| \le \varepsilon c_2$$

for each $(I,\phi,\varepsilon) \in K \times S^1 \times [0,1]$.

Step 4 Let us consider the *slow time* $\tau = \varepsilon t$; if we use a prime for the derivative with respect to τ we obtain that P and J are the solutions of the following differential equations:

$$P' = G(P) + R(P,\phi,\varepsilon); \tag{3.12}$$

$$J' = G(J) \; . \tag{3.13}$$

We estimate the difference between the solution of the averaged equations (3.13) and the function $P = P(t)$, that is solution of the equations (3.12). Let us set $Z(\tau) = P(t(\tau)) - J(t(\tau))$, then, assuming that I,P,J remain in the domain K, we have

$$\left|Z'(\tau)\right| \le a\,|Z(\tau)| + b \tag{3.14}$$

where $a = |G|_1$ and $b = \varepsilon c_2$.

Let us suppose that $|Z(0)| = c$; then by Gronwall's lemma we obtain

$$|Z(\tau)| \leq (c + b\tau)e^{a\tau}, \tag{3.15}$$

because the right hand side of (3.15) is the solution of the Cauchy problem

$$Z' = aZ + b; \qquad Z(0) = c \ .$$

We observe that

$$|c| = |Z(0)| = |P(0) - J(0)| = |I(0) + \varepsilon h(I(0), \phi(0)) - J(0)| = \varepsilon|h(I(0), \phi(0))| < \varepsilon|h|_0 \ .$$

Step 5 If we set $c_3 = |h|_0$ we obtain $|P(I, \phi, \varepsilon) - I| \leq \varepsilon c_3$. On the other hand from (3.14) we obtain

$$|P(t) - J(t)| = |Z(\tau)| \leq \varepsilon c_4$$

with $c_4 = (c_3 + c_2 T)e^{aT}$ for all t such that $0 \leq \varepsilon t \leq T$; we conclude that

$$|I(t) - J(t)| \leq |I(t) - P(t)| + |P(t) - J(t)| \leq \varepsilon(c_3 + c_4) \qquad \forall t \in [0, T/\varepsilon] \ .$$

□

3.3.2 Multi-frequency systems and resonances

Let us consider a system with more than one frequency, where the frequencies $\omega = \omega(I)$ depend on the slow variables $I \in B$. The main difficulty when dealing with such systems is the possible presence of resonances between the frequencies:

DEFINITION 10. *Let us consider a system like (3.2); we say that the frequency vector $\omega = (\omega_1, \ldots, \omega_n)$ is* resonant *if there exists a vector $k = (k_1, \ldots, k_n)$ with integer coefficients such that $\langle k, \omega \rangle = 0$. The vector k is called the* resonance number.

DEFINITION 11. *A point $I \in B$ is called* resonant *if the vector $\omega(I)$ is resonant. We define, for each resonant number k, the* resonant hypersurface

$$\Gamma_k = \{I \in B : \langle k, \omega(I) \rangle = 0\} \ .$$

DEFINITION 12. *We say that the domain is* non-resonant *if for each $I \in B$ the condition*

$$|\langle k, \omega(I) \rangle| > c^{-1}|k|^{-\nu} \qquad \text{(strong incommensurability)} \tag{3.16}$$

holds for some constants c, ν and for all the vectors $k \neq 0$ with integer components such that the harmonic with phase $\langle k, \phi \rangle$ appears in the Fourier expansion of the function f in equations (3.2).

If the domain B is non-resonant, then we can apply the averaging principle and we obtain an accuracy of the order ε over times of the order $1/\varepsilon$ for the results.

We can also replace the strong incommensurability condition (3.16) with the weaker condition $\langle k, \omega(I) \rangle \neq 0$, but we have to take into account a possible loss in accuracy.

REMARK 5. Note that in most cases in a multi-frequency system there are no non-resonant domains because the incommensurability conditions is generally violated on an everywhere dense set.

Then we have to face the problem of crossing the resonant surfaces. We notice that some resonances are more important than others, in particular the effects of the resonances decrease with their orders (we say that the number $|k| = \sum_{i=1}^{n} |k_i|$ is the order of the resonance defined by $k \in \mathbb{Z}^n$).

We observe that in a multi-frequency system there is the possibility of a *capture in resonance* for the solutions; in such cases the averaged system is not representative of the behavior of the projection of the basis B of the solutions of the full equations. Anyway the capture in resonance can happen only for an exceptional set of initial conditions.

A case that has been intensively studied is the case of two frequencies only. Let us consider the system

$$\begin{cases} \dot{I} = \varepsilon f(I,\phi,\varepsilon) \\ \dot{\phi}_1 = \omega_1(I) + \varepsilon g_1(I,\phi,\varepsilon) \\ \dot{\phi}_2 = \omega_2(I) + \varepsilon g_2(I,\phi,\varepsilon) \end{cases} \tag{3.17}$$

and let us assume that the functions at the right hand sides of (3.17) are analytic. The following result holds:

THEOREM 12. *Suppose that*

$$\left(\omega_1 \frac{\partial \omega_2}{\partial I} - \omega_2 \frac{\partial \omega_1}{\partial I}\right) f > c_1^{-1} > 0$$

holds true. Then the difference between the slow motion in the perturbed system, $I(t)$, and in the averaged system, $J(t)$, remains small over time $1/\varepsilon$; if $I(0) = J(0)$, then

$$|I(t) - J(t)| \leq c_2\sqrt{\varepsilon}, \qquad 0 \leq t \leq 1/\varepsilon \, .$$

The case with more that two frequencies has been studied less because the arrangement of the resonant surfaces in the phase space is more complicate. In the case of two frequencies the main effect is a passage through a single resonance, while in the case of more than two frequencies we have to take into account the tangency to the resonant surfaces and the addition of the effects of several resonances (see [2] for a more detailed explanation).

However there exist some theorems that justify in a statistical sense the applicability of the averaging methods also in the multi-frequency case: we give the statement of one of them. Let us assume that for almost all the values of the integrals of the unperturbed problem, the unperturbed motion on the common level set is ergodic; then we have

THEOREM 13. (Anosov) For each $\rho > 0$, the measure of the set of initial conditions (in a compact subset of the phase space) for which the error made in describing the exact motion by averaging exceeds ρ, i.e.,

$$mes\{(I_0,\phi_0) : \max_{0 \leq t \leq 1/\varepsilon} |I(t) - J(y)| > \rho \text{ if } I(0) = J(0) = I_0\},$$

tends to zero as $\varepsilon \to 0$.

3.3.3 Averaging in Hamiltonian systems

We consider the study of small Hamiltonian perturbations of a completely integrable Hamiltonian system, that was called by Poincarè [64] *the fundamental problem of dynamics*. Let us set

$$H(I,\phi,\varepsilon) = H_0(I) + \varepsilon H_1(I,\phi,\varepsilon)$$

where $H_1(I,\phi,\varepsilon)$ is a 2π-periodic function in the variable ϕ. The Hamiltonian $H_0(I)$ of the unperturbed system depends only on the action variables I so that the equations of the unperturbed motion are

$$\begin{cases} \dot{I} &= 0 \\ \dot{\phi} &= \dfrac{\partial H_0}{\partial I} \end{cases}$$

and the equations of the perturbed motion are

$$\begin{cases} \dot{I} = -\varepsilon \dfrac{\partial H_1}{\partial \phi} \\ \dot{\phi} = \dfrac{\partial H_0}{\partial I} + \varepsilon \dfrac{\partial H_1}{\partial I} \ . \end{cases}$$

PROPOSITION 3. *In a Hamiltonian system with n degrees of freedom and n frequencies, evolution of slow variables does not occur, in the sense that the averaged system has the form* $\dot{J} = 0$.

Proof. The integral $\int_0^{2\pi} [-(\partial H_1/\partial\phi)]\ d\phi$ yields zero, because it is the increment of the periodic function H_1 over its period. □

REMARK 6. The averaged equation is equal to the differential equation for I in the unperturbed system.

3.3.4 Averaging in Hamiltonian systems in the degenerate case

Often we encounter *properly degenerate* problems. in which the unperturbed Hamiltonian depends only on a part of the action variables and some of the unperturbed frequencies vanish identically:

$$H = H_0(I_1,\ldots,I_r) + \varepsilon H_1(I,\phi,\varepsilon) \qquad r < n \ . \tag{3.18}$$

As the angles ϕ_j with $j > r$ are slow variables the averaging method can represent the evolution of the system if it is done only on the fast phases ϕ_i with $i \le r$.

We have the following

PROPOSITION 4. *In a Hamiltonian system with n degrees of freedom, with Hamiltonian of the form (3.18), we have* $r < n$ *fast frequencies and the variables canonically conjugate to the fast phases are first integrals of the averaged system.*

We say that the degeneracy of the system is *simple* if the number of the fast phases is smaller by only one unit than the number of the degrees of freedom of the system: in this case the averaged system is integrable.

As the Hamiltonian of the Solar System planets is degenerate, we can deduce from Theorem 4 the following

COROLLARY 1. *(*Lagrange's theorem*) The semimajor axes of the Solar system planets do not present a secular evolution, up to terms of the first order in the perturbing masses (the ratio of the masses of the planets with respect to the mass of the Sun).*

3.4 Perturbative formulation of the restricted three body problem

In 1772, while studying the three body problem Sun-Earth-Moon, Leonard Euler thought to simplify the problem by assuming that the two more massive bodies had a circular motion around their common center of mass and that the Moon moved under the influence of the two bodies without perturbing their motion: this is called *circular restricted three body problem*. Even with these simplifications, the three body problem turns out to be not integrable, nevertheless the restricted circular problem has been used successfully for qualitative investigations of the motion of the Solar System bodies; the approximation that is made disregarding the influence of the third body makes even more sense if this body is an asteroid.

Let us consider a restricted 3-body problem Sun-planet-asteroid (we allow also elliptic orbits for the Sun and the planet): the gravitational attraction of the asteroid on the other two bodies is neglected and the equations to solve, in an inertial reference frame $Oy_1y_2y_3$, are

$$\begin{cases} \ddot{y} = k^2\left\{\dfrac{y_\odot - y}{|y_\odot - y|^3} + \mu\dfrac{y' - y}{|y' - y|^3}\right\} \\ \ddot{y}' = k^2\mu\dfrac{y_\odot - y'}{|y_\odot - y'|^3} \\ \ddot{y}_\odot = k^2\mu\dfrac{y' - y_\odot}{|y' - y_\odot|^3} \end{cases}$$

where $y, y', y_\odot$ are the positions of the asteroid, the planet and the Sun respectively, k^2 is Gauss's constant and $\mu = m'/m_\odot$ that is the ratio between the mass of the planet and the mass of the Sun.

The last two equations are independent from the first and the motion of the Sun and of the planet is completely determined as solution of a 2-body problem. Therefore the problem consists only in determining the motion of the asteroid around the Sun.

Perturbation theory is used to compute an approximation of the solution of this non-integrable problem.

In a heliocentric reference frame $O'x_1x_2x_3$, oriented like $Oy_1y_2y_3$, the equations of motion for the asteroid are

$$\ddot{x} = k^2\Big\{-\frac{x}{|x|^3} + \mu\frac{x'-x}{|x'-x|^3} - \mu\frac{x'}{|x'|^3}\Big\} \tag{3.19}$$

where $x = y - y_\odot$ and $x' = y - y'$.

The right hand side of equations (3.19) is the gradient of a *potential* $U = U_0 + \mu U_1$ where

$$U_0 = \frac{k^2}{|x|} \qquad U_1 = k^2\Big[\frac{1}{|x-x'|} - \frac{<x,x'>}{|x'|^3}\Big]$$

($<,>$ is the usual Euclidean scalar product) so that equations (3.19) can be written in the form

$$\ddot{x} = \nabla_x(U_0 + \mu U_1)\ .$$

We observe that the mass of Jupiter, the most massive planet in the Solar System, is only about $1/1000$ of the mass of the Sun, so that we can regard μ as a small parameter: hence we are dealing with a perturbed problem of the integrable two body motion of the asteroid.

3.5 The secular evolution of Main Belt Asteroids

Let us consider a Solar System model with the Sun, $N-2$ planets and an asteroid and let the mass of the asteroid be negligible, so that we have a *restricted* problem. Let us suppose in addition that the mass of the planets is small if compared to the mass of the Sun, then we have $N-2$ small perturbative parameters $\mu_i, i = 1 \dots N-2$.

We assume that the motion of the planets is completely determined and that there are no collisions between them or with the Sun.

We can apply the averaging principle in order to study the qualitative behavior of the orbits of Main Belt Asteroids (MBAs) assuming that *no mean motion resonances with low order occur* occurs between the asteroid and the planets in the model. This means that there exists $\varepsilon > 0$ not too small and a positive integer M not too large such that for each pair $(a(t), a_i(t))$, composed by the semimajor axes of the osculating orbits of the asteroid and the planet i $(i = 1 \dots N-2)$ we have

$$\Big|\ p\,[a(t)]^{3/2} - q\,[a_i(t)]^{3/2}\ \Big| > \varepsilon$$

for each pair of positive integers $p,q \leq M$ and for each t in the considered time span.

REMARK 7. As our purpose is not a study of the structure of the mean motion resonances we shall not give further details or any estimates on the size of ε and M.

We can study the motion of the asteroid using the heliocentric Delaunay variables (L,G,Z,ℓ,g,z), defined by

$$\begin{cases} L = k\sqrt{a} \\ G = k\sqrt{a(1-e^2)} \\ Z = k\sqrt{a(1-e^2)}\cos I \end{cases} \qquad \begin{cases} \ell = n(t-t_0) \\ g = \omega \\ z = \Omega \end{cases}$$

where $\{a,e,I,\omega,\Omega,\ell\}$ is the set of the Keplerian elements, k is Gauss's constant, n is the mean motion and t_0 is the time of passage at perihelion; the same variables with a suffix $(L_i,G_i,Z_i,\ell_i,g_i,z_i)$ $i=1\ldots N-2$ can be used to describe the motion of the planets.

Delaunay's variables, like the Keplerian elements, describe the evolution of the osculating orbit of the asteroid, that is of the trajectory that the asteroid would describe in a heliocentric reference frame, given its position and velocity at a time t, if only the Sun were present. For negative values of the Keplerian energy of the asteroid the osculating orbits are ellipses; we shall consider only such cases.

The Hamiltonian can be written as

$$H = -\frac{k^2}{2L^2} - R \tag{3.20}$$

where $-k^2/(2L^2)$ is the *unperturbed term*, describing the two body motion of the asteroid around the Sun, and R is the *perturbing function* defined by

$$R = \sum_{i=1}^{N-2} \mu_i R_i; \quad R_i = k^2\left[\frac{1}{|x-x_i|} - \frac{< x,x_i >}{|x_i|^3}\right]; \quad i = 1\ldots N-2 \tag{3.21}$$

in which $\langle , \rangle$ is the Euclidean scalar product and x and x_i are the position vectors of the asteroid and of all the planets in a heliocentric reference frame.

Note that each R_i is the sum of a *direct term* $k^2/|x-x_i|$, due to the direct interaction between the planet i and the asteroid, and an *indirect term* $-k^2 < x,x_i > /|x_i|^3$, representing the effects on the motion of the asteroid caused by the interaction between the Sun and the planet i.

If we set $\mathfrak{E}_{\mathcal{D}} = (L,G,Z,\ell,g,z)$ we can write Hamilton's equations as

$$\dot{\mathfrak{E}}_{\mathcal{D}} = \mathfrak{J}\,(\nabla_{\mathfrak{E}_{\mathcal{D}}} H)^t \tag{3.22}$$

where $\mathfrak{J}$ is the 6×6 matrix

$$\begin{bmatrix} O_3 & -I_3 \\ I_3 & O_3 \end{bmatrix}$$

composed by 3×3 zero and identity matrices, and

$$(\nabla_{\mathfrak{E}_{\mathcal{D}}} H)^t = \left(\frac{\partial H}{\partial \mathfrak{E}_{\mathcal{D}}}\right)^t$$

is the transposed vector of the partial derivatives of the Hamiltonian H with respect to $\mathfrak{E}_{\mathcal{D}}$.

3.5.1 The averaged equations

In this case we can consider averaged equations obtained by the integral average of the right hand side of (3.22) over the mean anomalies $\ell, \ell_1, .., \ell_{N-2}$ of the asteroid and the planets.

In the expression of the perturbing function (3.21) the effect of each planet is independently taken into account: each R_i is a function of the coordinates and the masses of the asteroid and one planet only. We shall study the case of only one perturbing planet and we shall use a prime for the quantities related to this planet: the perturbation of all the planets, up to the first order in the perturbing masses μ_i, will be obtained as the sum of the contribution of each planet.

Let us consider a three body restricted problem Sun-planet-asteroid; If we take into account the reduced set of Delaunay's variables $\mathcal{E}_{\mathcal{D}} = (G, Z, g, z)$, the averaged equations of motion for the asteroid can be written in the following form:

$$\dot{\widetilde{\mathcal{E}}}_{\mathcal{D}} = -\mathcal{J}\, \overline{\nabla_{\mathcal{E}_{\mathcal{D}}} R}^{\,t} \tag{3.23}$$

where $\widetilde{\mathcal{E}}_{\mathcal{D}} = (\widetilde{G}, \widetilde{Z}, \tilde{g}, \tilde{z})$ are averaged Delaunay variables, $\mathcal{J}$ is the 4×4 matrix

$$\mathcal{J} = \begin{bmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{bmatrix}$$

composed by 2×2 zero and identity matrices, and $\overline{\nabla_{\mathcal{E}_{\mathcal{D}}} R}^{\,t}$ is the transposed vector of the integral average over (ℓ, ℓ') of the partial derivatives of the perturbing function R with respect to $\mathcal{E}_{\mathcal{D}}$

$$\overline{\nabla_{\mathcal{E}_{\mathcal{D}}} R} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \nabla_{\mathcal{E}_{\mathcal{D}}} R \, d\ell \, d\ell' \; ; \qquad \nabla_{\mathcal{E}_{\mathcal{D}}} R = \frac{\partial R}{\partial \mathcal{E}_{\mathcal{D}}} \; .$$

As the variable ℓ becomes cyclic in this framework, then the corresponding conjugate variable L is constant.

A more explicit form for the averaged equations of motions (3.23) is

$$\begin{cases} \dot{\widetilde{G}} = \overline{\dfrac{\partial R}{\partial g}} \\ \dot{\widetilde{Z}} = \overline{\dfrac{\partial R}{\partial z}} = 0 \end{cases} \qquad \begin{cases} \dot{\tilde{g}} = -\overline{\dfrac{\partial R}{\partial G}} \\ \dot{\tilde{z}} = -\overline{\dfrac{\partial R}{\partial Z}} \; . \end{cases} \tag{3.24}$$

The solutions of equations (3.23) or (3.24) are representative of the solutions of the full equations of motion if there are no mean motion resonances between the asteroid and the planet and if no close approaches occur between them.

If the derivatives of R with respect to Delaunay's variables are regular functions we can use the *theorem of differentiation under the integral sign* [19] to exchange the derivatives and the integrals in (3.23); then the averaged equations take the form

$$\dot{\widetilde{E}}_{\mathcal{D}} = -\mathcal{J}\, (\nabla_{\mathcal{E}_{\mathcal{D}}} \overline{R})^{\,t}$$

where

$$\overline{R} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} R\, d\ell\, d\ell' = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\mu k^2}{|x-x'|}\, d\ell\, d\ell' \tag{3.25}$$

(μ is the ratio between the mass of the planet and the mass of the Sun); in fact the average of the indirect part of the perturbation is zero (see Appendix A for a proof of this statement).

Remark 8. From the form of the equations (3.23) we can deduce that if we decrease the perturbation by rescaling the perturbing parameters μ_i, we do not change the evolution trajectories but we only rescale the time needed to that evolution.

3.5.2 An integrable problem

The averaged problem (3.23) has been studied in [78] assuming that all the orbits of the planets in the model are circular and coplanar, following the example of [39].

This assumption makes the averaged dynamics integrable as the problem has three degrees of freedom and we have

Proposition 5. *The following three quantities are first integrals independent and in involution:*

1. *the semimajor axis a;*
2. *the third component of the angular momentum $Z = k\sqrt{a(1-e^2)}\cos I$;*
3. *the averaged perturbing function*

$$\overline{R} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} R\, d\ell\, d\ell' \quad (\text{Kozai integral}). \tag{3.26}$$

Proof. The functional independence of these three quantities is evident; we shall prove that they are in involution. We use the symbol $\{,\}$ for the Poisson brackets; that is, given two functions $f_1(\ell,g,z,L,G,Z)$ and $f_2(\ell,g,z,L,G,Z)$ we have

$$\{f_1,f_2\} = \left(\frac{\partial f_1}{\partial \ell}\frac{\partial f_2}{\partial L} - \frac{\partial f_1}{\partial L}\frac{\partial f_2}{\partial \ell}\right) + \left(\frac{\partial f_1}{\partial g}\frac{\partial f_2}{\partial G} - \frac{\partial f_1}{\partial G}\frac{\partial f_2}{\partial g}\right) + \left(\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial Z} - \frac{\partial f_1}{\partial Z}\frac{\partial f_2}{\partial z}\right).$$

As $a = L^2/k^2$ it follows immediately that $\{a,Z\} = 0$. Furthermore we have

$$\{a,\overline{R}\} = -\frac{\partial a}{\partial L}\frac{\partial \overline{R}}{\partial \ell} = 0$$

because we have eliminated the dependence on the mean anomaly ℓ in $\overline{R}$ by averaging over it.

We evaluate the last Poisson bracket:

$$\{Z,\overline{R}\} = -\frac{\partial Z}{\partial Z}\frac{\partial \overline{R}}{\partial z} = 0$$

because the perturbing function itself, even without averaging, does not depend on the longitude of the Node z if we consider the planets on circular coplanar orbits.

□

REMARK 9. Note that the assumption on the symmetry of the orbits of the planets (circular orbits) is essential in order to be able to integrate the problem.

We also observe that Lidov and Ziglin [43] made an analytical study of the restricted circular twice-averaged three body problem Sun-planet-asteroid in the case in which the orbit of the asteroid is uniformly close to the orbit of the planet.

Chapter 4

Averaging on planet crossing orbits: circular coplanar case

We take into account the possibility of intersections between the orbit of an asteroid an the orbits of the planets in the framework of a restricted problem. Let us give the following

DEFINITION 13. *We say that an asteroid is* planet crossing *if its orbit crosses the orbit of some planet during its evolution.*

Starting from 1898, the year in which the asteroid Eros (433) was discovered, a new asteroid population has been detected: the *Near Earth Asteroids* (NEAs). We say that an asteroid is a NEA if its osculating perihelion q satisfies the relation

$$q = a(1-e) < 1.3AU \ .$$

In the last years, thanks to the new observational techniques, the number of known NEAs has increased very rapidly, and up to now we know the existence of about 1800 of such asteroids.

As the orbits of the NEAs are close to the one of the Earth and of the inner planets, generally these asteroids are planet crossing.

4.1 Difficulties arising with planet crossing orbits

When we consider a planet crossing asteroid at the time of intersection of the orbits, the averaged perturbing function $\overline{R}$ is the integral of an unbounded function that is convergent because $1/|x-x'|$ has a first order polar singularity in the values $\overline{\ell}, \overline{\ell}'$ corresponding to a collision. The derivatives at the right hand side of (3.23) have second order polar singularities in $\overline{\ell}, \overline{\ell}'$, hence equations (3.23) do not make sense in this case because the integrals over ℓ, ℓ' of these derivatives are divergent and the classical averaging principle cannot be applied.

The study of the dynamics of NEAs requires the development of theories that can give statistical informations on the evolution of these objects, because the evolution starting from a single initial condition is not representative for long time spans, due to the chaoticity of the orbits; thus it is desirable to have an averaging method also in this case.

In this chapter we describe a generalization of the classical averaging principle, first presented in [28], that is suitable for planet crossing orbits in the framework of a Solar System with the planets on circular and coplanar orbits. We shall refer to this problem as to the *circular coplanar case*.

4.2 Geometry of the node crossing

We give a description of the possible geometric configurations in the plane $(e\cos\omega, e\sin\omega)$ when a node crossing occurs.

Recall that the ascending and descending node crossings with a planet with semimajor axis a' are characterized by the vanishing of the following expressions respectively:

$$d_{nod}^{+} = \frac{a(1-e^2)}{1+e\cos\omega} - a'; \qquad d_{nod}^{-} = \frac{a(1-e^2)}{1-e\cos\omega} - a';$$

that are called *nodal distances* (and can be negative).

As we observed in Chapter 3, in the averaged problem with the planets on circular coplanar orbits we have three integrals of motions: the semimajor axis a, the *Kozai integral* $\overline{H} = H_0 - \overline{R}$ and the *z-component of the angular momentum* $Z = k\sqrt{a(1-e^2)}\cos I$. The Z integral allows to determine the evolution of the averaged inclination $I(t)$ if we know $e(t)$; if we also know the evolution of the averaged perihelion argument $\omega(t)$ we can determine $\Omega(t)$ by a simple quadrature of $\partial\overline{R}/\partial Z$, that does not depend on Ω. From the expression of the integral Z we deduce the maximum value of the averaged inclination and eccentricity:

$$I_{max} = I|_{e=0} = \arccos\frac{Z}{k\sqrt{a}}; \qquad e_{max} = e|_{I=0} = \frac{\sqrt{k^2 a - Z^2}}{k\sqrt{a}} .$$

For a given value of the semimajor axis a we can represent the level lines of the averaged Hamiltonian, on which the averaged solutions evolve, in the plane $(\xi,\eta) := (e\cos\omega, e\sin\omega)$. We define the *Kozai domain*

$$W = \{(\xi,\eta) : \xi^2 + \eta^2 \le e_{max}^2\},$$

where the averaged dynamics is confined.

In the (ξ,η) reference plane the node crossing lines with the planets are circles: they are defined by

$$\Gamma^{+}(i) = \{(\xi,\eta) : d_{nod}^{+}(i) = 0\}; \qquad \Gamma^{-}(i) = \{(\xi,\eta) : d_{nod}^{-}(i) = 0\};$$

where i is the index of the planet.

At the ascending node crossing with the planet i the equation to be considered is

$$1 - \xi^2 - \eta^2 = \frac{a_i'}{a}(1+\xi) .$$

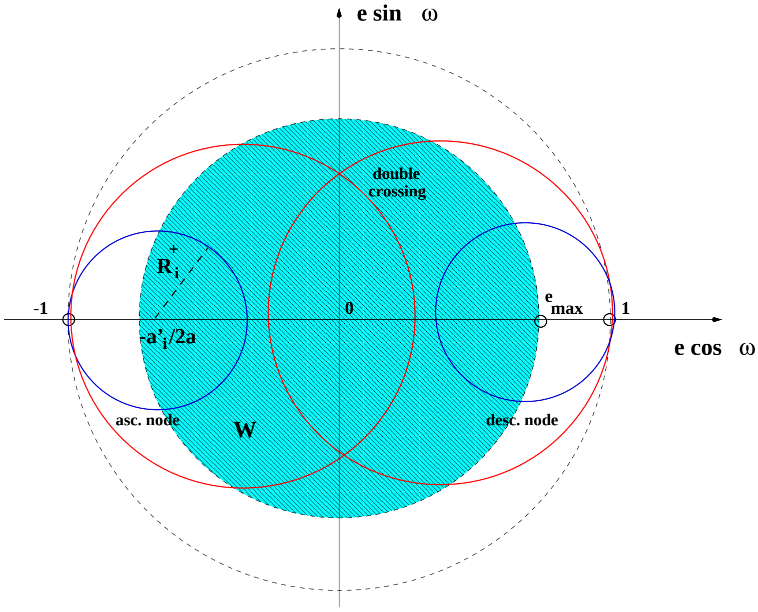

Figure 4.1: The Kozai domain $\{(e\cos\omega, e\sin\omega) : 0 \le e \le e_{max}, \omega \in \mathbb{R}\}$ is represented in the figure by the set W. We plot also the four circles corresponding to ascending and descending node crossing with two planets (they have their centers shifted respectively on the left and on the right). An additional exterior circle corresponding to the boundary for closed orbits ($e = 1$) is drawn.

After the coordinate change $X = \xi + a_i'/(2a)$; $Y = \eta$ we obtain

$$X^2 + Y^2 = \left(1 - \frac{a_i'}{2a}\right)^2 ,$$

that is, in the (ξ, η) plane, the equation of a circle of radius $R_i^+ = 1 - a_i'/(2a)$, with center in $(\xi_i^+, \eta_i^+) = (-a_i'/(2a), 0)$ (see Figure 4.1).

By the previous calculations we have

$$\begin{cases} d_{nod}^+(i) > 0 & \text{inside } \Gamma^+(i) \\ d_{nod}^+(i) < 0 & \text{outside } \Gamma^+(i) \ . \end{cases}$$

In a similar way we can prove that the equation $d_{nod}^-(i) = 0$ represents a circle of radius $R_i^- = R_i^+ = 1 - a_i'/(2a)$, with center in $(\xi_i^-, \eta_i^-) = (+a_i'/(2a), 0)$; furthermore we have

$$\begin{cases} d_{nod}^-(i) > 0 & \text{inside } \Gamma^-(i) \\ d_{nod}^-(i) < 0 & \text{outside } \Gamma^-(i) \ . \end{cases}$$

Definition 14. *A* double (node) crossing *is a crossing between the orbit of the asteroid and the orbit of a planet at both the ascending and descending node.*

By the symmetry of the circles $\{d^+_{nod}(i) = 0\}$ and $\{d^-_{nod}(i) = 0\}$, for each index i, we can deduce that a double crossing is possible only when $\omega = \pi/2$ or $\omega = 3\pi/2$ (see Figure 4.1). We obtain the following condition on the ratio of the semimajor axes a, a'_i:

$$-\frac{a'_i}{2a} \geq -\frac{1}{2} \qquad \text{that is} \qquad a \geq a'_i \ ,$$

and in particular we obtain that *there cannot exist Athen asteroids* (see Chapter 7) *that have a double crossing with the Earth.*

DEFINITION 15. *A* simultaneous crossing *is a crossing of the orbit of the asteroid and the orbits of two planets at the same time.*

We note that if we call a'_1, a'_2 the semimajor axes of the orbits of two different planets, we cannot have a simultaneous crossing at the ascending node of both planets (this would imply $a'_1 = a'_2$ in this model). By a similar argument we cannot have a simultaneous crossing at the descending node of both planets. On the other hand we have a simultaneous crossing at the ascending node with one planet and at descending node with the other one if

$$a'_1 = \frac{a(1-e^2)}{1+e\cos\omega} \qquad \text{and} \qquad a'_2 = \frac{a(1-e^2)}{1-e\cos\omega} \ ,$$

that is if

$$\frac{a'_1}{a'_2} = \frac{1-e\cos\omega}{1+e\cos\omega} \ .$$

In this framework we cannot have crossings of different type from the ones presented above (like *triple crossing*, etc.).

4.3 Description of the osculating orbits

We consider a model with three bodies only: Sun, planet, asteroid. We set the x_1 axis along the line of the nodes, pointing towards the ascending mutual node, and the x_2 axis is chosen on the orbital plane of the planet (see Figure 4.2). The equations defining the osculating orbits $P(u) = (p_1(u), p_2(u), p_3(u))$ and $P'(u') = \big(p'_1(u'), p'_2(u'), p'_3(u')\big)$ of the asteroid and the planet are

$$\begin{cases} p_1 = a[(\cos u - e)\cos\omega - \beta\sin u\sin\omega] \\ p_2 = a[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\cos I \\ p_3 = a[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\sin I \end{cases} \qquad \begin{cases} p'_1 = a'\cos u' \\ p'_2 = a'\sin u' \\ p'_3 = 0 \end{cases} \tag{4.1}$$

where u, u' are the eccentric anomalies and $\beta = \sqrt{1-e^2}$. These orbits are respectively an ellipse and a circle.

The distance between a point on an orbit and a point on the other one, appearing at the denominator of the direct term of the perturbing function, is defined by its square as

$$\mathfrak{D}^2(u,u') = (p_1 - p'_1)^2 + (p_2 - p'_2)^2 + (p_3 - p'_3)^2 =$$

$$= a^2(1-e\cos u)^2 + a'^2 - 2aa'\{\cos u'[(\cos u - e)\cos\omega - \beta\sin u\sin\omega] +$$
$$+\sin u'\cos I[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\} \ .$$

We introduce the function $\mathrm{D}(\ell,\ell')$, which is implicitly defined by

$$\mathrm{D}\left(\ell(u),\ell'(u')\right) = \mathfrak{D}(u,u')$$

and by Kepler's equations

$$\ell = u - e\sin u; \qquad \ell' = u' \tag{4.2}$$

for the asteroid and the planet (the latter has a simpler form because the orbit of the planet is circular).

We define the values of the anomalies $\overline{u},\overline{u}'$ corresponding to the mutual ascending node: we immediately notice that $\overline{u}' = 0$, while from

$$a(1-e\cos\overline{u}) = \frac{a\beta^2}{1+e\cos\omega}$$

we obtain

$$\cos\overline{u} = \frac{\cos\omega + e}{(1+e\cos\omega)}; \qquad \sin\overline{u} = -\frac{\beta\sin\omega}{(1+e\cos\omega)}$$

(the sign of $\sin\overline{u}$ has been chosen in such a way that it is opposite to the sign of $\sin\omega$).

The equations defining the anomalies $\overline{u}_1,\overline{u}_1'$, corresponding to the mutual descending node, are

$$\overline{u}_1' = \pi; \qquad \cos\overline{u}_1 = \frac{e-\cos\omega}{(1-e\cos\omega)}; \qquad \sin\overline{u}_1 = \frac{\beta\sin\omega}{(1-e\cos\omega)} \ .$$

In the following we shall study the case of the ascending node crossings, but the same methods are suitable to deal also with the descending ones and with double crossings: we shall give the formulas to be applied in these cases at the end of this chapter and in Appendix B.

4.4 Weak averaged solutions

The idea of the generalization of the averaging principle in [28] starts from the fact that if there are no crossings between the orbits, by the *theorem of differentiation under the integral sign* [19], the averaged equations of motion (3.23) are equivalent to Hamilton's equations (4.3) with the averaged perturbing function $\overline{R}$:

$$\dot{\overline{\mathcal{E}}}_{\mathcal{D}} = -\mathcal{J}\left[\nabla_{\mathcal{E}_{\mathcal{D}}}\overline{R}\right]^t \tag{4.3}$$

where

$$\overline{R} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} R\, d\ell\, d\ell' \, .$$

We write equations (4.3) in a more explicit form:

$$\begin{cases} \dot{\tilde{G}} = \dfrac{\partial \overline{R}}{\partial g} \\ \dot{\tilde{Z}} = \dfrac{\partial \overline{R}}{\partial z} = 0 \end{cases} \qquad \begin{cases} \dot{\tilde{g}} = -\dfrac{\partial \overline{R}}{\partial G} \\ \dot{\tilde{z}} = -\dfrac{\partial \overline{R}}{\partial Z}\,; \end{cases} \tag{4.4}$$

We shall skip the 'tilde' over the averaged variables in the following, to avoid the use of a heavy notation.

We shall prove that when the orbits intersect each other it is possible to define piecewise smooth solutions of equations (4.4), that we call *weak averaged solutions*, and we shall see that the loss of regularity corresponds exactly to the crossing configurations of the orbits: in fact we shall give a twofold meaning to the right hand sides of (4.4) at the node crossing, corresponding to the two limit values of the derivatives coming from inside and outside the circle that represents the ascending node crossing with the planet in the plane (ξ,η).

Note that the weak averaged solutions correspond to the classical averaged solutions as far as their trajectories in the reduced phase space (ξ,η) do not pass through a node crossing line.

We also observe that the exchange of the differential and integral operators in (4.4) is not essential for a theoretical definition of the weak solutions (they could anyway be defined as the limits of the solutions of (3.23) coming from both sides of the node crossing lines) but, as we shall see, this operation is necessary to obtain analytic formulas for the discontinuity of the average of the derivatives of R, that are not defined on the node crossing lines, and to define the semianalytic procedure to compute the weak solutions.

4.5 The Wetherill function

Let $\{P(\overline{u}), P'(\overline{u}')\}$ be the ascending mutual node. We consider the two straight lines $\mathrm{r}(\ell)$ and $\mathrm{r}'(\ell')$, tangent in $P(\overline{u})$ and $P'(\overline{u}')$ to the orbits of the asteroid and of the planet (see Figure 4.2); they can be parametrized by the mean anomalies ℓ,ℓ' so that $P(u(t))$ and $\mathrm{r}(\ell(t))$ have the same velocities (derivatives with respect to t) in $P(\overline{u})$ and $P'(u'(t))$ and $\mathrm{r}'(\ell'(t))$ have the same velocities in $P'(\overline{u}')$:

$$\begin{cases} r_1 = \overline{x}_1 - \mathcal{F}k \\ r_2 = \overline{x}_2 + \mathcal{G}\cos I k \\ r_3 = \overline{x}_3 + \mathcal{G}\sin I k \end{cases} \qquad \begin{cases} r_1' = \overline{x}_1' \\ r_2' = \overline{x}_2' + a'k' \\ r_3' = \overline{x}_3' \end{cases} \tag{4.5}$$

where $k = \ell - \overline{\ell}, k' = \ell' - \overline{\ell}'$, and $\overline{\ell},\overline{\ell}'$ are the values of the mean anomalies corresponding to $\overline{u},\overline{u}'$ (so that $\overline{\ell}' = 0$). We have used the following notations

$$\mathcal{F} \;=\; \frac{\sin\overline{u}\cos\omega + \beta\cos\overline{u}\sin\omega}{1 - e\cos\overline{u}} = \frac{ae\sin\omega}{\beta}\,;$$

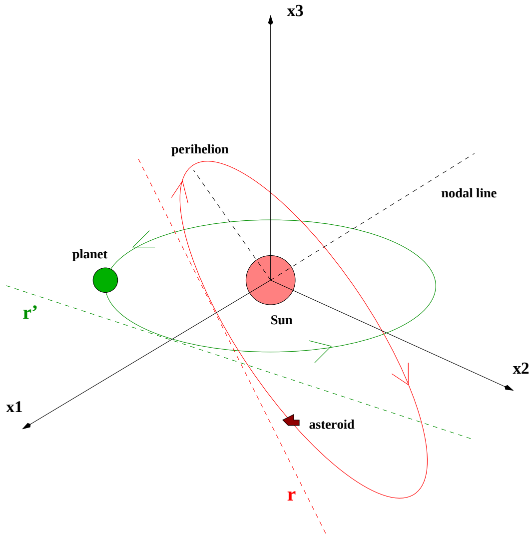

Figure 4.2: The straight lines r, r' represent Wetherill's approximation at the ascending node for the two osculating orbits of the asteroid and the planet.

$$\mathcal{G} = \frac{-\sin\bar{u}\sin\omega + \beta\cos\bar{u}\cos\omega}{1 - e\cos\bar{u}} = \frac{a(1+e\cos\omega)}{\beta}$$

and

$$\bar{x}_1 = \frac{a\beta^2}{1+e\cos\omega}; \qquad \bar{x}'_1 = a'; \qquad \bar{x}_2 = \bar{x}_3 = \bar{x}'_2 = \bar{x}'_3 = 0\,.$$

REMARK 10. Taylor's series developments of equations (4.1) in a neighborhood of the points $\bar{u}, \bar{u}'$ correspond, up to the first order in $v = u - \bar{u}, v' = u' - \bar{u}'$, to the Taylor's series developments in the same variables of equations (4.5):

$$\begin{cases} p_1 = a\beta^2(1+e\cos\omega)^{-1} - \mathcal{F}(1-e\cos\bar{u})\,v + o(v) \\ p_2 = \mathcal{G}\cos I(1-e\cos\bar{u})\,v + o(v) \\ p_3 = \mathcal{G}\sin I(1-e\cos\bar{u})\,v + o(v) \end{cases} \qquad \begin{cases} p'_1 = a' \\ p'_2 = a'v' \\ p'_3 = 0 \end{cases} \tag{4.6}$$

From Kepler's equations we deduce

$$v = k(1-e\cos\bar{u})^{-1} + O(k)$$

in a neighborhood of $k = 0$, so that

$$\begin{cases} p_1 = a\beta^2(1+e\cos\omega)^{-1} - \mathcal{F}\,k + O(k^2) \\ p_2 = \mathcal{G}\cos I\,k + O(k^2) \\ p_3 = \mathcal{G}\sin I\,k + O(k^2)\,. \end{cases}$$

Definition 16. *We call* Wetherill function *[77] the approximated distance function* d, *whose square is defined by*

$$\begin{aligned} \mathrm{d}^2(\ell,\ell') &= (r_1-r_1')^2+(r_2-r_2')^2+(r_3-r_3')^2 = \\ &= a'^2k'^2+[\mathcal{F}^2+\mathcal{G}^2]k^2-2kk'[\mathcal{G}a'\cos I]-2d_{nod}^+\mathcal{F}k+(d_{nod}^+)^2 \end{aligned}$$

with $k=\ell-\overline{\ell}, k'=\ell'$.

Note that d^2 is a quadratic form in the variables k,k': it is homogeneous when there is a crossing at the ascending node. We can write it more concisely as

$$\mathrm{d}^2(\ell,\ell') = d^2(\kappa) = \kappa^t\mathcal{A}\kappa + B^t\kappa + (d_{nod}^+)^2$$

where

$$\kappa = (k',k)\ ; \qquad B = 2(B_1,B_2)\ ; \qquad \mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\ ;$$

with components

$$\begin{cases} B_1 = 0 \\ B_2 = -d_{nod}^+\mathcal{F}\ ; \end{cases} \qquad \begin{cases} A_{11} = a'^2 \\ A_{12} = A_{21} = -\mathcal{G}a'\cos I \\ A_{22} = [\mathcal{F}^2+\mathcal{G}^2]\ . \end{cases}$$

For later use we define

$$\mathfrak{d}^2(u,u') = \mathrm{d}^2(\ell(u),\ell'(u'))\ .$$

The geometry of Wetherill's straight lines is strictly related to the degeneracy of the matrix $\mathcal{A}$, in fact we have

Lemma 1. *The matrix $\mathcal{A}$ is always* positive definite *if $I>0$. If $I=0$ we have degeneracy of $\mathcal{A}$ if and only if the straight lines* r,r' *are parallel: in this case $\mathcal{A}$ is* positive semi-definite.

Proof. $\mathcal{A}$ is a symmetric 2×2 matrix and it is positive definite if and only if its principal invariants, the trace $tr(\mathcal{A})$ and the determinant $det(\mathcal{A})$, are positive. By a direct computation we have

$$\begin{cases} tr(\mathcal{A}) = a'^2+\mathcal{F}^2+\mathcal{G}^2 \\ \\ det(\mathcal{A}) = a'^2\left(\mathcal{F}^2+\mathcal{G}^2\sin^2 I\right)\ . \end{cases}$$

From the above expressions we deduce that $tr(\mathcal{A})>0$ (we are considering only bounded orbits, so that $0\le e<1$); furthermore

$$det(\mathcal{A}) = 0 \quad \Longleftrightarrow \quad \begin{cases} I = 0 \\ e\sin\omega = 0\ , \end{cases}$$

that corresponds to the straight lines r,r' being parallel. □

Definition 17. *We call* tangent crossings *the crossing orbital configurations for which* $det(\mathcal{A}) = 0$.

The assumption that the inclination I of the asteroid is different from zero during its whole time evolution, or at least in a neighborhood of each crossing between the orbits, implies that *no tangent crossings occur*.

4.6 Kantorovich's method of singularity extraction

We shall describe Kantorovich's method of singularity extraction (see [15]) that allows to improve the stability of the numerical computation of the integrals when the integrand function $f_1(x)$ is unbounded in the neighborhood of one or more points.

Kantorovich's method consists in searching for a function $f_2(x)$ whose primitive has an analytic expression in terms of elementary functions and such that the difference $f_1(x) - f_2(x)$ is more regular than $f_1(x)$ (for example it is bounded or even continuous).

It is then convenient to split the computation as follows

$$\int f_1(x)\,dx = \int [f_1(x) - f_2(x)]\,dx + \int f_2(x)\,dx$$

so that the singularity has moved to the second term, that can be better handled.

This method can help us to study the regularity properties of the averaged perturbing function $\overline{R}$ defined in (3.25); we shall use the inverse of the Wetherill function $1/\mathrm{d}$ to extract the principal part from the direct term of the perturbing function.

The function D is 2π-periodic in both variables ℓ, ℓ' and this property can be used to shift the integration domain

$$\mathbb{T}^2 = \{(\ell, \ell') : -\pi \le \ell \le \pi, -\pi \le \ell' \le \pi\}$$

in a suitable way, so that the crossing values $(\overline{\ell}, 0)$ will be always internal points of this domain.

We shall prove that in computing the derivatives of $\overline{R}$ with respect to the variables $\mathcal{E}_{\mathcal{D}}$, for instance the G-derivative, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2}\frac{\partial}{\partial G}\overline{R} = \int_{\mathbb{T}^2} \frac{\partial}{\partial G}\left[\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}}\right] d\ell\, d\ell' + \frac{\partial}{\partial G}\int_{\mathbb{T}^2} \frac{1}{\mathrm{d}}\, d\ell\, d\ell'; \tag{4.7}$$

namely we shall prove the validity of the hypotheses of the theorem of differentiation under the integral sign to exchange the symbols of integral and derivative in front of the *remainder function* $1/\mathrm{D} - 1/\mathrm{d}$. The average of the remainder function is then differentiable as it is derivable with continuity with respect to all the variables $\mathcal{E}_{\mathcal{D}}$. Therefore we shall need only to study the regularity properties of the last term of the sum in (4.7), which is easier to handle.

Note that we use Kantorovich's method of singularity extraction in a wider extent: the derivatives of the remainder function still have a polar singularity in $(\overline{\ell}, 0)$, but it is of order one, so that the integrals over ℓ, ℓ' of these derivatives are convergent.

4.7 Integration of $1/\mathrm{d}$

We shall discuss the analytic method to integrate $1/\mathrm{d}$ over the torus $\mathbb{T}^2 = \{(\ell,\ell') : -\pi \leq \ell \leq \pi, -\pi \leq \ell' \leq \pi\}$ assuming that $(\overline{\ell},0)$ is an internal point of this domain.

We move the ascending node crossing point $(\overline{\ell},0)$ to the origin of the reference system by the variable change

$$\tau_{\overline{\ell},0} : \; (\ell,\ell') \longrightarrow (k,k') = (\ell-\overline{\ell},\ell') \tag{4.8}$$

and we set

$$\overline{\mathbb{T}}^2 = \tau_{\overline{\ell},0}\left[\mathbb{T}^2\right] = \{(\ell-\overline{\ell},\ell') : (\ell,\ell') \in \mathbb{T}^2\} \; .$$

Then we perform another variable change to eliminate the linear terms in the quadratic form $d^2(\kappa)$ defined by (4.7). The inverse of the transformation used for this purpose is

$$\Xi^{-1} : \psi \longrightarrow \kappa = \mathcal{T}\,\psi + S \tag{4.9}$$

where $S = (S_1,S_2) \in \mathbb{R}^2,\; \psi = (y',y) \in \mathbb{R}^2$ are the new variables and $\mathcal{T}$ is a 2×2 real-valued invertible matrix.

Setting to zero the coefficients of the linear terms of the quadratic form in the new variables ψ we obtain the equations

$$2\,\mathcal{A}S + B = 0$$

whose solutions are

$$S_1 = \frac{B_2 A_{12}}{det(\mathcal{A})} = \frac{d^+_{nod}\,\mathcal{F}\,\mathcal{G}a'\cos I}{det(\mathcal{A})}\,;$$

$$S_2 = -\frac{B_2 A_{11}}{det(\mathcal{A})} = \frac{d^+_{nod}\,\mathcal{F}\,a'^2}{det(\mathcal{A})}\; .$$

We can determine the non-degenerate matrix $\mathcal{T}$ in order to obtain

$$\mathcal{T}^t\mathcal{A}\mathcal{T} = I_2 \tag{4.10}$$

where I_2 is the 2×2 identity matrix: from relation (4.10) we obtain $\mathcal{A} = \left(\mathcal{T}^{-1}\right)^t\mathcal{T}^{-1}$; let us set

$$\mathcal{R} = \mathcal{T}^{-1} = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}\; ; \tag{4.11}$$

we obtain the system

$$\begin{cases} \rho_1^2 + \rho_3^2 = A_{11} \\ \rho_1\rho_2 + \rho_3\rho_4 = A_{12} \\ \rho_2^2 + \rho_4^2 = A_{22}\; . \end{cases} \tag{4.12}$$

System (4.12) has infinitely many solutions; we notice that the relations $A_{11} > 0$ and $det(\mathcal{A}) > 0$ allows us to find four solutions $(\rho_1,\rho_2,0,\rho_4)$ with $\rho_3 = 0$:

$$\rho_1 = \pm\sqrt{A_{11}}; \qquad \rho_2 = \frac{A_{12}}{\rho_1}; \qquad \rho_4 = \pm\sqrt{\frac{det(\mathcal{A})}{A_{11}}}\; . \tag{4.13}$$

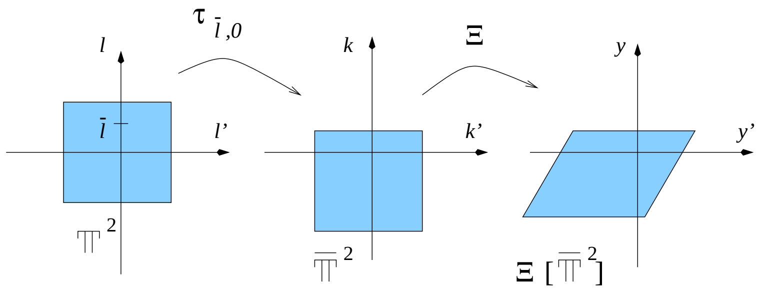

Figure 4.3: Description of the transformations of the integration domain $\mathbb{T}^2$ with the two coordinate changes (4.8), (4.9) used to bring the squared Wetherill function $\mathrm{d}^2(\ell,\ell')$ into the form $y^2+y'^2+(d^+_{min})^2$ in the new variables (y',y). Note that $\overline{\ell}'=0$ implies that $\overline{\mathbb{T}}^2$ is symmetric with respect to the k axis.

We select the signs in the above expressions and we call $(\tau,\sigma,0,\rho)$ the particular solution defined by

$$\tau = a'; \qquad \sigma = -\mathcal{G}\cos I; \qquad \rho = \frac{1}{a'}\sqrt{det(\mathcal{A})}; \tag{4.14}$$

so that we can write

$$\mathcal{T} = \frac{1}{\sqrt{det(\mathcal{A})}}\begin{pmatrix} \rho & -\sigma \\ 0 & \tau \end{pmatrix} = \begin{pmatrix} 1/\tau & -\sigma/\tau\rho \\ 0 & 1/\rho \end{pmatrix} . \tag{4.15}$$

The coordinate change

$$\Xi : \kappa \longrightarrow \psi = \mathcal{R}\,[\kappa - S]\ , \tag{4.16}$$

where

$$\mathcal{R} = \mathcal{T}^{-1} = \begin{pmatrix} \tau & \sigma \\ 0 & \rho \end{pmatrix} ,$$

brings $d^2(\kappa)$ into the form

$$d^2\left(\Xi^{-1}(\psi)\right) = y^2 + y'^2 + (d^+_{min})^2$$

in the new variables ψ, with

$$d^+_{min} \;\; = |d^+_{nod}|\left\{1 - \frac{a'^2\mathcal{F}^2}{det(\mathcal{A})}\right\}^{1/2} = |d^+_{nod}|\left[\frac{a'^2\mathcal{G}^2\sin^2 I}{det(\mathcal{A})}\right]^{1/2} . \tag{4.17}$$

The domain $\overline{\mathbb{T}}^2$ is transformed into a parallelogram with two sides parallel to the y' axis (see Figure 4.3).

REMARK 11. Note that d^+_{min} is the minimal distance between the straight lines r and r' (see also the remark after equation (9.16) in Chapter 9).

Using the variable changes (4.8), (4.16) and the transformation to polar coordinates, whose inverse is

$$\Pi^{-1} : \begin{pmatrix} r \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} y' \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

we obtain

$$\begin{aligned} \int_{\mathbb{T}^2} \frac{1}{\mathrm{d}}\, d\ell\, d\ell' &= \frac{1}{\sqrt{det(\mathcal{A})}} \int_{\Xi\left[\overline{\mathbb{T}}^2\right]} \frac{1}{\sqrt{y^2 + y'^2 + (d^+_{min})^2}}\, dy\, dy' = \\ &= \frac{1}{\sqrt{det(\mathcal{A})}} \int_{\mathfrak{T}} \frac{r}{\sqrt{r^2 + (d^+_{min})^2}}\, dr\, d\theta \end{aligned} \tag{4.18}$$

where $\Xi^{-1}\left[\Pi^{-1}(\mathfrak{T})\right] = \overline{\mathbb{T}}^2$.

Let us describe the domain $\mathfrak{T}$ in details. We define the straight lines that bound the integration domain $\Xi\left[\overline{\mathbb{T}}^2\right]$ as

$$\begin{aligned} \mathrm{r}_1 &= \{(y,y') : y' = \frac{\sigma}{\rho} y + \tau(\pi - S_1)\}; \qquad \mathrm{r}_2 = \{(y,y') : y = \rho(\pi - \overline{\ell} - S_2)\}; \\ \mathrm{r}_3 &= \{(y,y') : y' = \frac{\sigma}{\rho} y - \tau(\pi + S_1)\}; \qquad \mathrm{r}_4 = \{(y,y') : y = -\rho(\pi + \overline{\ell} + S_2)\}\,. \end{aligned}$$

The intersections of these lines with the y axis are

$$y_1 = -\rho(\pi + \overline{\ell} + S_2); \qquad y_2 = \rho(\pi - \overline{\ell} - S_2);$$

while the intersections with the y' axis are

$$\begin{aligned} y'_1 &= \lambda_3(y_1) = -\sigma(\pi + \overline{\ell} + S_2) - \tau(\pi + S_1) \\ y'_2 &= \lambda_3(0) = -\tau(\pi + S_1) \\ y'_3 &= \lambda_3(y_2) = \sigma(\pi - \overline{\ell} - S_2) - \tau(\pi + S_1) \\ y'_4 &= \lambda_1(y_1) = -\sigma(\pi + \overline{\ell} + S_2) + \tau(\pi - S_1) \\ y'_5 &= \lambda_1(0) = \tau(\pi - S_1) \\ y'_6 &= \lambda_1(y_2) = \sigma(\pi - \overline{\ell} - S_2) + \tau(\pi - S_1) \end{aligned}$$

where $\lambda_1(y) = (\sigma/\rho)y + \tau(\pi - S_1)$ and $\lambda_3(y) = (\sigma/\rho)y - \tau(\pi + S_1)$.

We can then decompose the domain $\mathfrak{T}$ into four parts (see Figure 4.4)

$$\mathfrak{T} = \bigcup_{j=1}^{4} \left\{(r,\theta) \in \mathbb{R}^2 \;:\; \theta_j \le \theta \le \theta_{j+1} \text{ and } 0 \le r \le r_j(\theta)\right\}$$

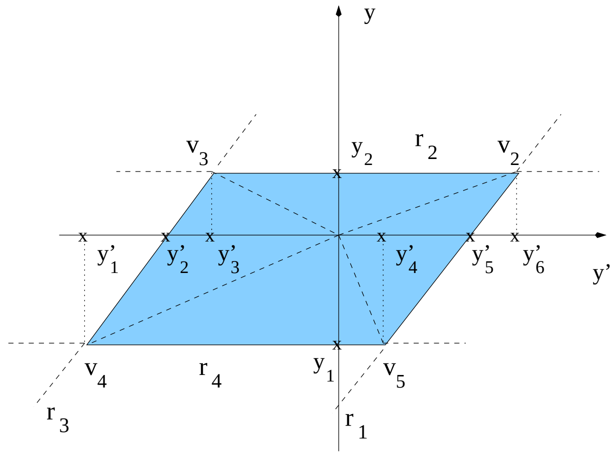

Figure 4.4: We show the decomposition of the integration domain used to compute the last integral in (4.18) in polar coordinates.

where $r_j(\theta)$, with $j = 1 \ldots 4$, represent the lines r_j delimiting $\Xi[\overline{\mathbb{T}}^2]$ in polar coordinates:

$$r_1(\theta) = \frac{\rho\tau(\pi - S_1)}{\rho\cos\theta - \sigma\sin\theta}; \qquad r_2(\theta) = \frac{\rho(\pi - \overline{\ell} - S_2)}{\sin\theta};$$

$$r_3(\theta) = \frac{-\rho\tau(\pi + S_1)}{\rho\cos\theta - \sigma\sin\theta}; \qquad r_4(\theta) = \frac{-\rho(\pi + \overline{\ell} + S_2)}{\sin\theta};$$

while $\theta_1 = \theta_5 - 2\pi$ and θ_l, with $l = 2 \ldots 5$, are the counter-clockwise angles between the y'-axis and the vertexes v_l seen from the origin of the axes (see Figure 4.4):

$$0 < \theta_2 < \theta_3 < \pi < \theta_4 < \theta_5 < 2\pi;$$

$$\tan\theta_2 = \frac{\rho(\pi - \overline{\ell} - S_2)}{\sigma(\pi - \overline{\ell} - S_2) + \tau(\pi - S_1)}; \quad \tan\theta_3 = \frac{\rho(\pi - \overline{\ell} - S_2)}{\sigma(\pi - \overline{\ell} - S_2) - \tau(\pi + S_1)};$$

$$\tan\theta_4 = \frac{\rho(\pi + \overline{\ell} + S_2)}{\sigma(\pi + \overline{\ell} + S_2) + \tau(\pi + S_1)}; \quad \tan\theta_5 = \frac{\rho(\pi + \overline{\ell} + S_2)}{\sigma(\pi + \overline{\ell} + S_2) - \tau(\pi - S_1)}.$$

Using the previous decomposition for $\mathfrak{T}$ and integrating in the r variable the last expression in (4.18) we obtain

$$\int_{\mathbb{T}^2} \frac{1}{\mathrm{d}} d\ell\, d\ell' = \frac{1}{\sqrt{det(\mathcal{A})}} \cdot \left\{ \sum_{j=1}^{4} \int_{\theta_j}^{\theta_{j+1}} \sqrt{(d_{min}^{+})^2 + r_j^2(\theta)}\, d\theta - 2\pi d_{min}^{+} \right\}. \tag{4.19}$$

Note that the integrals in (4.19) are elliptic and the integrand functions are bounded so that these integrals are differentiable functions of the orbital elements. We shall see that the loss of regularity of the averaged perturbing function is due only to the term d_{min}^{+}.

4.8 Boundedness of the remainder function

When there is a crossing at the ascending node, then from the equations of the orbits (4.1) and from Kepler's equations (4.2) we deduce that Taylor's development of $\mathcal{D}^2(\kappa) = \mathrm{D}^2(\ell,\ell')$ in a neighborhood of $\kappa = (0,0)$ is given by

$$\mathcal{D}^2(\kappa) = d^2(\kappa) + O(|\kappa|^3) \tag{4.20}$$

where $O(|\kappa|^3)$ is an infinitesimal of the same order as $|\kappa|^3$ for $|\kappa| \to 0$. We prove the following

LEMMA 2. *If there is an ascending node crossing between the orbits, there exist a neighborhood $\mathcal{U}_0$ of $\kappa = (0,0)$ and two positive constants B_1, B_2 such that*

$$B_1 d^2(\kappa) \le \mathcal{D}^2(\kappa) \le B_2 d^2(\kappa) \qquad \forall \kappa \in \mathcal{U}_0 \, .$$

Proof. First we notice that for $d^+_{nod} = 0$ we have $d^2(\kappa) = \kappa^t \mathcal{A} \kappa$, where $\mathcal{A}$ is positive definite, hence there exist two positive constants C_1, C_2 such that

$$C_1 |\kappa|^2 \le \kappa^t \mathcal{A} \kappa \le C_2 |\kappa|^2 \qquad \forall \kappa \in \mathbb{R}^2. \tag{4.21}$$

Using the relations (4.20) and (4.21) we obtain

$$\lim_{|\kappa| \to 0} \frac{\mathcal{D}^2(\kappa)}{d^2(\kappa)} = 1 \, ,$$

that implies the existence of the neighborhood $\mathcal{U}_0$ and of the constants B_1, B_2 as in the statement of the lemma. □

We prove the following result:

PROPOSITION 6. *The remainder function $1/\mathrm{D} - 1/\mathrm{d}$ is bounded even if there is an ascending node crossing.*

Proof. If there are no crossings between the orbits the remainder function is trivially bounded, in fact $\mathrm{D}(\ell,\ell') > 0$ for each $(\ell,\ell') \in \mathbb{T}^2$ and the minimal value of $\mathrm{d}(\ell,\ell')$ is d^+_{min} that, for $I \neq 0$, can be zero only if $d^+_{nod} = 0$ (see equation (4.17)).

If there is a crossing at the ascending node we have to investigate the local behavior of the remainder function in a neighborhood of $(\ell,\ell') = (\bar{\ell}, 0)$, where both D and d can vanish. The boundedness of the remainder function can be shown using the previous lemma: we know that there exists a neighborhood $\mathcal{U}_0$ and a positive constant B_1 such that the relation

$$\mathcal{D}(\kappa) \ge \sqrt{B_1} d(\kappa)$$

holds for each $\kappa \in \mathcal{U}_0$. It follows that in this neighborhood the remainder function can be bounded in the following way:

$$\begin{aligned} \left|\frac{1}{\mathcal{D}(\kappa)} - \frac{1}{d(\kappa)}\right| &= \frac{|d^2(\kappa) - \mathcal{D}^2(\kappa)|}{d(\kappa)\mathcal{D}(\kappa)[d(\kappa) + \mathcal{D}(\kappa)]} \leq \\ &\leq \frac{1}{\sqrt{B_1}[1+\sqrt{B_1}]} \cdot \frac{|d^2(\kappa) - \mathcal{D}^2(\kappa)|}{|\kappa|^3} \cdot \frac{|\kappa|^3}{d^3(\kappa)} \ . \end{aligned}$$

We observe that $|d^2(\kappa) - \mathcal{D}^2(\kappa)| = O(|\kappa|^3)$ and that by (4.21) there is a positive constant C_1 such that $d^2(\kappa) \geq C_1|\kappa|^2$, so that there exists a constant $L > 0$ such that

$$\left|\frac{1}{\mathcal{D}(\kappa)} - \frac{1}{d(\kappa)}\right| \leq L \qquad \forall \kappa \in \mathcal{U}_0 \ .$$

□

REMARK 12. Although the remainder function $1/\mathrm{D} - 1/\mathrm{d}$ is bounded, it is not continuous in $(\ell, \ell') = (\bar{\ell}, 0)$ when there is a crossing at the ascending node; this can be seen, for instance, by computing the limits of this function along the straight lines $k' = \lambda k$ ($\lambda \in \mathbb{R}$) as $k \to 0$.

4.9 Formulas for descending node crossings

Let us assume that a descending node crossing occurs between the orbits, we can follow a similar procedure to define weak averaged solutions of equations (4.4).

We use the same reference frame as for the ascending crossing case, with the x-axis towards the mutual ascending node, in order to be able to use the same formulas also in case of double crossings.

We already observed in Section 4.3 that the values of the eccentric anomalies $\overline{u}_1, \overline{u}'_1$ corresponding to the descending node crossing are given by

$$\overline{u}'_1 = \pi; \qquad \cos\overline{u}_1 = \frac{e - \cos\omega}{1 - e\cos\omega}; \qquad \sin\overline{u}_1 = \frac{\beta\sin\omega}{1 - e\cos\omega};$$

we define $\overline{\ell}_1 = \overline{u}_1 - e\sin\overline{u}_1, \overline{\ell}'_1 = \pi$ as the corresponding values of the mean anomalies.

Let $\{P(\overline{u}_1), P'(\overline{u}'_1)\}$ be the descending mutual node. We consider the two straight lines $\mathrm{r}_1(\ell)$ and $\mathrm{r}'_1(\ell')$, tangent in $P(\overline{u}_1)$ and $P'(\overline{u}'_1)$ to the orbits of the asteroid and of the planet (see Figure 4.5); they can be parametrized by the mean anomalies ℓ_1, ℓ'_1 so that $P(u(t))$ and $\mathrm{r}_1(\ell(t))$ have the same velocities (derivatives with respect to t) in $P(\overline{u}_1)$ and $P'(u'(t))$ and $\mathrm{r}'_1(\ell'(t))$ have the same velocities in $P'(\overline{u}'_1)$:

$$\begin{cases} r_{1,1} = \overline{x}_{1,1} - \mathcal{F}_1 k_1 \\ r_{1,2} = \overline{x}_{1,2} + \mathcal{G}_1 \cos I k_1 \\ r_{1,3} = \overline{x}_{1,3} + \mathcal{G}_1 \sin I k_1 \end{cases} \qquad \begin{cases} r'_{1,1} = \overline{x}'_{1,1} \\ r'_{1,2} = \overline{x}'_{1,2} - a' k'_1 \\ r'_{1,3} = \overline{x}'_{1,3} \ . \end{cases} \tag{4.22}$$

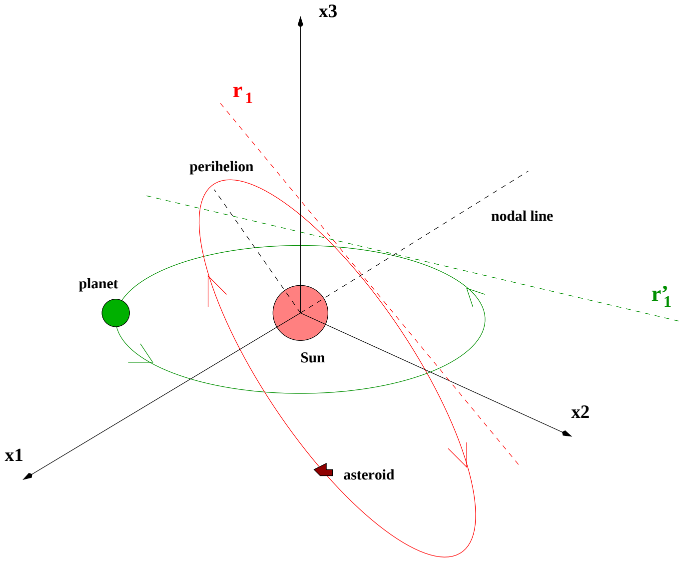

Figure 4.5: The straight lines r_1, r_1' represent Wetherill's approximation at the descending node for the two osculating orbits of the asteroid and the planet.

We have used the following notations

$$\mathcal{F}_1 = \frac{\sin\bar{u}_1\cos\omega + \beta\cos\bar{u}_1\sin\omega}{1-e\cos\bar{u}_1} = \frac{ae\sin\omega}{\beta} = \mathcal{F}\,;$$

$$\mathcal{G}_1 = \frac{-\sin\bar{u}_1\sin\omega + \beta\cos\bar{u}_1\cos\omega}{1-e\cos\bar{u}_1} = -\frac{a(1-e\cos\omega)}{\beta}\,;$$

and

$$\bar{x}_{1,1} = -\frac{a\beta^2}{1-e\cos\omega}; \qquad \bar{x}'_{1,1} = -a'; \qquad \bar{x}_{1,2} = \bar{x}_{1,3} = \bar{x}'_{1,2} = \bar{x}'_{1,3} = 0;$$

We can define a *Wetherill function*, similar to the one of the ascending node crossing case, by using straight lines tangent to the orbits at the descending mutual node. The square of this Wetherill function is given by

$$d_1^2 = a'^2 k_1'^2 + [\mathcal{F}_1^2 + \mathcal{G}_1^2]k_1^2 + 2k_1k_1'[\mathcal{G}_1 a'\cos I] + 2k_1 d_{nod}^- \mathcal{F}_1 + (d_{nod}^-)^2$$

where $k_1 = \ell - \bar{\ell}_1, k_1' = \ell' - \pi$ and

$$d_{nod}^- = \frac{a(1-e^2)}{1-e\cos\omega} - a'\,.$$

Remark 13. There are some differences between the general appearance of this formula and the one for the ascending node: they are due to the sign of the mixed quadratic monomial and of the linear term. The positive signs are due, respectively, to the minus sign in the expression of $r'_{1,2}$ and to the fact that $\bar{x}_{1,1} - \bar{x}'_{1,1} = -d_{nod}^-$.

We can write it more concisely as

$$\mathrm{d}_1^2(\ell,\ell') = d_1^2(\kappa_1) = \kappa_1^t \mathcal{A}_d \,\kappa_1 + B_d^t \kappa_1 + (d_{nod}^-)^2$$

where

$$\kappa_1 = (k_1', k_1)\ ; \qquad B_d = 2(B_1^-, B_2^-)\ ; \qquad \mathcal{A}_d = \begin{bmatrix} A_{11}^- & A_{12}^- \\ A_{21}^- & A_{22}^- \end{bmatrix}\ ;$$

with components

$$\begin{cases} B_1^- = 0 \\ B_2^- = d_{nod}^- \mathcal{F}_1 \end{cases} \qquad \begin{cases} A_{11}^- = a'^2 \\ A_{12}^- = A_{21}^- = \mathcal{G}_1 a' \cos I \\ A_{22}^- = [\mathcal{F}_1^2 + \mathcal{G}_1^2]\ . \end{cases}$$

The variable change to eliminate the linear terms in the quadratic form $d_1^2(\kappa_1)$ is defined by its inverse

$$\Xi_1^{-1} : \psi_1 \longrightarrow \kappa_1 = \mathcal{T}_1 \psi_1 + S_d$$

where $S_d = (S_1^-, S_2^-) \in \mathbb{R}^2$, $\psi_1 = (y_1', y_1) \in \mathbb{R}^2$ are the new variables and $\mathcal{T}_1$ is a 2×2 real-valued invertible matrix.

The equation to be solved in this case is

$$2\ \mathcal{A}_d S_d + B_d = 0 \tag{4.23}$$

whose solutions are

$$S_1^- = \frac{B_2^- A_{12}^-}{det(\mathcal{A}_d)} = \frac{d_{nod}^- \mathcal{F}_1 \mathcal{G}_1 a' \cos I}{det(\mathcal{A}_d)}\ ;$$

$$S_2^- = -\frac{B_2^- A_{11}^-}{det(\mathcal{A}_d)} = -\frac{d_{nod}^- \mathcal{F}_1 a'^2}{det(\mathcal{A}_d)}\ .$$

We obtain

$$\mathcal{T}_1 = \frac{1}{\sqrt{det(\mathcal{A}_d)}} \begin{pmatrix} \rho_1 & -\sigma_1 \\ 0 & \tau_1 \end{pmatrix} = \begin{pmatrix} 1/\tau_1 & -\sigma_1/\tau_1\rho_1 \\ 0 & 1/\rho_1 \end{pmatrix} \tag{4.24}$$

where

$$\tau_1 = a'; \qquad \sigma_1 = \mathcal{G}_1 \cos I; \qquad \rho_1 = \frac{1}{a'}\sqrt{det(\mathcal{A}_d)};$$

and

$$det(\mathcal{A}_d) = a'^2 \left(\mathcal{F}_1^2 + \mathcal{G}_1^2 \sin^2 I\right)\ .$$

The minimal distance between the straight lines in this case is

$$d_{min}^- = |d_{nod}^-| \left[1 - \frac{a'^2 \mathcal{F}_1^2}{det(\mathcal{A}_1)}\right]^{1/2} = |d_{nod}^-| \left[\frac{a'^2 \mathcal{G}_1^2 \sin^2 I}{det(\mathcal{A}_d)}\right]^{1/2}\ .$$

Chapter 5

Hamilton's equations with $\overline{H}$

5.1 The derivatives of the averaged perturbing function $\overline{R}$

Kantorovich's method is used to describe the singularities of the derivatives of the averaged perturbing function with respect to Delaunay's variables appearing in equations (4.4).

Note that by the *chain rule* we can write

$$\frac{\partial \overline{R}}{\partial \mathcal{E}_{\mathcal{D}}} = \frac{\partial \overline{R}}{\partial \mathcal{E}_{\mathcal{K}}} \frac{\partial \mathcal{E}_{\mathcal{K}}}{\partial \mathcal{E}_{\mathcal{D}}}$$

where $\mathcal{E}_{\mathcal{K}} = \{e, I, \omega, \Omega\}$ is a subset of the Keplerian elements of the asteroid and

$$\frac{\partial \mathcal{E}_{\mathcal{K}}}{\partial \mathcal{E}_{\mathcal{D}}} = \begin{bmatrix} \mathcal{M} & \mathcal{O}_2 \\ \mathcal{O}_2 & I_2 \end{bmatrix},$$

in which I_2 and $\mathcal{O}_2$ are the 2×2 identity and zero matrices, and

$$\mathcal{M} = -\frac{1}{k\sqrt{a}} \begin{bmatrix} \beta/e & 0 \\ -\mathrm{cotan} I/\beta & 1/(\beta \sin I) \end{bmatrix}.$$

Hence we can do the computations using the derivatives of $\overline{R}$ with respect to the Keplerian elements e, I, ω ($\overline{R}$ does not depend on Ω).

We shall not need to perform the splitting of Kantorovich's method to compute the derivative of $\overline{R}$ with respect to the inclination I; in fact we shall see that the derivative of $1/\mathfrak{D}$ with respect to I can be bounded by a function with a first order polar singularity in $\overline{u}, \overline{u}'$, so it is Lebesgue integrable over $\mathbb{T}^2$.

In the following we shall first prove that the derivatives of the remainder function $1/\mathrm{D} - 1/\mathrm{d}$ are always Lebesgue integrable over $\mathbb{T}^2$, and the integrals have finite values, even if the two orbits intersect each other. Hence the average of the remainder function is differentiable: indeed its derivatives can be computed by exchanging the position of the integral and differential operators as in (4.7). Then we shall see that, if there is an ascending node crossing, a discontinuous term appears in the derivatives of the average of $1/\mathrm{d}$, which is responsible of the discontinuity of the derivatives of $\overline{R}$. These derivatives admit two limit values at crossings (coming from the regions defined by $d^+_{nod} > 0$ and $d^+_{nod} < 0$).

As the properties we intend to prove are invariant by coordinate changes, we shall show them using the coordinates (u,u') instead of (ℓ,ℓ').

The derivatives of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$.

Let us set $\upsilon = (u,u')$ and $\nu = (v,v') = (u-\overline{u}, u'-\overline{u}')$. We apply Taylor's formula with the integral remainder to the vector functions $P(u), P'(u')$:

$$\begin{cases} P(u) = P(\overline{u}) + P_u(\overline{u})\,v + \displaystyle\int_{\overline{u}}^{u} (u-s)P_{ss}(s)\,ds \\ P'(u') = P'(\overline{u}') + P'_{u'}(\overline{u}')\,v' + \displaystyle\int_{\overline{u}'}^{u'} (u'-t)P'_{tt}(t)\,dt\ . \end{cases}$$

The functions defining the straight lines $r(u) = \mathrm{r}(\ell(u))$ and $r'(u') = \mathrm{r}'(\ell'(u'))$ have the same Taylor's developments, up to the first order in $|\nu| = \sqrt{v^2 + v'^2}$, as $P(u)$ and $P'(u')$ respectively, so that we can write

$$\begin{cases} r(u) = P(\overline{u}) + P_u(\overline{u})\,v + \displaystyle\int_{\overline{u}}^{u} (u-s)r_{ss}(s)\,ds \\ r'(u') = P'(\overline{u}') + P'_{u'}(\overline{u}')\,v' + \displaystyle\int_{\overline{u}'}^{u'} (u'-t)r'_{tt}(t)\,dt\ . \end{cases}$$

We prove the following

THEOREM 14. *If there is an ascending node crossing at $(u,u') = (\overline{u},\overline{u}')$, the derivatives of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$ with respect to e,ω can be bounded by functions having a first order polar singularity in $\overline{u},\overline{u}'$, so they are Lebesgue integrable over $\mathbb{T}^2$.*

Proof. We shall consider only the derivatives with respect to e: the proof for the other derivatives is similar. First we note that

$$\frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{D}(\upsilon)}\right] = -\frac{1}{2\mathfrak{D}^3(\upsilon)}\frac{\partial}{\partial e}\left[\mathfrak{D}^2(\upsilon)\right]\ ; \qquad \frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{d}(\upsilon)}\right] = -\frac{1}{2\mathfrak{d}^3(\upsilon)}\frac{\partial}{\partial e}\left[\mathfrak{d}^2(\upsilon)\right]\ .$$

Let us write $\langle\,,\rangle$ for the Euclidean scalar product. We have

$$\frac{\partial}{\partial e}\left[\mathfrak{D}^2(\upsilon)\right] = \mathfrak{D}^2_{e,0}(\upsilon) + \mathfrak{D}^2_{e,1}(\upsilon) + \mathfrak{D}^2_{e,2}(\upsilon) \tag{5.1}$$

where

$$\begin{aligned} \mathfrak{D}^2_{e,0}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[P(u) - P'(u')\right], P(\overline{u}) - P'(\overline{u}')\right\rangle\ ; \\ \mathfrak{D}^2_{e,1}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[P(u) - P'(u')\right], P_u(\overline{u})\,v - P'_{u'}(\overline{u}')\,v'\right\rangle\ ; \\ \mathfrak{D}^2_{e,2}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[P(u) - P'(u')\right], \int_{\overline{u}}^{u} (u-s)P_{ss}(s)\,ds - \int_{\overline{u}'}^{u'} (u'-t)P'_{tt}(t)\,dt\right\rangle\ ; \end{aligned}$$

and

$$\frac{\partial}{\partial e}\left[\mathfrak{d}^2(\upsilon)\right] = \mathfrak{d}^2_{e,0}(\upsilon) + \mathfrak{d}^2_{e,1}(\upsilon) + \mathfrak{d}^2_{e,2}(\upsilon) \tag{5.2}$$

where

$$\begin{aligned}
\mathfrak{d}^2_{e,0(\upsilon)} &= 2\left\langle \frac{\partial}{\partial e}\left[r(u) - r'(u')\right], P(\overline{u}) - P'(\overline{u}') \right\rangle ;\\
\mathfrak{d}^2_{e,1}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[r(u) - r'(u')\right], P_u(\overline{u})\,v - P'_{u'}(\overline{u}')\,v' \right\rangle ;\\
\mathfrak{d}^2_{e,2}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[r(u) - r'(u')\right], \int_{\overline{u}}^{u}(u-s)r_{ss}(s)\,ds - \int_{\overline{u}'}^{u'}(u'-t)r'_{tt}(t)\,dt \right\rangle .
\end{aligned}$$

If we set the *crossing conditions* $P(\overline{u}) = P'(\overline{u}')$ we obtain

$$\mathfrak{D}^2_{e,0}(\upsilon) = \mathfrak{d}^2_{e,0}(\upsilon) = 0$$

and, in particular, the constant terms in Taylor's developments of $\partial\mathfrak{D}^2/\partial e$ and $\partial\mathfrak{d}^2/\partial e$ vanish.

The terms defined by $\mathfrak{D}^2_{e,2}$ and $\mathfrak{d}^2_{e,2}$ are at least infinitesimal of the second order with respect to $|\nu|$ as $\upsilon \to (\overline{u}, \overline{u}')$, so that the first order terms in $|\nu|$ at crossing can be given only by $\mathfrak{D}^2_{e,1}$ and $\mathfrak{d}^2_{e,1}$.

Using the theorems on the integrals depending on a parameter we obtain

$$\begin{aligned}
&\frac{\partial}{\partial e}\left[\int_{\overline{u}}^{u}(u-s)P_{ss}(s)\,ds - \int_{\overline{u}'}^{u'}(u'-t)P'_{tt}(t)\,dt\right] =\\
&\int_{\overline{u}}^{u}(u-s)\frac{\partial P_{ss}}{\partial e}(s)\,ds - \frac{\partial\overline{u}}{\partial e}P_{uu}(\overline{u})\,v - \int_{\overline{u}'}^{u'}(u'-t)\frac{\partial P'_{tt}}{\partial e}(t)\,dt + \frac{\partial\overline{u}'}{\partial e}P'_{u'u'}(\overline{u}')\,v' ;\\
&\frac{\partial}{\partial e}\left[\int_{\overline{u}}^{u}(u-s)r_{ss}(s)\,ds - \int_{\overline{u}'}^{u'}(u'-t)r'_{tt}(t)\,dt\right] =\\
&\int_{\overline{u}}^{u}(u-s)\frac{\partial r_{ss}}{\partial e}(s)\,ds - \frac{\partial\overline{u}}{\partial e}r_{uu}(\overline{u})\,v - \int_{\overline{u}'}^{u'}(u'-t)\frac{\partial r'_{tt}}{\partial e}(t)\,dt + \frac{\partial\overline{u}'}{\partial e}r'_{u'u'}(\overline{u}')\,v' ;
\end{aligned}$$

so that these two expressions are at least infinitesimal of the first order with respect to $|\nu|$. As these terms are multiplied by first order terms in the expressions of $\mathfrak{D}^2_{e,1}$ and $\mathfrak{d}^2_{e,1}$, they give rise to at least second order terms.

We can conclude that the first order terms in the expressions (5.1) and (5.2) are equal and they are given by

$$2\left\langle \frac{\partial}{\partial e}\left[P(\overline{u}) - P'(\overline{u}')\right] - \left[\frac{\partial\overline{u}}{\partial e}P_u(\overline{u}) - \frac{\partial\overline{u}'}{\partial e}P'_{u'}(\overline{u}')\right], P_u(\overline{u})\,v - P'_{u'}(\overline{u}')\,v' \right\rangle ;$$

therefore the asymptotic developments of the e-derivatives of $\mathfrak{D}^2(\upsilon)$ and $\mathfrak{d}^2(\upsilon)$ in a neighborhood of $\upsilon = (\overline{u}, \overline{u}')$ are

$$\frac{\partial}{\partial e}\left[\mathfrak{D}^2(\upsilon)\right] = \alpha v + \beta v' + \mathfrak{r}_{\mathfrak{D}}(\upsilon); \qquad \frac{\partial}{\partial e}\left[\mathfrak{d}^2(\upsilon)\right] = \alpha v + \beta v' + \mathfrak{r}_{\mathfrak{d}}(\upsilon);$$

where α, β are independent on u, u' and $\mathfrak{r}_{\mathfrak{D}}(\upsilon)$, $\mathfrak{r}_{\mathfrak{d}}(\upsilon)$ are infinitesimal of the second order with respect to $|\nu|$ as $\upsilon \to (\overline{u}, \overline{u}')$.

Using the decomposition

$$\left[\frac{1}{\mathfrak{D}^3} - \frac{1}{\mathfrak{d}^3}\right] = \left[\frac{1}{\mathfrak{D}} - \frac{1}{\mathfrak{d}}\right]\left[\frac{1}{\mathfrak{D}^2} + \frac{1}{\mathfrak{D}\mathfrak{d}} + \frac{1}{\mathfrak{d}^2}\right] ,$$

the boundedness of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$ and lemma 2 (that also hold in the (u, u') coordinates), we conclude that there exist two constants $L_1, L_2 > 0$ such that

$$\left|\frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{D}(\upsilon)}\right] - \frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{d}(\upsilon)}\right]\right| = \frac{1}{2}\left|\left\{\left[\frac{1}{\mathfrak{D}^3(\upsilon)} - \frac{1}{\mathfrak{d}^3(\upsilon)}\right](\alpha v + \beta v') + \right.\right.$$

$$\left.\left. + \frac{1}{\mathfrak{D}^3(\upsilon)}\mathfrak{r}_{\mathfrak{D}}(\upsilon) - \frac{1}{\mathfrak{d}^3(\upsilon)}\mathfrak{r}_{\mathfrak{d}}(\upsilon)\right\}\right| \leq L_1 \frac{1}{|\nu|} + L_2$$

in a neighborhood of $\upsilon = (\overline{u}, \overline{u}')$ and the theorem is proven.

□

Singularities of the $\{e, \omega\}$-derivatives of the average of $1/\mathrm{d}$.

As $det(\mathcal{A}) > 0$ and $(\overline{\ell}, 0)$ is in the interior part of $\mathbb{T}^2$, we have $d^2_{min} + r^2_j(\theta) > 0$ for each $\theta \in [\theta_j, \theta_{j+1}]$ and for each $j = 1 \ldots 4$. Then we can use again the theorem of differentiation under the integral sign and compute, for instance, the derivative of the average of $1/\mathrm{d}$ with respect to e as

$$\frac{\partial}{\partial e}\int_{\mathbb{T}^2} \frac{1}{\mathrm{d}}\, d\ell\, d\ell' = \frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right] \cdot \left\{\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}} \sqrt{(d^+_{min})^2 + r^2_j(\theta)}\, d\theta - 2\pi d^+_{min}\right\} +$$

$$+ \left[\frac{1}{\sqrt{det(\mathcal{A})}}\right] \cdot \left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}} \frac{\frac{\partial}{\partial e}[(d^+_{min})^2 + r^2_j(\theta)]}{\sqrt{(d^+_{min})^2 + r^2_j(\theta)}}\, d\theta - 2\pi \frac{\partial}{\partial e} d^+_{min}\right\} . \tag{5.3}$$

We have similar formulas for the derivatives with respect to ω, obtained simply by substitution of the partial derivative operators.

The discontinuities present in the terms

$$\frac{\partial}{\partial e} d^+_{min}; \qquad \frac{\partial}{\partial \omega} d^+_{min};$$

are responsible of the discontinuities in the derivatives of the averaged perturbing function that produce a sort of *crests* in the surfaces representing this function (see Figure 5.1) and cause the loss of regularity in its level lines, where the weak averaged solutions lie.

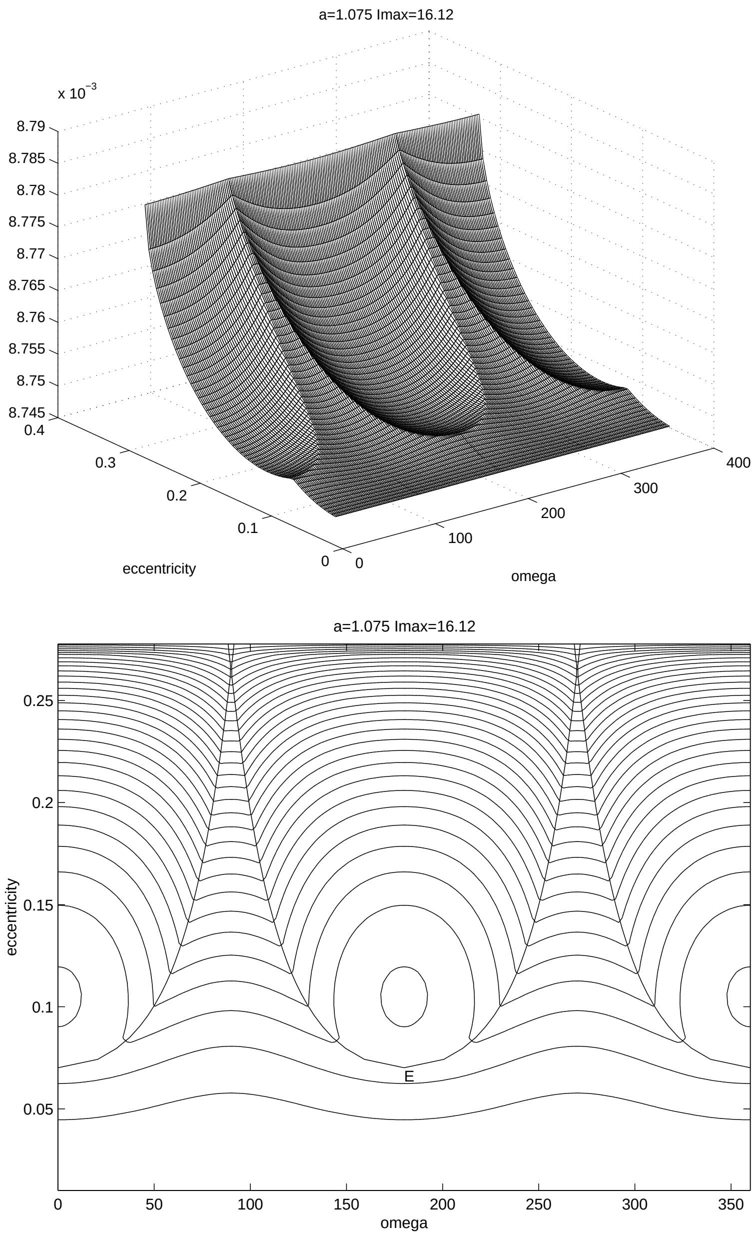

Figure 5.1: We draw the graphic of the averaged perturbing function (top) and its level lines (bottom) in the plane (ω, e) for the Near Earth Asteroid 2000 CO_{101} (ω is in degrees in this figure). The loss of regularity at the node crossing lines with the Earth is particularly evident for this object.

5.1.1 Singularities of the I-derivative of $\overline{R}$

We observe that we do not need to perform the splitting of Kantorovich's method to compute the derivative with respect to I of the integral average of $1/\mathfrak{D}$, in fact we have the following

PROPOSITION 7. *If there is an ascending node crossing, the derivative of $1/\mathfrak{D}$ with respect to the mutual inclination I can be bounded by a function with a first order polar singularity in $\overline{u},\overline{u}'$, so it is Lebesgue integrable over $\mathbb{T}^2$.*

Proof. We have

$$\frac{\partial}{\partial I}\left[\frac{1}{\mathfrak{D}(\upsilon)}\right] = -\frac{1}{2\mathfrak{D}^3(\upsilon)}\frac{\partial}{\partial I}\left[\mathfrak{D}^2(\upsilon)\right],$$

so we need to prove that $\partial\mathfrak{D}^2/\partial I$ is an infinitesimal of the second order with respect to $|\nu|$ as $\upsilon \to (\overline{u},\overline{u}')$. For this purpose we only need to check the vanishing of the term

$$2\left\langle \frac{\partial}{\partial I}\left[P(\overline{u}) - P'(\overline{u}')\right] - \left[\frac{\partial \overline{u}}{\partial I}P_u(\overline{u}) - \frac{\partial \overline{u}'}{\partial I}P'_{u'}(\overline{u}')\right], P_u(\overline{u})\,v - P'_{u'}(\overline{u}')\,v' \right\rangle \tag{5.4}$$

that formally represents the first order terms in the derivative of $\mathfrak{D}^2(\upsilon)$ with respect to I. as we can see from similar computations in Theorem 14. The expression in (5.4) vanishes because $P(\overline{u}), P'(\overline{u}'), \overline{u}$ and $\overline{u}'$ do not depend on I, in fact

$$\cos\overline{u}\sin\omega + \beta\sin\overline{u}\cos\omega = 0 \ .$$

□

5.1.2 The averaged equations and their discontinuities

In this section we describe in details the averaged equations of motion: we also put in evidence the discontinuities of these equations giving analytical formulas for them. This is done by means of Kantorovich's decomposition, that allows us to give an *operative definition* of the weak averaged solutions.

REMARK 14. It is possible that for very peculiar choices of the initial conditions the definition of a weak averaged solution is not unique but there are two ways of continuing the solution. This happens when the level line of the averaged Hamiltonian, that constrains the orbit in the phase space (ω, e), reaches a node crossing line being tangent to it. This is possible when the crossing point corresponds to a local minimum of the averaged perturbing function restricted to the node crossing line (see Chapter 6).

Description of the differential equations.
Let us consider the perturbing function

$$R = k^2\mu\frac{1}{\mathrm{D}} \qquad\qquad (\mu = m'/m_\odot)$$

as a function of the mean anomalies ℓ, ℓ'. The averaged equations that we want to solve, in terms of the orbital elements, are

$$\begin{aligned}
\dot{\omega} &= -\frac{\partial\overline{R}}{\partial G} = -\frac{\partial\overline{R}}{\partial e}\Big[\frac{-\beta}{k\sqrt{a}e}\Big] + \frac{\partial\overline{R}}{\partial I}\Big[\frac{cotgI}{k\sqrt{a}\beta}\Big] = \\
&= k^2\mu\Big\{\Big[\frac{\beta}{k\sqrt{a}e}\Big]\Big[\frac{\partial}{\partial e}\int_T \frac{1}{\mathrm{d}}\,d\ell\, d\ell' + \int_T \frac{\partial}{\partial e}\Big(\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}}\Big)\,d\ell\, d\ell'\Big] - \\
&\quad -\Big[\frac{cotgI}{k\sqrt{a}\beta}\Big]\Big[\int_T \frac{\partial}{\partial I}\Big(\frac{1}{\mathrm{D}}\Big)\,d\ell\, d\ell'\Big]\Big\}\ ; \\
\dot{G} &= \frac{\partial\overline{R}}{\partial\omega} = \frac{\partial}{\partial\omega}\int_T \frac{1}{\mathrm{d}}\,dl\, dl' + \int_T \frac{\partial}{\partial\omega}\Big(\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}}\Big)\,d\ell\, d\ell'\ ; \\
\dot{\Omega} &= -\frac{\partial\overline{R}}{\partial Z} = -\frac{\partial\overline{R}}{\partial I}\Big[\frac{-1}{k\sqrt{a}\beta\sin I}\Big] = \Big[\frac{1}{k\sqrt{a}\beta\sin I}\Big]\int_T \frac{\partial}{\partial I}\Big(\frac{1}{\mathrm{D}}\Big)\,d\ell\, d\ell'\ .
\end{aligned}$$

Discontinuities of the equations.

We present the formulas for the discontinuities of the derivatives of $\overline{R}$.

Let

$$\frac{\partial^+}{\partial e};\ \frac{\partial^+}{\partial\omega};\qquad \text{and}\qquad \frac{\partial^-}{\partial e};\ \frac{\partial^-}{\partial\omega};$$

be the partial derivative operators applied in the regions of the space where $d^+_{nod} > 0$ and $d^+_{nod} < 0$ respectively, that is the partial derivatives of the restriction of a function to these domains.

We define the operator 'Diff' to describe the differences in the right hand sides of equations (4.3) at $d^+_{nod} = 0$ when we pass from a region where $d^+_{nod} > 0$ to a region where $d^+_{nod} < 0$; this definition is given taking into account the existence of a continuous extension of the derivatives to the boundaries of both the sets $\{d^+_{nod} > 0\} \cap W$ and $\{d^+_{nod} < 0\} \cap W$, where W is the Kozai domain. We have

$$\begin{aligned}
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial G}\right) &= \frac{k\mu(1-e^2)}{2\pi\sqrt{det(\mathcal{A})}\cdot e\sqrt{a}}\cdot\Big[1 - \frac{a^2a'^2e^2\sin^2\omega}{(1-e^2)det(\mathcal{A})}\Big]^{\frac{1}{2}}\cdot\mathrm{Diff}\left(\frac{\partial|d^+_{nod}|}{\partial e}\right)\ ; \\
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega}\right) &= -\frac{k^2\mu}{2\pi\sqrt{det(\mathcal{A})}}\cdot\Big[1 - \frac{a^2a'^2e^2\sin^2\omega}{(1-e^2)det(\mathcal{A})}\Big]^{\frac{1}{2}}\cdot\mathrm{Diff}\left(\frac{\partial|d^+_{nod}|}{\partial\omega}\right)\ ; \\
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial Z}\right) &= 0\,;
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Diff}\left(\frac{\partial}{\partial e}|d^+_{nod}|\right) &= \Big[\frac{\partial^+|d^+_{nod}|}{\partial e} - \frac{\partial^-|d^+_{nod}|}{\partial e}\Big] = \frac{2a(\cos\omega(1+e^2)+2e)}{(1+e\cos\omega)^2}\,; \\
\mathrm{Diff}\Big(\frac{\partial}{\partial\omega}|d^+_{nod}|\Big) &= \Big[\frac{\partial^+|d^+_{nod}|}{\partial\omega} - \frac{\partial^-|d^+_{nod}|}{\partial\omega}\Big] = -\frac{2a(1-e^2)e\sin\omega}{(1+e\cos\omega)^2}\ .
\end{aligned}$$

REMARK 15. In the last formula the quantities

$$\frac{\partial^+|d^+_{nod}|}{\partial e};\qquad \frac{\partial^-|d^+_{nod}|}{\partial e};\qquad \frac{\partial^+|d^+_{nod}|}{\partial \omega};\qquad \frac{\partial^-|d^+_{nod}|}{\partial \omega};$$

have to be considered as the extensions of these functions to the points of the common boundary $d^+_{nod} = 0$.

5.2 Different kind of node crossings

In this section we shall briefly discuss the remaining two cases: *crossings at the descending node* and *double crossings*.

5.2.1 Descending node crossings

Let us assume that a descending node crossing occurs between the orbits: we can define weak averaged solutions of equations (3.23) following a procedure similar to the one described in the previous sections.

We shall use the inverse of the Wetherill function $1/\mathrm{d}_1$ to extract the principal part from the direct term of the perturbing function. We observe that the point $(\overline{\ell}_1, \pi)$ is on the boundary of the torus $\mathbb{T}^2 = \{(\ell,\ell') : -\pi \le \ell \le \pi, -\pi \le \ell' \le \pi\}$: we use the periodicity of the function D and consider the shifted domain

$$\tau_{\pi,0}\left(\mathbb{T}^2\right) = \{(\ell,\ell') : -\pi \le \ell \le \pi, 0 \le \ell' \le 2\pi\}\ .$$

so that the crossing value $(\overline{\ell}_1, \pi)$ is generally an internal point.

In computing the derivatives of $\overline{R}$ with respect to the variables $\mathcal{E}_{\mathcal{D}}$, for instance the G-derivative, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2}\frac{\partial}{\partial G}\overline{R} = \int_{\tau_{\pi,0}(\mathbb{T}^2)} \frac{\partial}{\partial G}\left[\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}_1}\right] d\ell\, d\ell' + \frac{\partial}{\partial G}\int_{\tau_{\pi,0}(\mathbb{T}^2)} \frac{1}{\mathrm{d}_1}\, d\ell\, d\ell'\ . \qquad (5.5)$$

Discontinuities in the derivatives: in case of crossing at the descending node, we have the following formulas:

$$det(\mathcal{A}_d) = \frac{a^2 a'^2}{(1-e^2)} \cdot (\sin^2 I(1-2e\cos\omega + e^2) + e^2\cos^2 I \sin^2\omega)\,;$$

$$\mathrm{Diff}_1\left(\frac{\partial}{\partial e}|d^-_{nod}|\right) = \frac{\partial^+_1|d^-_{nod}|}{\partial e} - \frac{\partial^-_1|d^-_{nod}|}{\partial e} = -\frac{2a(\cos\omega(1+e^2)-2e)}{(1-e\cos\omega)^2};$$

$$\mathrm{Diff}_1\left(\frac{\partial}{\partial \omega}|d^-_{nod}|\right) = \frac{\partial^+_1|d^-_{nod}|}{\partial \omega} - \frac{\partial^-_1|d^-_{nod}|}{\partial \omega} = \frac{2a(1-e^2)e\sin\omega}{(1-e\cos\omega)^2};$$

where

$$\frac{\partial_1^+}{\partial e};\ \frac{\partial_1^+}{\partial \omega}; \qquad \text{and} \qquad \frac{\partial_1^-}{\partial e};\ \frac{\partial_1^-}{\partial \omega};$$

are the partial derivative operators applied to the restriction to the regions of the space where $d_{nod}^- > 0$ and $d_{nod}^- < 0$ respectively, and the operator 'Diff$_1$' describes the differences in the right hand sides of equations (4.3) at $d_{nod}^- = 0$, when we pass from a region where $d_{nod}^- > 0$ to a region where $d_{nod}^- < 0$.

5.2.2 Double crossings

In the case of double crossings we use both the inverse of the Wetherill functions $1/\mathrm{d}$ and $1/\mathrm{d}_1$ to extract the principal part from the direct term of the perturbing function. We observe that the point $(\overline{\ell}, 0)$ is on the boundary of $\tau_{\pi,0}\left(\mathbb{T}^2\right)$ and the point $(\overline{\ell}_1, \pi)$ is on the boundary of $\mathbb{T}^2$; we use again the periodicity of D and consider the shifted domain

$$\tau_{\pi/2,0}\left(\mathbb{T}^2\right) = \{(\ell,\ell') : -\pi \le \ell \le \pi, -\pi/2 \le \ell' \le 3\pi/2\},$$

so that the crossing values $(\overline{\ell}, 0)$ and $(\overline{\ell}_1, \pi)$ are generally internal points.

In computing the derivatives of $\overline{R}$ with respect to the variables $\mathcal{E}_{\mathcal{D}}$, for instance the G-derivative, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2}\frac{\partial}{\partial G}\overline{R} = \int_{\tau_{\pi/2,0}(\mathbb{T}^2)} \frac{\partial}{\partial G}\left[\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}} - \frac{1}{\mathrm{d}_1}\right] d\ell\, d\ell' + \frac{\partial}{\partial G}\int_{\tau_{\pi/2,0}(\mathbb{T}^2)} \frac{1}{\mathrm{d}}\, d\ell\, d\ell' + $$
$$+\frac{\partial}{\partial G}\int_{\tau_{\pi/2,0}(\mathbb{T}^2)} \frac{1}{\mathrm{d}_1}\, d\ell\, d\ell' \,.$$

5.3 Derivatives of $\overline{R}$ along the node crossing lines

For each point $(\overline{\omega}, \overline{e})$ on a node crossing line in the phase space we consider the unit vector $\tau_{\overline{\omega},\overline{e}}$, tangent to the line at that point and we study the regularity properties of the averaged Hamiltonian $\overline{H}$ restricted to the node crossing line. It comes out that the Hamiltonian is derivable in each point $(\overline{\omega}, \overline{e})$ of this curve along the direction of $\tau_{\overline{\omega},\overline{e}}$, the tangent vector to the curve in $(\overline{\omega}, \overline{e})$.

The following result is easily shown:

Lemma 3. *Let $F(x,y) \in C^\infty(\Omega;\mathbb{R})$ where $\Omega \in \mathbb{R}^2$ is an open domain. Let us assume that $\nabla F(x,y) \neq 0$ for each $(x,y) \in \Omega$ and let us consider the curve γ defined by the equation $F(x,y) = 0$. If for each in $(\overline{x},\overline{y}) \in \Omega$ we call $\tau_{\overline{x},\overline{y}}$ the unit vector tangent to γ in $(\overline{x},\overline{y})$ we have that the function $|F(x,y)|$ is derivable in $(\overline{x},\overline{y})$ along the direction defined by $\tau_{\overline{x},\overline{y}}$, and we have*

$$\frac{\partial |F|}{\partial \tau_{\overline{x},\overline{y}}}(\overline{x},\overline{y}) = 0\,.$$

Proof. Take $(\overline{x},\overline{y}) \in \Omega$ and assume that $\partial_x F(\overline{x},\overline{y}) \neq 0$, where ∂_x is the partial derivative with respect to x. We can write a vector tangent to γ in $(\overline{x},\overline{y})$ as

$$\tau_{\overline{x},\overline{y}} = (1, \beta(\overline{x},\overline{y})) \qquad \text{with} \qquad \beta(\overline{x},\overline{y}) = -\frac{\partial_x F(\overline{x},\overline{y})}{\partial_y F(\overline{x},\overline{y})}\,.$$

The derivative of $|F|$ in $(\overline{x},\overline{y})$, along the direction of $\tau_{\overline{x},\overline{y}}$, is defined by

$$\frac{\partial |F|}{\partial \tau_{\overline{x},\overline{y}}}(\overline{x},\overline{y}) = \lim_{h\to 0}\left\{\frac{|F(\overline{x}+h,\overline{y}+h\beta(\overline{x},\overline{y}))| - |F(\overline{x},\overline{y})|}{h}\right\} = \lim_{h\to 0}\left\{\frac{|F(\overline{x}+h,\overline{y}+h\beta(\overline{x},\overline{y}))|}{h}\right\}\,.$$

Furthermore we have

$$\begin{aligned} &F(\overline{x}+h,\overline{y}+h\beta(\overline{x},\overline{y})) = F(\overline{x},\overline{y}) + h\partial_x F(\overline{x},\overline{y}) + h\beta(\overline{x},\overline{y})\partial_y F(\overline{x},\overline{y}) + o(h) = \\ &= h\left\{\partial_x F(\overline{x},\overline{y}) + \beta(\overline{x},\overline{y})\partial_y F(\overline{x},\overline{y})\right\} + o(h) = \\ &= h\left\{\partial_x F(\overline{x},\overline{y}) - \frac{\partial_x F(\overline{x},\overline{y})}{\partial_y F(\overline{x},\overline{y})}\partial_y F(\overline{x},\overline{y})\right\} + o(h) = o(h)\,. \end{aligned}$$

The proof under the assumption that $\partial_y F(\overline{x},\overline{y}) \neq 0$ is quite similar.

□

Let us consider the smooth function d^+_{nod}, whose zeros define the ascending node crossing line. We observe that the gradient of d^+_{nod} is never zero in the interior part of the Kozai domain: in fact

$$\begin{aligned} \frac{\partial}{\partial e} d^+_{nod} &= -\frac{a(2e+(1+e^2)\cos\omega)}{(1+e\cos\omega)^2}\,; \\ \frac{\partial}{\partial \omega} d^+_{nod} &= \frac{ae(1-e^2)\sin\omega}{(1+e\cos\omega)^2}\,; \end{aligned}$$

so that if $\sin\omega = 0$ we have either

$$2e + (1+e^2)\cos\omega = (1+e)^2 > 0$$

or

$$2e + (1+e^2)\cos\omega = -(1-e)^2 < 0\,.$$

Let us fix a point $(\overline{\omega},\overline{e})$ on the node crossing line. We can prove, following the steps of Section 5.1, that the only term that can be responsible of a discontinuity at $(\overline{\omega},\overline{e})$ in the derivatives of the averaged perturbing function along the direction $\tau_{\overline{\omega},\overline{e}}$ tangent to $(\overline{\omega},\overline{e})$ is

$$\frac{\partial |d^+_{nod}|}{\partial \tau_{\overline{\omega},\overline{e}}}\,;$$

by the previous lemma this derivative exists in $(\overline{\omega},\overline{e})$ and it is zero.

The proof for the derivatives of $\overline{R}$ along the directions tangent to the descending node lines is quite similar and will be omitted.