The limiting curve

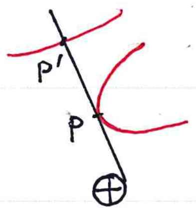
We know that in D_2 the number of solutions of (I) is even (i.e. 2 sols)

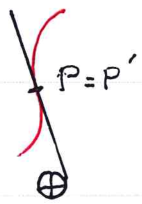
Our aim is to divide $\mathcal{D}$ into points with the same # of sols (1 sol or 3 sols)

DEF $\sigma = \mathcal{S} \cap \mathcal{D}$ portion of the singular curve in $\mathcal{D}$

In $\mathcal{D}$ the sols of (I) are 1 or 3.

A point in σ has multiplicity ≥ 2, therefore we have 2 possibilities:





DEF (residual points) Fix $\gamma \neq 1$ and let $(\bar{\xi}, \bar{\Psi})$ correspond to $P \in \sigma$.

If $F_{\xi\xi}(\mathcal{C}, \bar{\xi}, \bar{\Psi}) \neq 0$ we call residual point related to P the point P' lying on the same obs line and the same level curve of $\mathcal{C}^{(\gamma)}$ as P.

If $F_{\xi\xi}(\mathcal{C}, \bar{\xi}, \bar{\Psi}) = 0$ we call P a self-residual point.

HINT We agree that $(x,y) = (q,0)$ is the residual point related to $(x,y) = (q,0)$

DEF Let $\gamma \neq 1$. The limiting curve $\mathcal{L}$ is the set composed by all the residual points related to the points in σ.

REMARK The symmetry of σ and of the level curves of $\mathcal{C}^{(\gamma)}$ implies the symmetry of $\mathcal{L}$.

LEMMA (separating property)

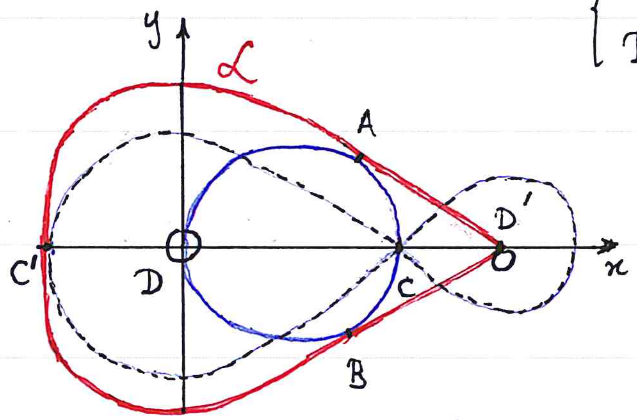
For $\gamma \neq 1$ the limiting curve $\mathcal{L}$ is a connected, simple, continuous curve, separating $\mathcal{D}$ into two connected regions $\mathcal{D}_1, \mathcal{D}_3$:

$\mathcal{D}_3$ contains the whole portion σ of the singular curve.

If $\gamma < 1$ then $\mathcal{L}$ is a closed curve; if $\gamma > 1$ it is unbounded.

SKETCH of the PROOF $\quad \gamma < 1 \quad$ Take $C, D \in \sigma$ such that

$$\begin{cases} C = \text{saddle point with } x > 0 \\ D = \text{center of the Sun} \end{cases}$$



Take the related residual points C', D'

D' = observer position

Take A, B, unique points with mult. 3

Consider the arcs $\widehat{CA}$, $\widehat{AD}$, that are portions of σ, with $y \geq 0$, connecting C to A, A to D.

By the continuity of the roots of (I), $\exists$ two continuous curves $\widehat{C'A}$, $\widehat{AD'}$ connecting C' to A, A to D', composed by the residual points related to the points of $\widehat{CA}$, $\widehat{AD}$ respectively.

$\widehat{C'A}$ and $\widehat{AD'}$ are portions of $\mathcal{L}$ with $y \geq 0$.

We can make a similar construction in the half-plane $y \leq 0$ and we obtain $\widehat{C'B}$, $\widehat{BD'}$.

We obtain the whole $\mathcal{L}$ by joining these 4 curves.

HINT σ connected, closed, simple, continuous

$\Longrightarrow$ $\mathcal{L}$ has the same properties.

$\mathcal{L}$ divides the reference plane into 2 c.c. : $\mathcal{D}_1$ bounded, $\mathcal{D}_3$ unbounded.

$\mathcal{D}_3$ contains σ inside, in fact there cannot be intersections between $\mathcal{L}$ and σ except for A, B, and C', D' are external to σ.

PROPOSITION (transversality)

The level curves of $\mathcal{C}^{(\gamma)}_{(x,y)}$ cross $\mathcal{L}$ transversely, except for at most the two self-residual points and the points where $\mathcal{L}$ meets the x-axis.

SKETCH of the PROOF

If $(\mathcal{C}, \tilde{\varsigma}, \tilde{\psi})$ is s.t. $F(\mathcal{C}, \tilde{\varsigma}, \tilde{\psi}) = F_\varsigma(\mathcal{C}, \tilde{\varsigma}, \tilde{\psi}) = 0$

and
$$\det \begin{bmatrix} F_{\mathcal{C}}(\mathcal{C}, \tilde{\varsigma}, \tilde{\psi}) & 0\ (= F_\varsigma) \\ F_{\varsigma\mathcal{C}}(\mathcal{C}, \tilde{\varsigma}, \tilde{\psi}) & F_{\varsigma\varsigma}(\mathcal{C}, \tilde{\varsigma}, \tilde{\psi}) \end{bmatrix} = \frac{\tilde{\varsigma}}{9} F_{\varsigma\varsigma}(\mathcal{C}, \tilde{\varsigma}, \tilde{\psi}) \neq 0$$

then $\exists\ \psi \longmapsto (\mathcal{C}_S(\psi), \varsigma_S(\psi))$ defined in a neighborhood of $\tilde{\psi}$

such that $\mathcal{C}_S(\tilde{\psi}) = \mathcal{C}$, $\varsigma_S(\tilde{\psi}) = \tilde{\varsigma}$

and $F(\mathcal{C}_S(\psi), \varsigma_S(\psi), \psi) = F_\varsigma(\mathcal{C}_S(\psi), \varsigma_S(\psi), \psi) = 0$.

In particular, $\psi \longmapsto (\varsigma_S(\psi), \psi)$

is a local parametrization of S, in fact, for each given ψ we meet the level curve $\mathcal{C}^{(\gamma)}(x,y) = \mathcal{C}_S(\psi)$ in a point of S.

If $(\bar{\varsigma}, \bar{\psi})$ corresponds to $P' \in \mathcal{L}$ that is neither self-residual, nor with $y=0$, then

$$F_\psi\left(\mathcal{C}_{\mathcal{S}}(\bar{\psi}), \bar{\varsigma}, \bar{\psi}\right) = 3\,\frac{q^4 \bar{\varsigma}}{\pi^5} \sin\bar{\psi} \neq 0$$

and relation $\quad F(\mathcal{C}_{\mathcal{S}}(\psi), \varsigma, \psi) = 0 \quad$ implicitely defines a map $\psi \longmapsto \varsigma_{\mathcal{L}}(\psi)$ in a nbhd of $\tilde{\psi}$ such that

$$\varsigma_{\mathcal{L}}(\bar{\psi}) = \bar{\varsigma} \qquad \text{and} \qquad F\left(\mathcal{C}_{\mathcal{S}}(\psi), \varsigma_{\mathcal{L}}(\psi), \psi\right) = 0.$$

The map $\quad \psi \longmapsto \left(\varsigma_{\mathcal{L}}(\psi), \psi\right)$ is a local perametrization of $\mathcal{L}$.

Moreover, $\quad \varsigma_{\mathcal{L}}(\psi) \neq \varsigma_{\mathcal{S}}(\psi)$.

We can compute $\varsigma'_{\mathcal{L}}(\psi)$ and $\varsigma'_{\mathcal{C}}(\psi)$, where the map

$$\psi \longmapsto \left(\varsigma_{\mathcal{C}}(\psi), \psi\right) \text{ is a local perametrization of } \mathcal{C}^{(\gamma)} = \mathcal{C}_{\mathcal{S}}(\bar{\psi}),$$

that exists because the map $\varsigma_{\mathcal{C}}(\psi)$ is implicitely defined by

$F\left(\mathcal{C}_{\mathcal{S}}(\bar{\psi}), \varsigma, \psi\right) = 0$ and $F_\varsigma\left(\mathcal{C}_{\mathcal{S}}(\bar{\psi}), \bar{\varsigma}, \bar{\psi}\right) \neq 0$ because P' is not self-residual.

In this way we can show that

$$\boxed{\varsigma'_{\mathcal{L}}(\bar{\psi}) \neq \varsigma'_{\mathcal{C}}(\bar{\psi})}$$

for the points P' that are neither self-residual, nor with $y=0$.

$\blacksquare$

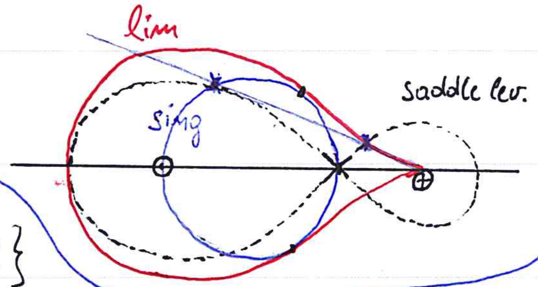
$\boxed{\text{THEOREM}}$ (limiting property)

For $\gamma \neq 1$ the limiting curve $\mathcal{L}$ divides $\mathcal{D}$ into 2 c.c. $\mathcal{D}_1, \mathcal{D}_3$:
the points of $\mathcal{D}_1$ are the unique sols of the corresponding (I) problem;
the points of $\mathcal{D}_3$ are sols of (I) problems with 3 sols, and the other 2 sols also lie in $\mathcal{D}_3$.

SKETCH of the PROOF

$\boxed{\gamma \leq 0}$

Have in mind this case!



Take $\gamma \neq 1$.
Choose $\mathcal{C}$ s.t. $\mathcal{M}_{\mathcal{C}} = \{(x,y) : G^{(\gamma)}(x,y) = \mathcal{C}\}$
is $\neq \emptyset$. Then $\mathcal{M}_{\mathcal{C}}$ is either contained in $\mathcal{D}$ or lying outside $\mathcal{D}$.

~~For $\gamma \leq 0$, $\mathcal{D} = \mathbb{R}^2 \setminus \{(q,0)\}$. In general~~ consider the case $\mathcal{M}_{\mathcal{C}}$ inside $\mathcal{D}$.

There is a unique solution at opposition if

$$\begin{cases} \gamma \leq 0 & \mathcal{C} < 0 \\ 0 < \gamma < 1 & \mathcal{C} \leq 0 \\ \gamma > 1 & \mathcal{C} > 0 \end{cases}$$

therefore, for each $\gamma \neq 1$, there is only 1 sol at opposition in $\mathcal{D}$.
This solution lies in $\mathcal{D}_1$.

If $\mathcal{M}_{\mathcal{C}} \cap \mathcal{S} = \emptyset$, then $\mathcal{M}_{\mathcal{C}} \subset \mathcal{D}_1$, in fact
$\mathcal{S} \subset \mathcal{D}_3$ and $\boxed{\mathcal{M}_{\mathcal{C}} \cap \mathcal{L} \neq \emptyset \iff \mathcal{M}_{\mathcal{C}} \cap \mathcal{S} \neq \emptyset}$.

Therefore, a necessary condition so that M_φ enters into D_3 for increasing values of ψ is that $\boxed{M_\varphi \cap \mathfrak{S} \neq \emptyset}$ (*)

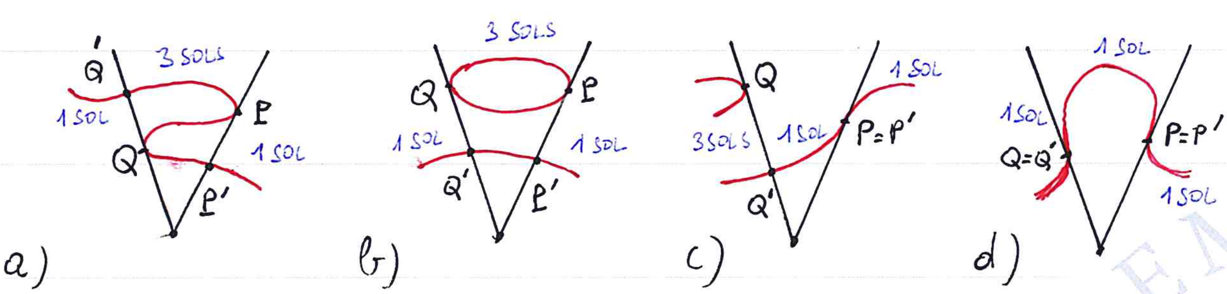
Assume (*) holds.

- We exclude that M_φ passes from D_1 to D_3 in a point $P' \in \mathcal{L} \cap \{y=0\}$ using the symmetry of M_φ.

- Let φ be a regular value of $G^{(\gamma)}$ and $\psi_P \in (0,\pi)$ the smallest value of $\psi > 0$ s.t. the observation line defined by ψ_P meets $M_\varphi \cap \mathfrak{S}$; we call P the intersection point and P' the residual point relative to P.

- By symmetry, $\exists$ at least another point $Q \in M_\varphi \cap \mathfrak{S}$. We choose Q such that, denoting by ψ_Q the related angle, in the sector $\{(\rho,\psi) : \psi_P < \psi < \psi_Q\}$ there are not points of $M_\varphi \cap \mathfrak{S}$. Let Q' be the residual point relative to Q.

<u>POSSIBLE CASES</u>



<u>REMARK</u> We use the fact that M_φ is a 1-D compact manifold without boundary, and the intersections of M_φ with a given direction line cannot be more than 3.

- In cases a), b), By the transversality property, M_φ crosses $\mathcal{L}$ and passes from D_1 to D_3 in P'; then M_φ comes back to D_1 through Q'.

- case a) : the curve $\widehat{P'Q'}$ is contained in D_3 and passes through P, Q
- case b) : $\widehat{P'Q'}$ is in D_3 but P, Q belong to another c.c. ; however, $P, Q \in D_3$ too because $\mathcal{O} \subset D_3$, therefore also the loop joining P, Q is in D_3.

- cases c), d) : M_φ cannot cross $\mathcal{L}$, otherwise there would be a portion of M_φ in D_3 whose points would correspond to an (I) problem with only 1 sol. Then it would be like that in a whole ngbhd, which is a contradiction.
- we use a similar density argument for the stationary values of $C^{(\gamma)}$.