On the geometry of two Keplerian orbits with a common focus: results and open problems

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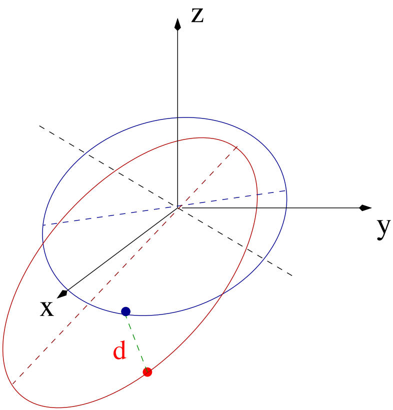
The Keplerian distance function

Let (E_j, v_j), $j = 1, 2$ be the orbital elements of two celestial bodies on Keplerian orbits with a common focus:
E_j represents the trajectory of a body,
v_j is a parameter along it.

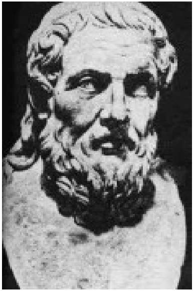
Set $V = (v_1, v_2)$. For a given two-orbit configuration $\mathcal{E} = (E_1, E_2)$, we introduce the Keplerian distance function

$$\mathbb{T}^2 \ni V \mapsto d(\mathcal{E}, V) = |\mathcal{X}_1 - \mathcal{X}_2|.$$

We are interested in the local minimum points of d and in particular in the absolute minimum d_{min}, called orbit distance, or MOID.



Is there still something that we do not know about distance of points on conic sections?



ἐθεώρουν σε σπεύδοντα μετασχεῖν τῶν πεπραγμένων ἡμῖν κωνικῶν (1)

(Apollonius of Perga, *Conics*, Book I)

(1) I observed you were quite eager to be kept informed of the work I was doing in conics.

Critical points of d^2

- The local minimum points of d can be found by computing all the critical points of d^2 (so that crossing points are also critical).

  How many are they?

- Apart from the case of two concentric coplanar circles, or two overlapping ellipses, d^2 has finitely many critical points...

  ... but they can be more than what we expect!

- There exist configurations with 12 critical points, and 4 local minima of d^2.
  This is thought to be the maximum possible, but a proof is not known yet.

Computation of the local minima

There are several papers in the literature about the computation of the MOID, e.g. Sitarski (1968), Dybczyński et al. (1986) and more recently Hedo et al. (2018), Baluev and Mikryukov (2019).

The following papers introduced algebraic methods to compute all the critical points of d^2:

- Kholshevnikov and Vassiliev, CMDA (1999), with *Gröbner bases*;
- Gronchi, SJSC (2002), CMDA (2005), with *resultant theory*.

They are based on a polynomial formulation of the problem, which gives some advantages.

Algebraic formulation

The critical points equations is

$$\nabla_V d^2(\mathcal{E}, V) = 0. \tag{1}$$

By the coordinate change

$$s = \tan(v_1/2)\,; \qquad t = \tan(v_2/2)$$

we obtain from (1) a system of 2 polynomials in 2 unknowns

$$\begin{cases} p(s,t) &= f_4(t)\,s^4 + f_3(t)\,s^3 + f_2(t)\,s^2 + f_1(t)\,s + f_0(t) = 0 \\ q(s,t) &= g_2(t)\,s^2 + g_1(t)\,s + g_0(t) = 0 \end{cases}$$

each with total degree 6; precisely $p(s,t)$ has degree 4 in s and degree 2 in t, while $q(s,t)$ has degree 2 in s and degree 4 in t.

Computation of the solutions

From elimination theory we know that p and q have a common solution if and only if

$$\mathtt{Res}(p,q,s)(t) = \det \mathtt{S}(t) = 0\,;$$

where

$$\mathtt{S}(t) = \begin{pmatrix} f_4 & 0 & g_2 & 0 & 0 & 0 \\ f_3 & f_4 & g_1 & g_2 & 0 & 0 \\ 0 & f_3 & g_0 & g_1 & g_2 & 0 \\ f_1 & 0 & 0 & g_0 & g_1 & g_2 \\ f_0 & f_1 & 0 & 0 & g_0 & g_1 \\ 0 & f_0 & 0 & 0 & 0 & g_0 \end{pmatrix}.$$

$R(t) = \mathtt{Res}(p,q,s)(t)$ is a polynomial with degree 20; it has a factor $(1+t^2)^2$ giving 4 imaginary roots.

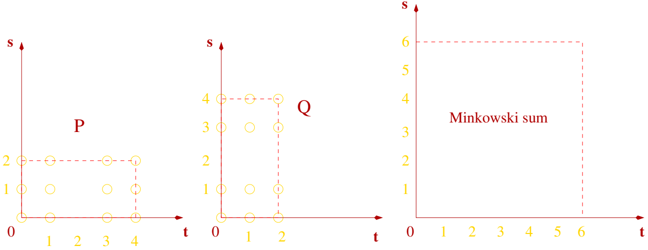
Maximal number of critical points

For the case of two bounded orbits we can prove the following:

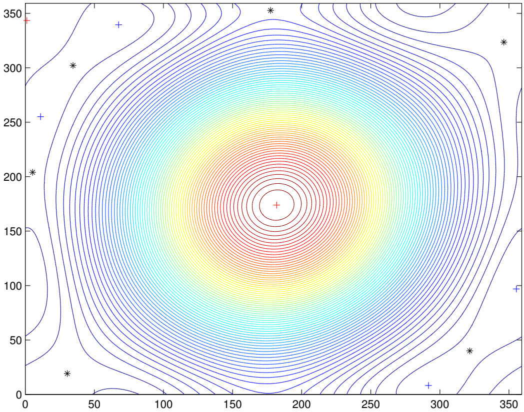
If there are finitely many critical points of d^2, then they are at most 16 in the general case and at most 12 if one orbit is circular.

The proof uses Bernstein's theorem, which says that an upper bound for the solutions in $\mathbb{C}^2$ is given by the mixed area of Newton's polygons of p e q:

$$\text{Mixed Area}(P, Q) = \text{Area}(P + Q) - \text{Area}(P) - \text{Area}(Q)$$



Example with 12 critical points, 4 minima



level curves of d^2, plane of the eccentric anomalies

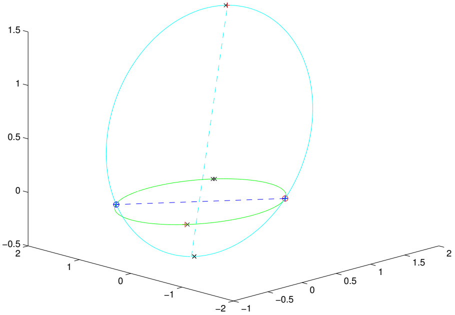
$$\begin{cases} + = max \\ + = min \\ * = saddle \end{cases}$$

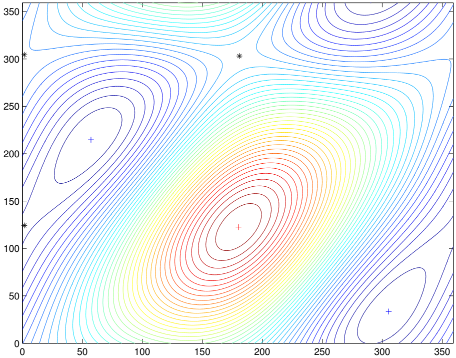
By Morse theory

$$\#(max) + \#(min) = \#(saddles)$$

Q	e_1	q	e_2	i_M	$\omega_M^{(1)}$	$\omega_M^{(2)}$
0.585	0.415	0.462	0.615	80.0°	8.0°	176.0°

Near-Earth asteroid 1999AN$_{10}$ (data from NEODyS, August 7, 2006)





u_2	u_1	distance	type
57.20790383	214.71613368	0.00020038	MINIMUM
305.39545534	33.46086711	0.00443519	MINIMUM
1.58119802	124.24652678	0.66236738	SADDLE
1.65988384	304.47441317	1.53697717	SADDLE
180.95186706	303.07116589	1.64610573	SADDLE
180.19171642	122.55726209	3.12113151	MAXIMUM

http://newton.spacedys.com/neodys

Conjecture

The following table gives a conjecture on the maximum number of critical points in case of bounded orbits:

$e_1 \neq 0$	$e_2 \neq 0$	12 points
$e_1 \neq 0$	$e_2 = 0$	10 points
$e_1 = 0$	$e_2 \neq 0$	10 points
$e_1 = 0$	$e_2 = 0$	8 points

This is still an open problem!

The local minimum distance maps

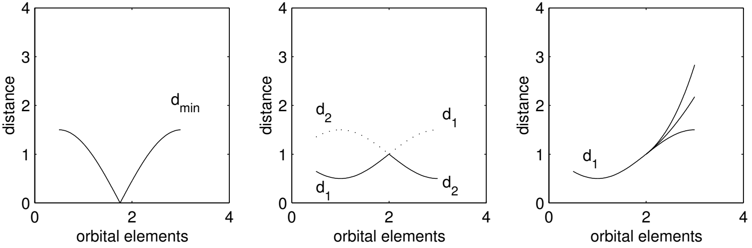
Gronchi and Tommei, DCDS-B (2007)

Let $V_h = V_h(\mathcal{E})$ be a local minimum point of $V \mapsto d^2(\mathcal{E}, V)$.
Consider the maps

$$\mathcal{E} \mapsto d_h(\mathcal{E}) = d(\mathcal{E}, V_h) \,,$$
$$\mathcal{E} \mapsto d_{min}(\mathcal{E}) = \min_h d_h(\mathcal{E}) \;.$$

The map $\mathcal{E} \mapsto d_{min}(\mathcal{E})$ gives the MOID.

Singularities of d_h and d_{min}



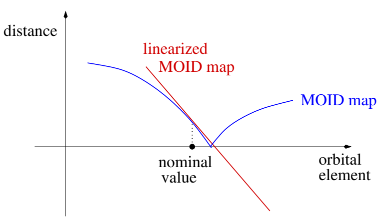
(i) d_h and d_{min} are not differentiable where they vanish;

(ii) two local minima can exchange their role as absolute minimum thus d_{min} loses its regularity without vanishing;

(iii) when a bifurcation occurs the definition of the maps d_h may become ambiguous after the bifurcation point.

Problems in computing the uncertainty of d_{min}

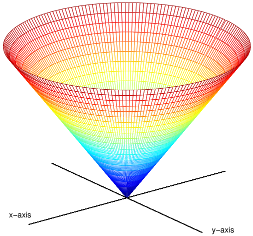
Given a nominal orbit configuration $\bar{\mathcal{E}}$, with its covariance matrix $\Gamma_{\bar{\mathcal{E}}}$, the covariance propagation of a function of $\mathcal{E}$, like d_{min}, is based on a linearization of the function near $\bar{\mathcal{E}}$.

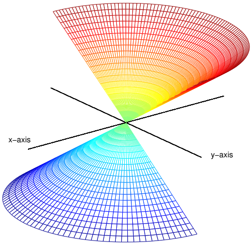


Remark: $d_{min}(\mathcal{E})$ is not smooth where it vanishes, thus usually the linearizzation of d_{min} in a neighborhood of the nominal orbit is not a good approximation (see fig. on the left)

Problem: can we give a sign to $d_{min}(\mathcal{E})$ so that its linearization becomes meaningful (see fig. on the right)?

Smoothing through change of sign



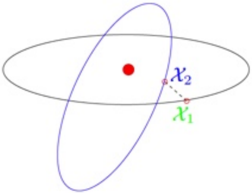


Toy problem:

$$f(x,y) = \sqrt{x^2+y^2} \qquad \tilde{f}(x,y) = \begin{cases} -f(x,y) & \text{for } x>0 \\ f(x,y) & \text{for } x<0 \end{cases}$$

Can we smooth the maps $d_h(\mathcal{E})$, $d_{min}(\mathcal{E})$
through a change of sign?

Local smoothing of d_h at a crossing singularity



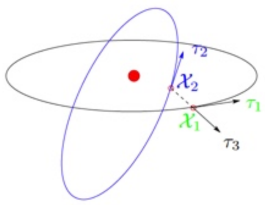
Smoothing d_h, the procedure for d_{min} is the same.

- Consider the points on the two orbits

$$\mathcal{X}_1^{(h)} = \mathcal{X}_1(E_1, v_1^{(h)}) \, ; \qquad \mathcal{X}_2^{(h)} = \mathcal{X}_2(E_2, v_2^{(h)}) \, .$$

corresponding to the local minimum point $V_h = (v_1^{(h)}, v_2^{(h)})$ of d^2;

Local smoothing of d_h at a crossing singularity



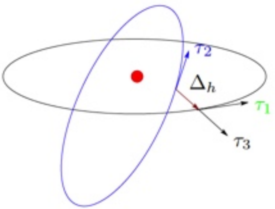
- introduce the tangent vectors to the trajectories E_1, E_2 at these points:

$$\tau_1 = \frac{\partial \mathcal{X}_1}{\partial v_1}(E_1, v_1^{(h)}) \,, \qquad \tau_2 = \frac{\partial \mathcal{X}_2}{\partial v_2}(E_2, v_2^{(h)}) \,,$$

and their cross product

$$\tau_3 = \tau_1 \times \tau_2;$$

Local smoothing of d_h at a crossing singularity



- define also

$$\Delta = \mathcal{X}_1 - \mathcal{X}_2, \qquad \Delta_h = \mathcal{X}_1^{(h)} - \mathcal{X}_2^{(h)}.$$

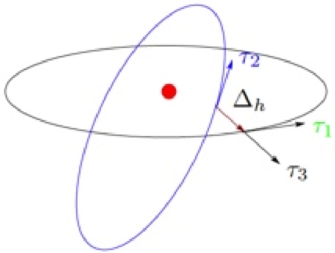
The vector Δ_h joins the points attaining a local minimum of d^2 and $|\Delta_h| = d_h$.

Note that $\Delta_h \times \tau_3 = 0$.

Smoothing the crossing singularity

smoothing rule:

$$\tilde{d}_h = \text{sign}(\tau_3 \cdot \Delta_h) d_h$$



$\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ is an analytic map in a neighborhood of most crossing configurations.

Uncertainty of the MOID

For a given orbit $\bar{\mathcal{E}}$, with its covariance matrix $\Gamma_{\bar{\mathcal{E}}}$, the covariance propagation formula

$$\Gamma_{\tilde{d}_{min}(\bar{\mathcal{E}})} = \left[\frac{\partial \tilde{d}_{min}}{\partial \mathcal{E}}(\bar{\mathcal{E}})\right] \Gamma_{\bar{\mathcal{E}}} \left[\frac{\partial \tilde{d}_{min}}{\partial \mathcal{E}}(\bar{\mathcal{E}})\right]^t$$

allows us to compute the covariance of the regularized MOID.

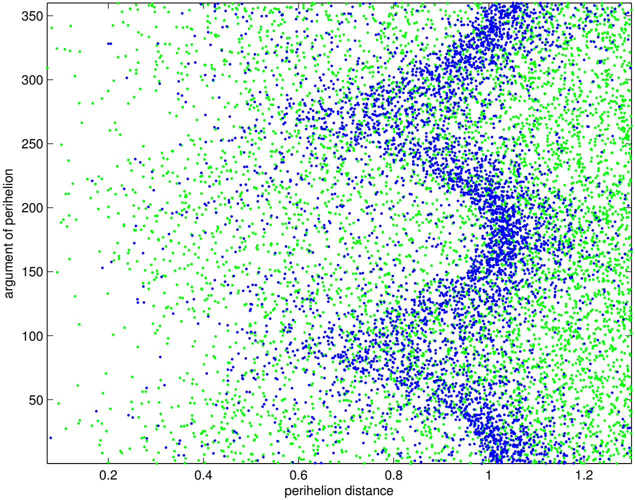
Using the orbit distance

to detect observational biases

in the discovery of NEAs

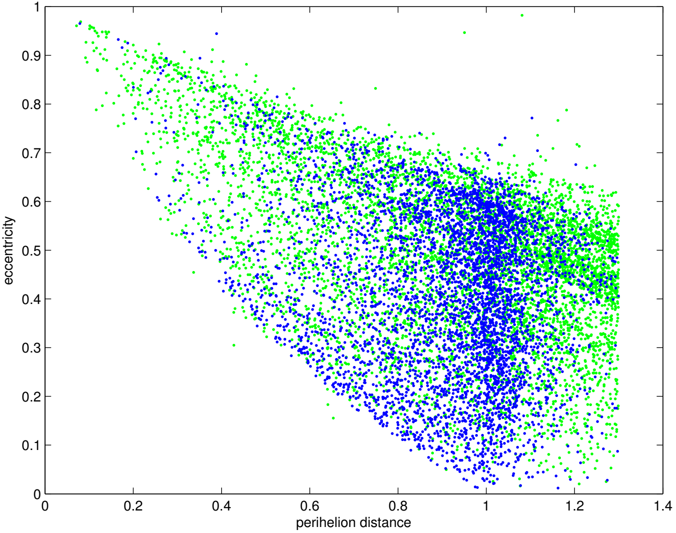
(q, ω) plot of all the known NEAs

Gronchi and Valsecchi, MNRAS (2014)



The blue dots are NEAs with $H > 22$.

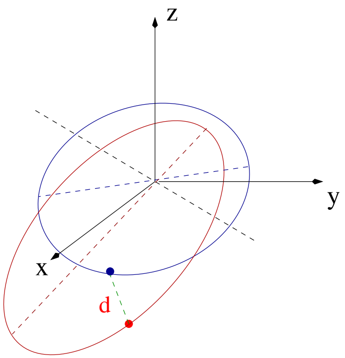
(q, e) plot of all the known NEAs



The blue dots are NEAs with $H > 22$.

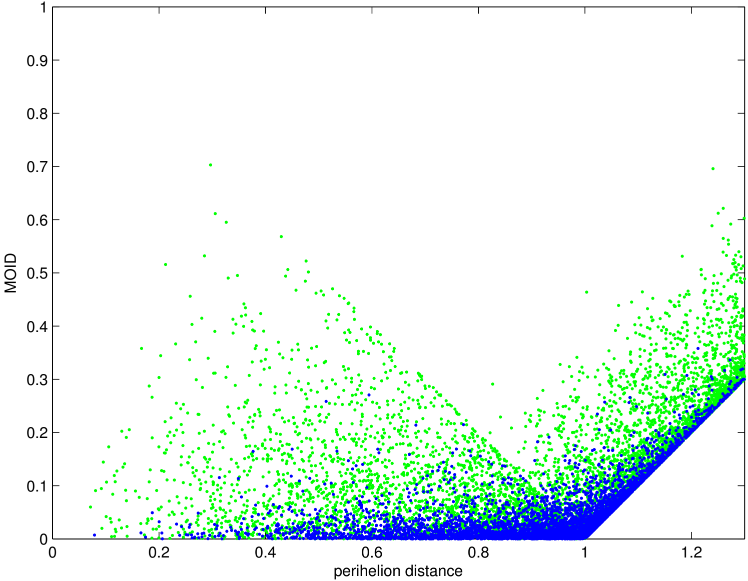
Geometry of ground-based observations

Consider the orbits of the Earth and of a NEA. We denote by d_{min} the MOID between the trajectories of these two bodies.



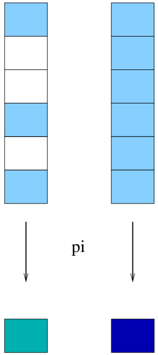
- Most NEAs with a small value of d_{min} are detected, sooner or later;
- small NEAs with a large value of d_{min} are likely to be unobserved.

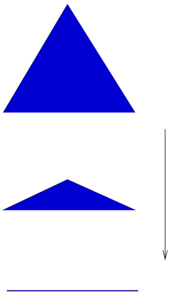
(q, d_{min}) plot of all the known NEAs



Projections

In all the previous plots we see projections on a plane of data from an N-dimensional space, with $N > 2$.





'Nothing was visible, nor could be visible, to us, except Straight Lines' (E. A. Abbot), *Flatland*.

The near-Earth asteroid class

We define the NEA class $\mathcal{N}$ as the set of cometary orbital elements $(q, e, I, \Omega, \omega)$ such that

$$q \in [0, q_{max}], \quad e \in [0, 1], \quad I \in [0, \pi], \quad \Omega \in [0, 2\pi], \quad \omega \in [0, 2\pi].$$

Here q is the perihelion distance and $q_{max} = 1.3$ au.

We use

$$q' = 1, \quad e' = 0, \quad I' = 0, \quad \Omega' = 0, \quad \omega' = 0$$

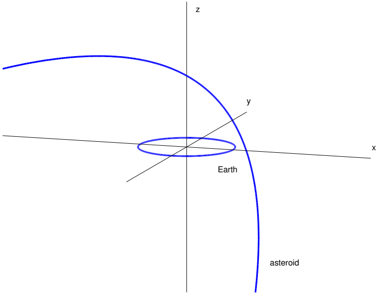
for the elements of the Earth.

Possible values of d_{min} as function of (q, ω)

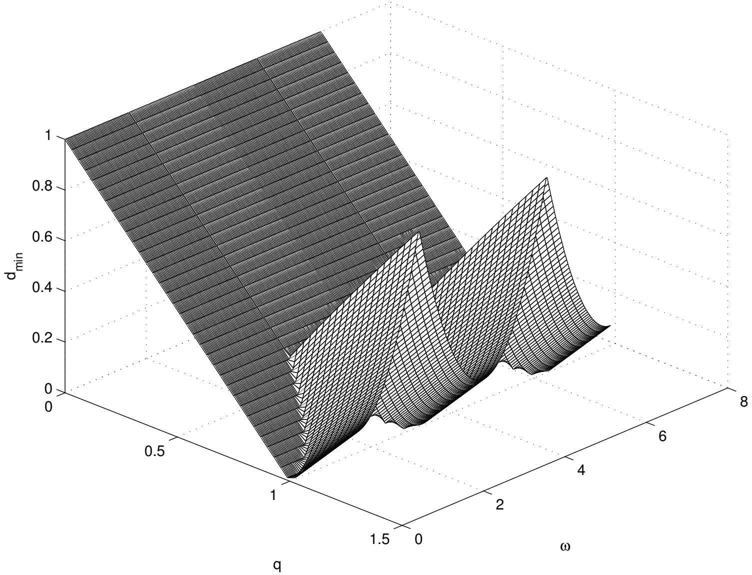
Let $\mathcal{D}_1 = \{(e, I) : 0 \leq e \leq 1, 0 \leq I \leq \pi\}$. For each choice of (q, ω), with $0 < q \leq q_{max}, 0 \leq \omega \leq 2\pi$, we have

$$\max_{(e,I)\in\mathcal{D}_1} d_{min} = \max\{q' - q, \delta(q, \omega)\}$$

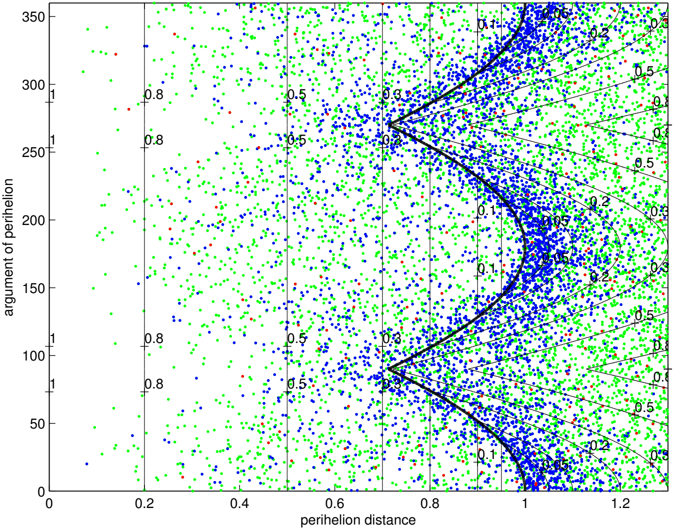
where $\delta(q, \omega)$ is the distance between the orbit of the Earth and a parabolic orbit $(e = 1)$ with $I = \pi/2$.



Maximal orbit distance as function of (q, ω)

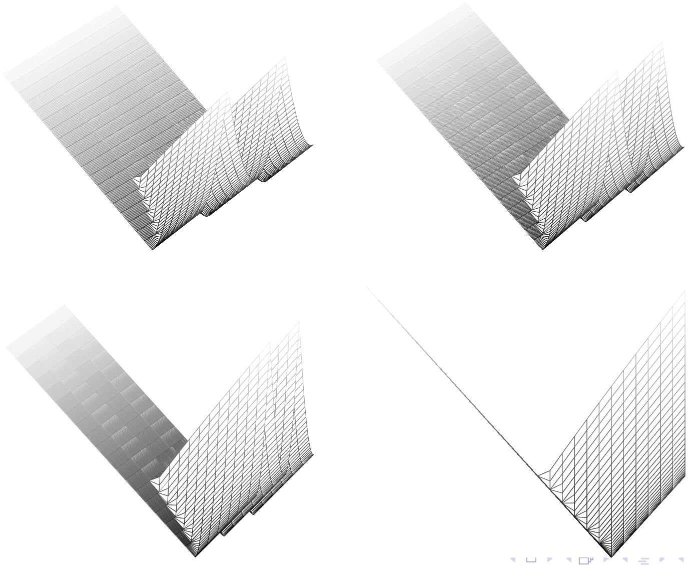


Distribution of NEAs in the plane (q, ω)

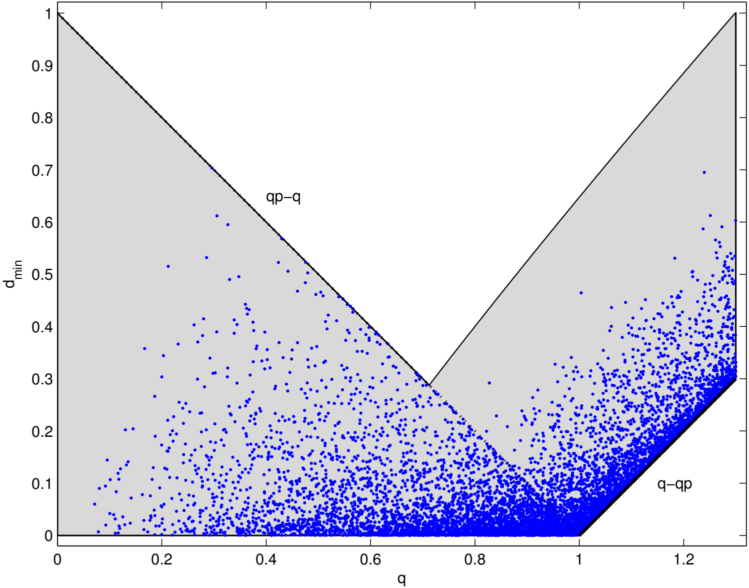


Blue dots are NEAs with $H > 22$, red dots with $H < 16$.

Distribution of NEAs in the plane (q, d_{min})



Distribution of NEAs in the plane (q, d_{min})

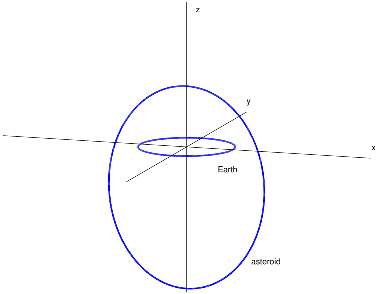


Possible values of d_{min} as function of (q, e)

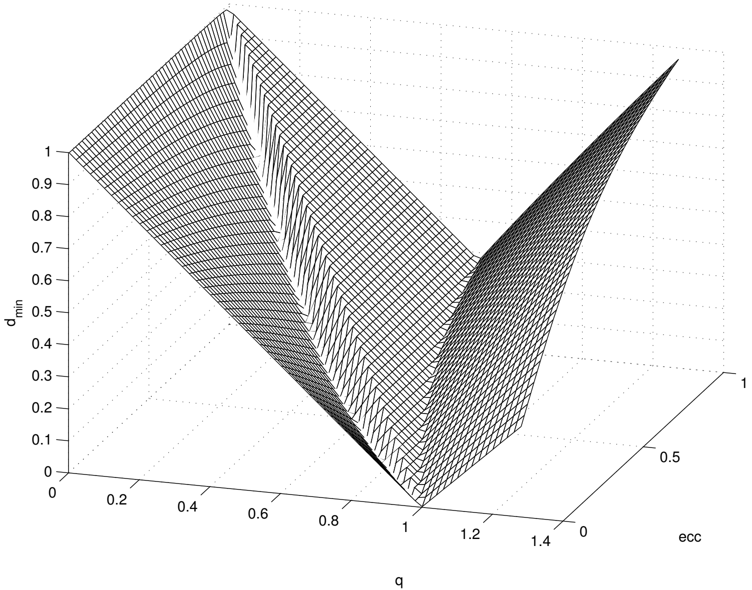
Let $\mathcal{D}_3 = \{(I, \omega) : 0 \le I \le \pi, 0 \le \omega \le 2\pi\}$. For each choice of (q, e) with $0 < q \le q_{max}, 0 \le e \le 1$ we have

$$\max_{(I,\omega)\in\mathcal{D}_3} d_{min} = \max\{\min\{q' - q, Q - q'\}, \delta(q, e)\}$$

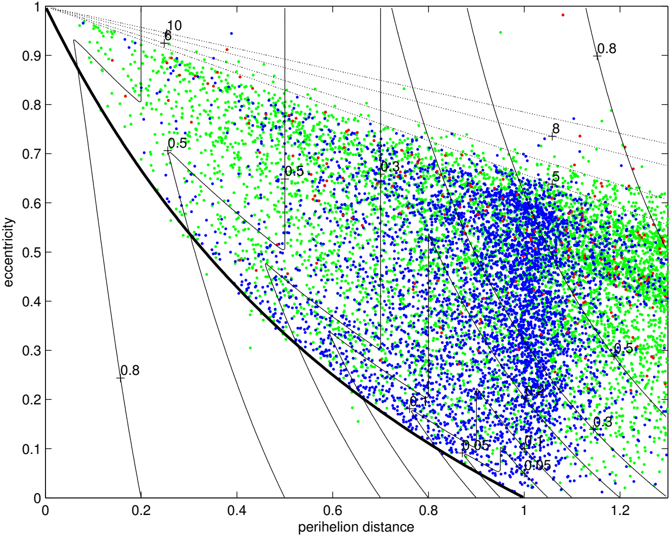
where $Q = q(1+e)/(1-e)$ is the aphelion distance and $\delta(q, e)$ is the distance between the orbit of the Earth and an orbit with $I = \pi/2, \omega = \pi/2$.



Maximal orbit distance as function of (q, e)



Distribution of NEAs in the plane (q, e)



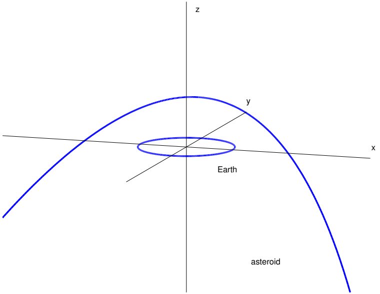
Blue dots are NEAs with $H > 22$, red dots with $H < 16$.

Possible values of d_{min} as function of (q, I)

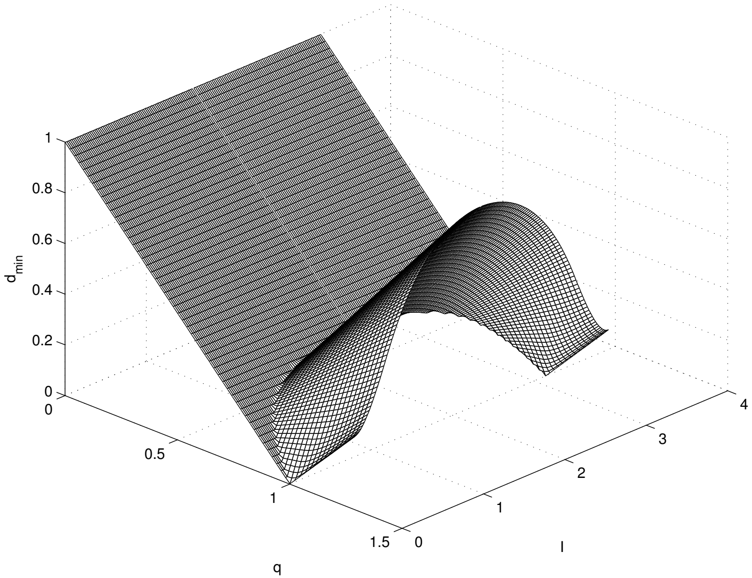
Let $\mathcal{D}_5 = \{(e, \omega) : 0 \leq e \leq 1, 0 \leq \omega \leq 2\pi\}$. For each choice of (q, I) with $0 < q \leq q_{max}, 0 \leq I \leq \pi$ we have

$$\max_{(e,\omega)\in\mathcal{D}_5} d_{min} = \max\{q' - q, \delta_I(q, I)\}$$

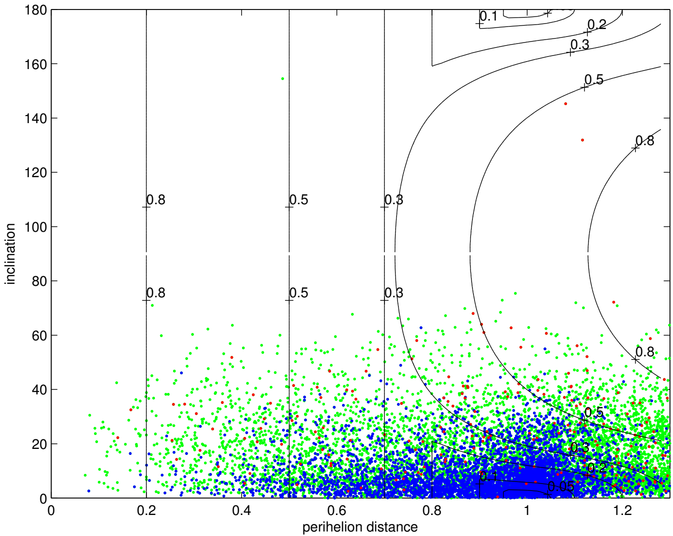
where $\delta_I(q, I)$ is the distance between the orbit of the Earth and an orbit with $e = 1, \omega = \pi/2$.



Maximal orbit distance as function of (q, I)

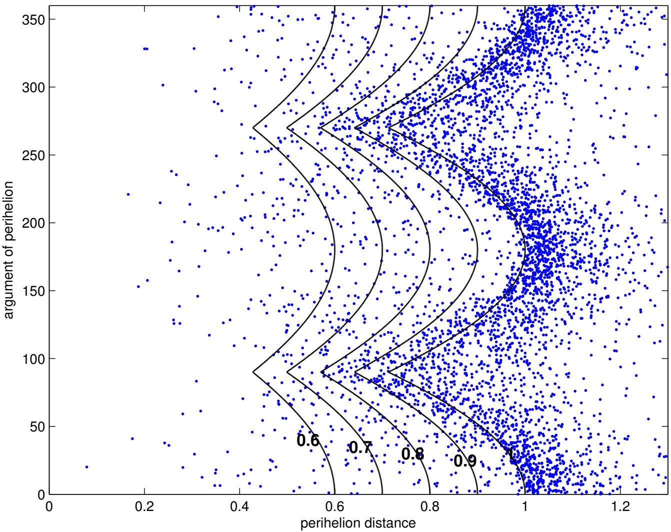


Distribution of NEAs in the plane (q, I)



Blue dots are NEAs with $H > 22$, red dots with $H < 16$.

Observing from inner-Earth circular orbits...

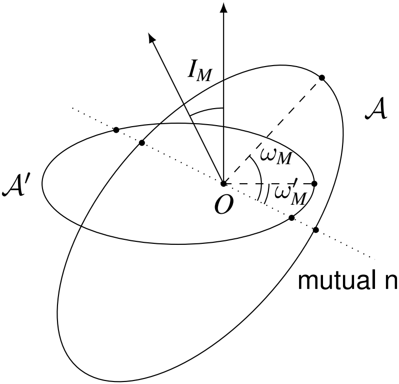


The eccentric case $e' \in (0,1)$

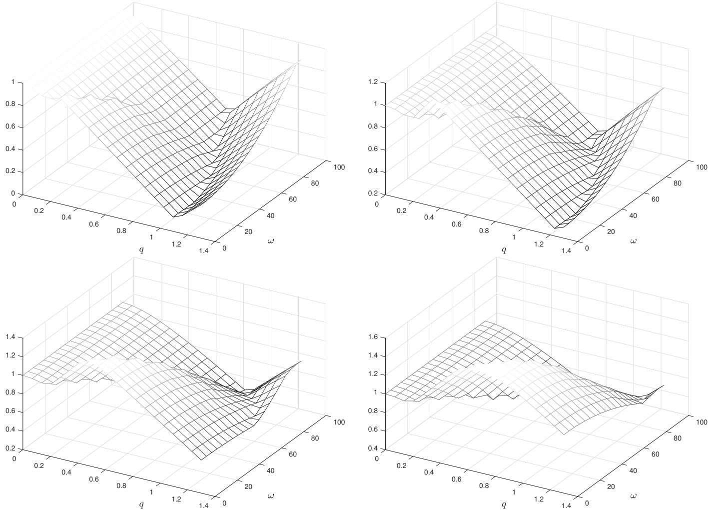
Problem: generalize this theory to the eccentric case $e' \in (0,1)$.

Gronchi and Niederman, CMDA (2020)

Mutual orbital elements: $\mathcal{E}_M = (q, e, q', e', I_M, \omega_M, \omega'_M)$



The eccentric case $e' \in (0,1)$



Graphic of $\max_{\widetilde{\mathcal{D}}_1} d_{\min}(q,\omega)$, with $\widetilde{\mathcal{D}}_1 = \{(e, I, \omega') : 0 \le e \le 1,\ 0 \le I \le \pi/2,\ 0 \le \omega' \le 2\pi\}$.
$e' = 0.1$ (top left), $e' = 0.2$ (top right), $e' = 0.3$ (bottom left), $e' = 0.4$ (bottom right). Here we set $q' = 1$.

The nodal distance

Let

$$r_+ = \frac{q(1+e)}{1+e\cos\omega}, \qquad r_- = \frac{q(1+e)}{1-e\cos\omega},$$
$$r'_+ = \frac{q'(1+e')}{1+e'\cos\omega'}, \qquad r'_- = \frac{q'(1+e')}{1-e'\cos\omega'}$$

and introduce the ascending and descending nodal distances:

$$d^+_{\rm nod} = r'_+ - r_+, \qquad d^-_{\rm nod} = r'_- - r_-.$$

The (minimal) nodal distance $\delta_{\rm nod}$ is the minimum between the absolute values of the ascending and descending nodal distances:

$$\delta_{\rm nod} = \min\{|d^+_{\rm nod}|, |d^-_{\rm nod}|\}. \qquad (2)$$

Note that $\delta_{\rm nod}$ does not depend on the mutual inclination I.

The nodal distance

The transformations

$$(\omega, \omega') \mapsto (\pi - \omega, \pi - \omega'), \qquad (\omega, \omega') \mapsto (\pi + \omega, \pi - \omega'),$$
$$(\omega, \omega') \mapsto (2\pi - \omega, \omega'), \qquad (\omega, \omega') \mapsto (\omega, 2\pi - \omega')$$

leave the values of δ_{nod} unchanged.

Therefore we get all the possible values of δ_{nod} even if we restrict ω, ω' to the following ranges:

$$0 \le \omega \le \pi/2, \qquad 0 \le \omega' \le \pi. \tag{3}$$

Linking configurations

We consider the following linking configurations between the trajectories $\mathcal{A}, \mathcal{A}'$:

- internal nodes: the nodes of $\mathcal{A}$ are internal to those of $\mathcal{A}'$, that is $d^+_{\rm nod}, d^-_{\rm nod} > 0$.
- external nodes: the nodes of $\mathcal{A}$ are external to those of $\mathcal{A}'$ (possibly located at infinity), that is $d^+_{\rm nod}, d^-_{\rm nod} < 0$.
- linked orbits: $\mathcal{A}$ and $\mathcal{A}'$ are topologically linked, that is $d^+_{\rm nod} < 0 < d^-_{\rm nod}$, or $d^-_{\rm nod} < 0 < d^+_{\rm nod}$;
- crossing orbits: $\mathcal{A}$ and $\mathcal{A}'$ have at least one point in common, that is $d^+_{\rm nod} d^-_{\rm nod} = 0$.

Linking configurations

Assume $q' > 0$ and $e' \in [0, 1)$ are given. We introduce the functions

$$\begin{aligned}
\delta_{\rm int}(q, e, \omega, \omega') &= \min\{d^+_{\rm nod}, d^-_{\rm nod}\},\\
\delta_{\rm ext}(q, e, \omega, \omega') &= \min\{-d^+_{\rm nod}, -d^-_{\rm nod}\},\\
\delta^{(i)}_{\rm link}(q, e, \omega, \omega') &= \min\{-d^+_{\rm nod}, d^-_{\rm nod}\},\\
\delta^{(ii)}_{\rm link}(q, e, \omega, \omega') &= \min\{d^+_{\rm nod}, -d^-_{\rm nod}\},\\
\delta_{\rm link}(q, e, \omega, \omega') &= \max\{\delta^{(i)}_{\rm link}, \delta^{(ii)}_{\rm link}\}.
\end{aligned}$$

Linking configurations

The linking configurations depend on the sign of these functions as described below. Given the vector (q, e, ω, ω'), we have

a) internal nodes if and only if $\delta_{\text{int}}(q, e, \omega, \omega') > 0$,

b) external nodes if and only if $\delta_{\text{ext}}(q, e, \omega, \omega') > 0$,

c) linked orbits if and only if $\delta_{\text{link}}(q, e, \omega, \omega') > 0$,

d) crossing orbits if and only if $\delta_{\text{int}} = \delta_{\text{ext}} = \delta_{\text{link}} = 0$ at (q, e, ω, ω').

Moreover,

$$\delta_{\text{nod}} = \max\{\delta_{\text{int}}, \delta_{\text{ext}}, \delta_{\text{link}}\}.$$

In the following we assume $q' > 0$ and $e' \in (0, 1)$ are given.

Optimal bounds for δ_{nod} when $e' \in (0, 1)$

Let

$$\mathcal{D}_1 = \{(e, \omega') : 0 \leq e \leq 1, 0 \leq \omega' \leq \pi\},$$
$$\mathcal{D}_2 = \{(q, \omega) : 0 < q \leq q_{\text{max}}, 0 \leq \omega \leq \pi/2\}.$$

For each choice of $(q, \omega) \in \mathcal{D}_2$ we have

$$\min_{(e,\omega')\in\mathcal{D}_1} \delta_{\text{nod}} = \max\left\{0, \ell^{\omega}_{\text{int}}, \ell^{\omega}_{\text{ext}}\right\},$$
$$\max_{(e,\omega')\in\mathcal{D}_1} \delta_{\text{nod}} = \max\left\{u^{\omega}_{\text{int}}, u^{\omega}_{\text{ext}}, u^{\omega}_{\text{link}}\right\},$$

Optimal bounds for δ_{nod} when $e' \in (0, 1)$

where[1]

$$\ell^{\omega}_{\text{int}}(q,\omega) = q' - \frac{2q}{1-\cos\omega}, \qquad \ell^{\omega}_{\text{ext}}(q,\omega) = q - Q', \qquad u^{\omega}_{\text{int}}(q,\omega) = p' - q,$$

$$u^{\omega}_{\text{ext}}(q,\omega) = \min\left\{\frac{2q}{1-\cos\omega} - \frac{p'}{1-\hat{\xi}'_*}, \; \frac{2q}{1+\cos\omega} - q'\right\},$$

with

$$\hat{\xi}'_* = \min\{\xi'_*, e'\}, \qquad \xi'_*(q,\omega) = \frac{4q\cos\omega}{p'\sin^2\omega + \sqrt{p'^2\sin^4\omega + 16q^2\cos^2\omega}},$$

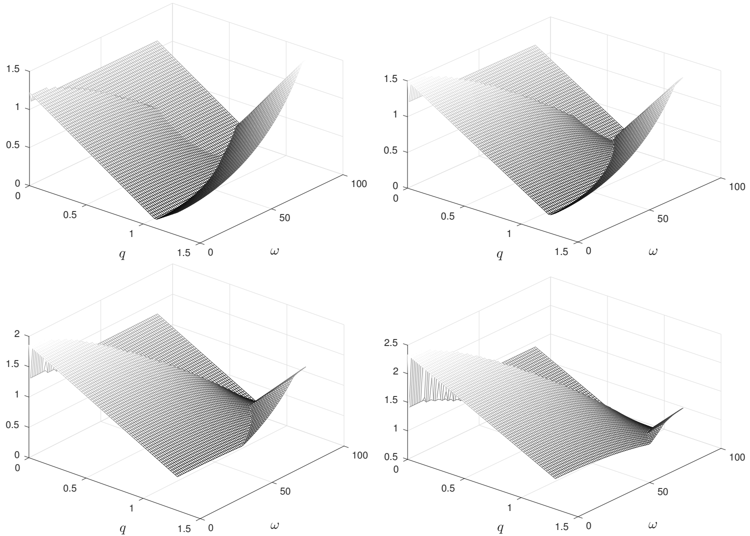
and

$$u^{\omega}_{\text{link}}(q,\omega) = \min\left\{Q' - \frac{q(1+\hat{e}_*)}{1+\hat{e}_*\cos\omega}, \; \frac{2q}{1-\cos\omega} - q'\right\}, \tag{4}$$

with

$$\hat{e}_* = \max\{0, \min\{e_*, 1\}\}, \qquad e_*(q,\omega) = \frac{2(p' - q(1-e'^2))}{q(1-e'^2) + \sqrt{q^2(1-e'^2)^2 + 4p'\cos^2\omega(p' - q(1-e'^2))}}.$$

[1] we admit infinite values for the considered functions



Graphics of $(q, \omega) \mapsto \max_{(e,\omega') \in \mathcal{D}_1} \delta_{\rm nod}(q, \omega)$ for $e' = 0.1$ (top left), $e' = 0.2$ (top right), $e' = 0.3$ (bottom left), $e' = 0.4$ (bottom right). Here we set $q' = 1$.

Optimal bounds for δ_{nod} when $e' = 0$

Set

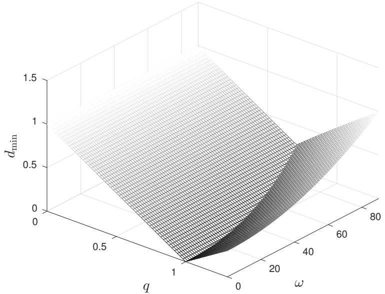
$$\mathcal{D}_1'' = \{e : 0 \leq e \leq 1\}, \quad \mathcal{D}_2 = \{(q,\omega) : 0 < q \leq q_{\text{max}}, 0 \leq \omega \leq \pi/2\}.$$

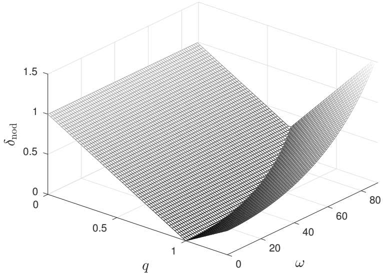
For each choice of $(q,\omega) \in \mathcal{D}_2$ we have

$$\min_{e\in\mathcal{D}_1''} \delta_{\text{nod}} = \max\left\{0, q' - \frac{2q}{1-\cos\omega}, q - q'\right\},$$

$$\max_{e\in\mathcal{D}_1''} \delta_{\text{nod}} = \max\left\{q' - q, \frac{2q}{1+\cos\omega} - q'\right\}.$$

Optimal bounds for $\delta_{\rm nod}$ when $e' = 0$





Left: $\max_{(e,I)\in\mathcal{D}_1'} d_{\rm min}(q,\omega)$, with $\mathcal{D}_1' = \{(e,I) : 0 \le e \le 1, 0 \le I \le \pi/2\}$.
Right: $\max_{e\in\mathcal{D}_1''} \delta_{\rm nod}(q,\omega)$.

The curves γ and β

(G. and Valsecchi 2014) introduced the curve γ, which separates the region in the plane (q, ω) where the trajectories maximizing $d_{\min}$ over $\mathcal{D}_1'$ have $e = 0$, from the region where such trajectories have $e = 1$, that is, γ is the set of points (q, ω) where $q' - q$ and $\delta(q, \omega)$, assume the same values. The equation of γ is

$$\begin{aligned} &2q^4 + 2q'(-5 + 7y)q^3 - 2q'^2(3y + 22)(y - 1)q^2 + \\ &+ q'^3(y^3 + 13y^2 + 9y - 27)q - 2q'^4y^3 = 0, \end{aligned}$$

with $y = \cos\omega$.

The curves γ and β

The equation of the curve analogous to γ for $\delta_{\rm nod}$ (for $e' = 0$) is

$$qy + 3q - 2q'y - 2q' = 0, \qquad (5)$$

that is easily obtained by equating $q' - q$ with $\frac{2q}{1+\cos\omega} - q'$. We denote by β the curve defined by (5). In Figure 1 we plot both curves for comparison.

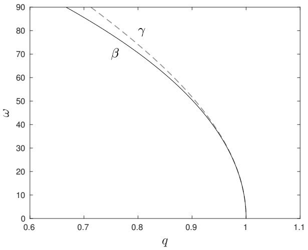


Figure: Comparison between the curves γ and β.

Optimal bounds for $\delta_{\rm nod}$ when $e' \in (0,1)$

Let

$$\mathcal{D}_3 = \{(\omega,\omega') : \omega \in [0,\pi/2], \omega' \in [0,\pi)\},$$
$$\mathcal{D}_4 = \{(q,e) : q \in (0,q_{\rm max}], e \in [0,1]\}.$$

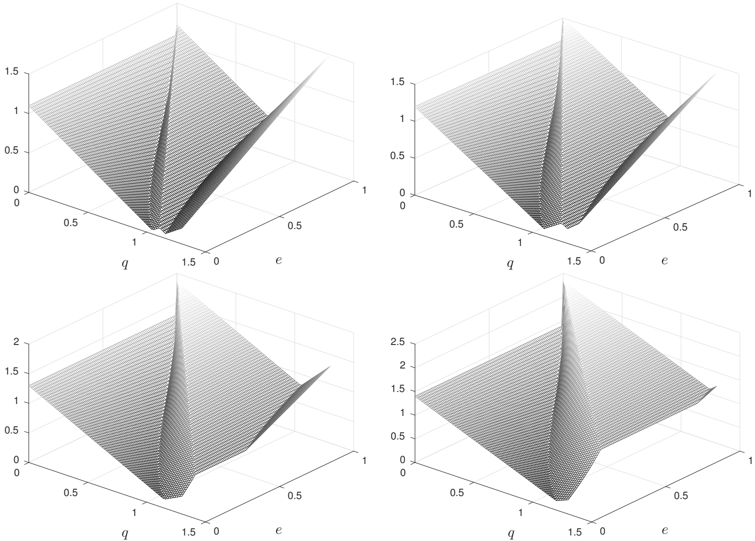
For each choice of $(q,e) \in \mathcal{D}_4$ we have

$$\min_{(\omega,\omega')\in\mathcal{D}_3} \delta_{\rm nod} = \max\left\{0, \ell^e_{\rm int}, \ell^e_{\rm ext}\right\},$$
$$\max_{(\omega,\omega')\in\mathcal{D}_3} \delta_{\rm nod} = \max\{u^e_{\rm link}, |p' - q(1+e)|\},$$

where[2]

$$\ell^e_{\rm int}(q,e) = q' - \frac{q(1+e)}{1-e}, \qquad \ell^e_{\rm ext}(q,e) = q - Q',$$
$$u^e_{\rm link}(q,e) = \min\left\{\frac{q(1+e)}{1-e} - q', Q' - q\right\}.$$

[2]here $\ell^e_{\rm int}(q,1) = -\infty$, and $u^e_{\rm link}(q,1) = Q' - q$.



Graphic of $(q,e) \mapsto \max_{(\omega,\omega')\in\mathcal{D}_3} \delta_{\rm nod}(q,e)$ for $e' = 0.1$ (top left), $e' = 0.2$ (top right), $e' = 0.3$ (bottom left), $e' = 0.4$ (bottom right). Here we set $q' = 1$.

Optimal bounds for $\delta_{\rm nod}$ when $e' \in (0, 1)$

Let

$$\mathcal{D}_5 = \{(e, \omega) : e \in [0, 1], \omega \in [0, \pi]\},$$
$$\mathcal{D}_6 = \{(q, \omega') : q \in [0, q_{\rm max}], \omega' \in [0, \pi/2]\}.$$

For each choice of $(q, \omega') \in \mathcal{D}_6$ we have

$$\min_{(e,\omega)\in\mathcal{D}_5} \delta_{\rm nod} = \max\{0, \ell_{\rm ext}^{\omega'}\},$$
$$\max_{(e,\omega)\in\mathcal{D}_5} \delta_{\rm nod} = \max\left\{u_{\rm link}^{\omega'}, u_{\rm ext}^{\omega'}\right\},$$

where

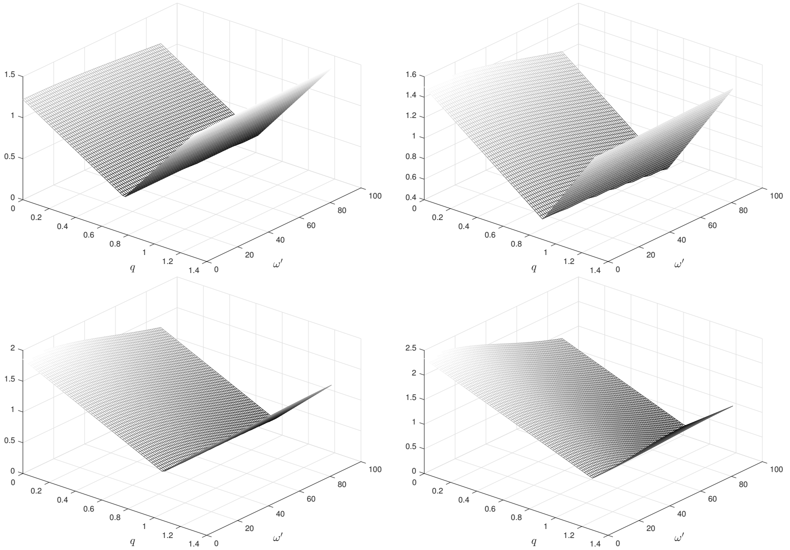
$$\ell_{\rm ext}^{\omega'}(q, \omega') = q - \frac{p'}{1 - e' \cos\omega'}, \qquad u_{\rm link}^{\omega'}(q, \omega') = \frac{p'}{1 - e' \cos\omega'} - q,$$

and

$$u_{\rm ext}^{\omega'}(q, \omega') = \frac{2q}{1 + \cos\omega_*} - \frac{p'}{1 + e' \cos\omega'},$$

with

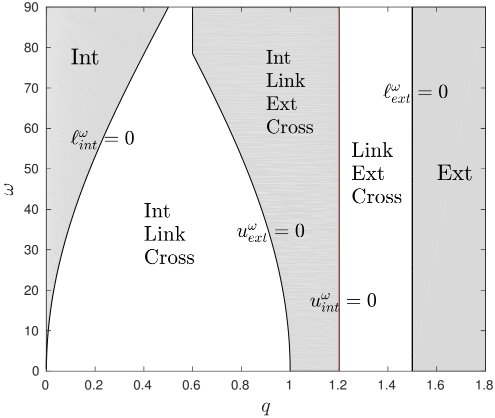
$$\cos\omega_* = \frac{p' e' \cos\omega'}{\sqrt{q^2(1 - e'^2\cos^2\omega')^2 + (p' e' \cos\omega')^2} + q(1 - e'^2\cos^2\omega')}.$$



Graphic of $(q, \omega') \mapsto \max_{(e,\omega)\in\mathcal{D}_5} \delta_{\rm nod}(q, \omega')$ for $e' = 0.1$ (top left), $e' = 0.2$ (top right), $e' = 0.3$ (bottom left), $e' = 0.4$ (bottom right). Here we set $q' = 1$.

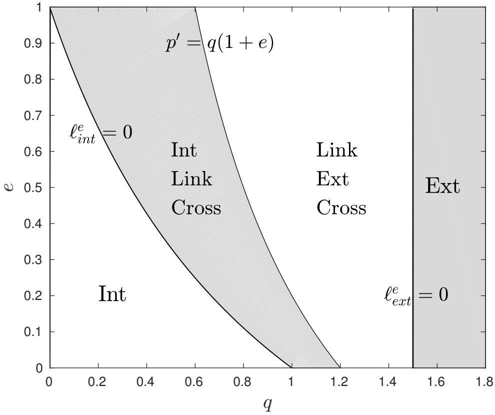
Linking conditions: (q, ω)

The zero level curves of $\ell^\omega_{\rm int}, \ell^\omega_{\rm ext}, u^\omega_{\rm int}, u^\omega_{\rm ext}$ divide the plane (q, ω) into regions where different linking configurations can occur. Moreover, $u^\omega_{\rm ext}(q, \omega) = 0$ is a piecewise smooth curve with only one component, a portion of which is a vertical segment with $q = p'/2$.



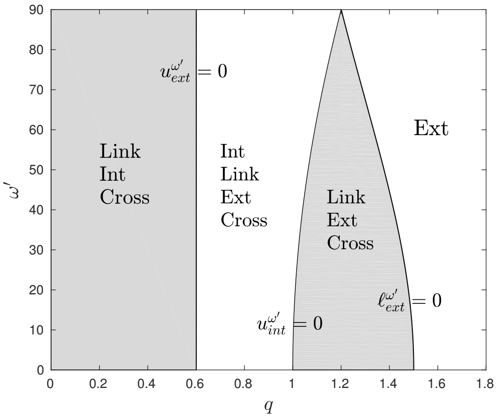
Linking conditions: (q, e)

The zero level curves of $\ell^e_{\rm int}, \ell^e_{\rm ext}, p' - q(1+e)$ divide the plane (q, e) into regions where different linking configurations can occur.

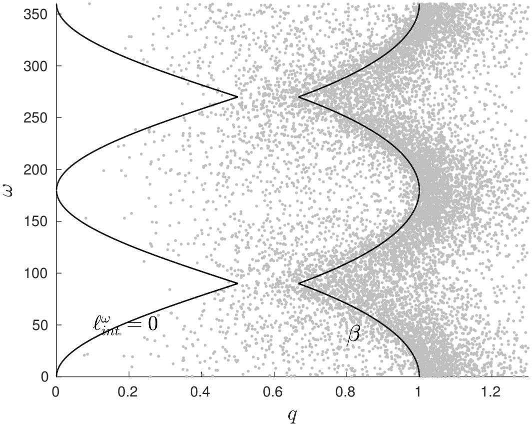


Linking conditions: (q, ω')

The zero level curves of $\ell^{\omega'}_{\text{ext}}, u^{\omega'}_{\text{int}}, u^{\omega'}_{\text{ext}}$ divide the plane (q, ω') into regions where different linking configurations can occur. Moreover, the curve $u^{\omega'}_{\text{ext}} = 0$ corresponds to the straight line $q = p'/2$.



Application to the known population of NEAs



Orbital distribution of the known NEAs (July 23, 2019) in the plane (q, ω). The gray dots correspond to faint asteroids ($H > 22$).

Thanks for your attention!

References (strongly biased)

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[3] Gronchi, G.F. and Tommei, G.: *'On the uncertainty of the minimal distance between two confocal Keplerian orbits'*, DCDS-B **7**/**4**, 755–778 (2007)

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