Lecture II: Charlier's theory

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Equations for preliminary orbits

From the geometry of the observations we have

$$r^2 = \rho^2 + 2q\rho\cos\epsilon + q^2 \qquad \text{(geometric equation).} \qquad (1)$$

From the two-body dynamics, both Laplace's and Gauss' methods yield an equation of the form

$$\mathcal{C}\,\frac{\rho}{q} = \gamma - \frac{q^3}{r^3} \qquad \text{(dynamic equation)}\,, \qquad (2)$$

with $\mathcal{C}, \gamma$ real parameters depending on the observations.

Preliminary orbits and multiple solutions

intersection problem:

$$\begin{cases} (q\gamma - \mathcal{C}\rho)r^3 - q^4 = 0 \\ r^2 - q^2 - \rho^2 - 2q\rho\cos\epsilon = 0 \\ r, \rho > 0 \end{cases} \tag{3}$$

reduced problem:

$$P(r) = 0\,, \qquad r > 0 \tag{4}$$

with

$$P(r) = \mathcal{C}^2 r^8 - q^2(\mathcal{C}^2 + 2\mathcal{C}\gamma\cos\epsilon + \gamma^2)r^6 + 2q^5(\mathcal{C}\cos\epsilon + \gamma)r^3 - q^8.$$

We investigate the *existence of multiple solutions of the intersection problem*.

Carl V. L. Charlier (1862-1934)

In 1910 Charlier gave a geometric interpretation of the occurrence of multiple solutions in preliminary orbit determination with Laplace's method, assuming geocentric observations ($\gamma = 1$).

'the condition for the appearance of another solution simply depends on the position of the observed body' (MNRAS, 1910)

Charlier's hypothesis: $\mathcal{C}, \epsilon$ *are such that a solution of the corresponding intersection problem with* $\gamma = 1$ *always exists*.

A *spurious solution* of (4) is a positive root $\bar{r}$ of $P(r)$ that is not a component of a solution $(\bar{r}, \bar{\rho})$ of (3) for any $\bar{\rho} > 0$.

We have:

- $P(q) = 0$, and $r = q$ corresponds to the observer position;
- $P(r)$ has always 3 positive and 1 negative real roots.

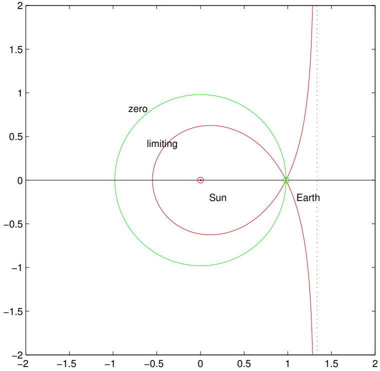
Let $P(r) = (r - q)P_1(r)$: then

$$P_1(q) = 2q^7\mathcal{C}[\mathcal{C} - 3\cos\epsilon].$$

If $P_1(q) < 0$ there are 2 roots $r_1 < q$, $r_2 > q$; one of them is spurious.
If $P_1(q) > 0$ both roots are either $< q$ or $> q$; they give us 2 different solutions of (3).

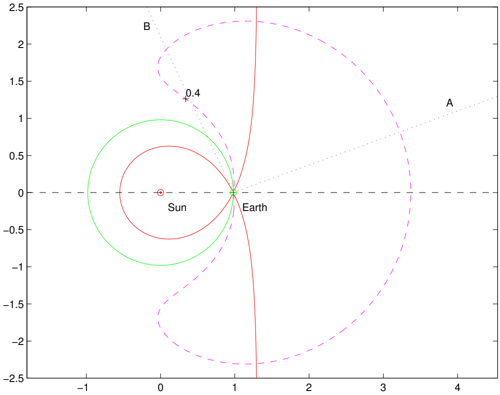
Zero circle and limiting curve

zero circle: $\mathcal{C} = 0$,
limiting curve: $\mathcal{C} - 3\cos\epsilon = 0$.



The green curve is the zero circle. The red curve is the limiting curve, whose equation in heliocentric rectangular coordinates (x, y) is

$$4 - 3\frac{x}{q} = \frac{q^3}{r^3}.$$



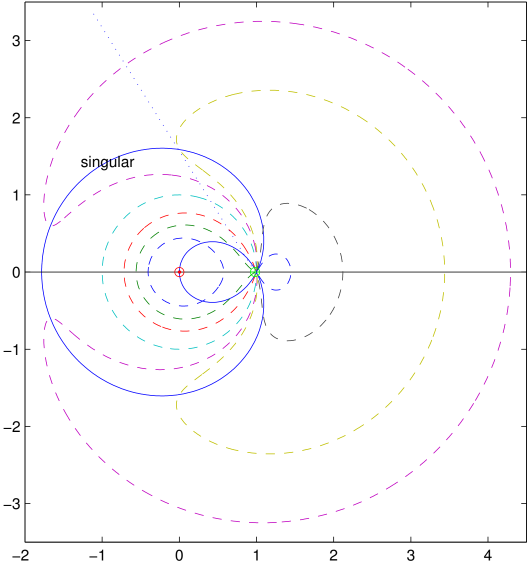
The position of the observed body corresponds to the intersection of the level curve $\mathcal{C}^{(1)}(x,y) = \mathcal{C}$ with the observation line (defined by ϵ), where $\mathcal{C}^{(1)} = \mathcal{C}^{(1)} \circ \Psi$,

$$\mathcal{C}^{(1)}(r,\rho) = \frac{q}{\rho}\left[1 - \frac{q^3}{r^3}\right]$$

and $(x,y) \mapsto \Psi(x,y) = (r,\rho)$ is the map from rectangular to bipolar coordinates.

Note that the position of the observed body defines an intersection problem.

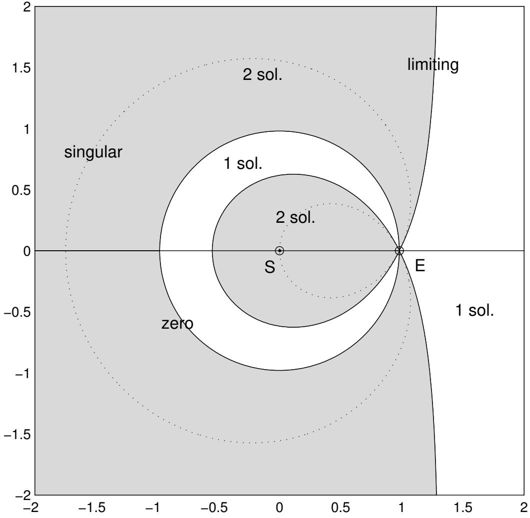
The singular curve



The singular curve is the set of tangency points between an observation line and a level curve of $\mathcal{C}^{(1)}$. It can be written as

$$4 - 3q\frac{x}{r^2} = \frac{r^3}{q^3}.$$

Multiple solutions: summary



Alternative solutions occurs in 2 regions: the interior of the limiting curve loop and outside the zero circle, on the left of the unbounded branches of the limiting curve.

See Gronchi, G.F.: CMDA **103/4** (2009)

Let $\gamma \in \mathbb{R}, \gamma \neq 1$. By the dynamic equation we define

$$\mathfrak{C}^{(\gamma)} = \mathcal{C}^{(\gamma)} \circ \Psi \,, \qquad \mathcal{C}^{(\gamma)}(r,\rho) = \frac{q}{\rho}\left[\gamma - \frac{q^3}{r^3}\right]$$

with $(x, y) \mapsto \Psi(x, y) = (r, \rho)$.

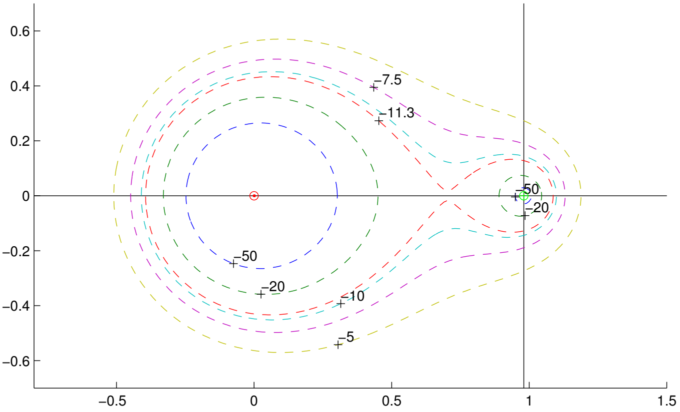
We also define the zero circle, with radius

$$r_0 = q/\sqrt[3]{\gamma}, \qquad \text{for } \gamma > 0.$$

Introduce the following assumption:

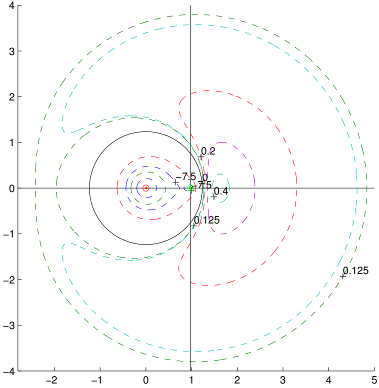
the parameters $\gamma, \mathcal{C}, \epsilon$ are such that the corresponding intersection problem admits at least one solution. (5)

Topology of the level curves of $\mathcal{C}^{(\gamma)}$

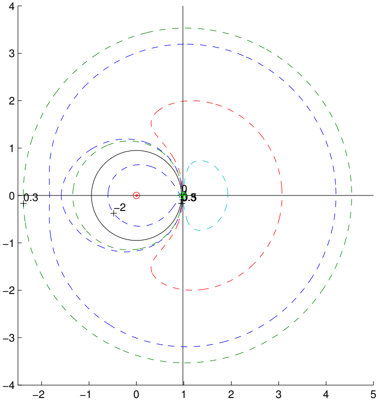


$\gamma \leq 0$

Topology of the level curves of $\mathcal{C}^{(\gamma)}$



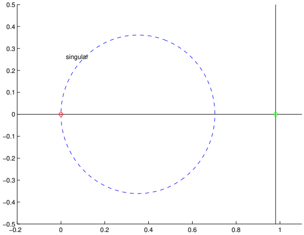
$0 < \gamma < 1$



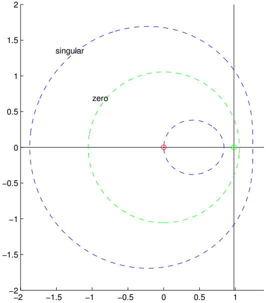
$\gamma > 1$

For $\gamma \neq 1$ we *cannot* define the limiting curve by Charlier's approach, in fact $P(q) \neq 0$. Nevertheless we can define the singular curve as the set

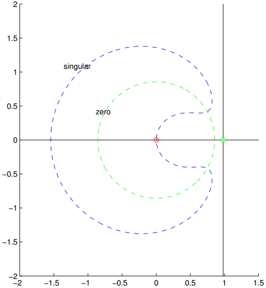
$$\mathcal{S} = \{(x,y) : \mathcal{G}(x,y) = 0\}\,, \qquad \mathcal{G}(x,y) = -\gamma r^5 + q^3(4r^2 - 3qx).$$



$\gamma \leq 0$



$0 < \gamma < 1$



$\gamma > 1$

An even or an odd number of solutions

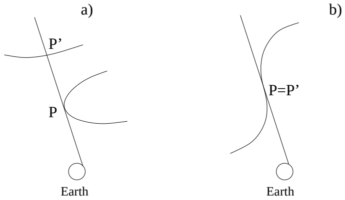
The solutions of an intersection problem (3) can not be more than 3. In particular, for $(\gamma, \mathcal{C}, \epsilon)$ fulfilling (5) with $\gamma \neq 1$,
if the number of solutions is even they are 2,
if it is odd they are either 1 or 3.
For $\gamma \neq 1$ we define the sets

$$\mathcal{D}_2(\gamma) = \begin{cases} \emptyset & \text{if } \gamma \leq 0 \\ \{(x,y) : r > r_0\} & \text{if } 0 < \gamma < 1 \\ \{(x,y) : r \leq r_0\} & \text{if } \gamma > 1 \end{cases}$$

and

$$\mathcal{D}(\gamma) = \mathbb{R}^2 \setminus (\mathcal{D}_2(\gamma) \cup \{(q,0)\}) .$$

Points in $\mathcal{D}_2(\gamma)$ corresponds to intersection problems with 2 solutions; points in $\mathcal{D}(\gamma)$ to problems with 1 or 3 solutions.



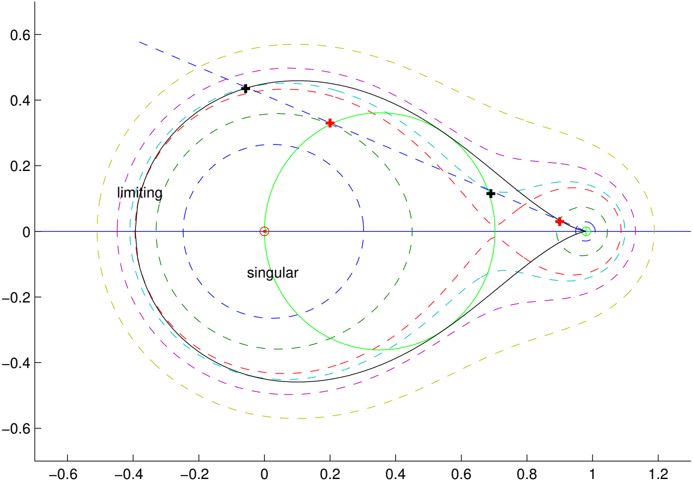
Fix $\gamma \neq 1$ and let $(\bar{\rho}, \overline{\psi})$ correspond to a point $\mathrm{P} \in \mathfrak{S} = \mathcal{S} \bigcap \mathcal{D}$. Let

$$F(\mathcal{C}, \rho, \psi) = \mathcal{C}\frac{\rho}{q} - \gamma + \frac{q^3}{r^3}, \qquad r = \sqrt{\rho^2 + q^2 + 2\rho q \cos\psi}$$

If $F_{\rho\rho}(\mathcal{C}, \bar{\rho}, \overline{\psi}) \neq 0$, we call residual point related to P the point $\mathrm{P}' \neq \mathrm{P}$ lying on the same observation line and the same level curve of $\mathcal{C}^{(\gamma)}(x, y)$, see Figure a).

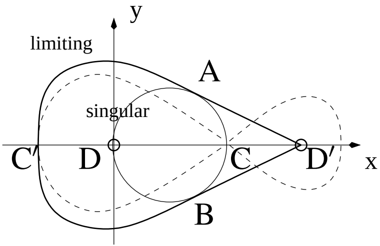
If $F_{\rho\rho}(\mathcal{C}, \bar{\rho}, \overline{\psi}) = 0$ we call P a self–residual point, see Figure b).

Let $\gamma \neq 1$. The limiting curve is the set composed by all the residual points related to the points in $\mathfrak{S}$.



The limiting curve

Separating property: *the limiting curve $\mathcal{L}$ separates $\mathcal{D}$ into two connected regions $\mathcal{D}_1, \mathcal{D}_3$: $\mathcal{D}_3$ contains the whole portion $\mathfrak{S}$ of the singular curve. If $\gamma < 1$ then $\mathcal{L}$ is a closed curve, if $\gamma > 1$ it is unbounded.*



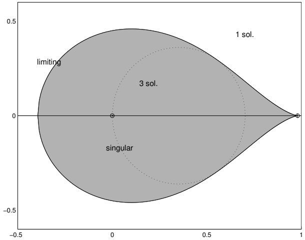
$(\gamma \leq 0)$

The limiting curve

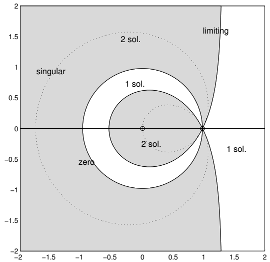
Transversality: *the level curves of* $\mathcal{C}^{(\gamma)}(x,y)$ *cross* $\mathcal{L}$ *transversely, except for at most the two self–residual points and for the points where* $\mathcal{L}$ *meets the* x*-axis.*

Limiting property: For $\gamma \neq 1$ the limiting curve $\mathcal{L}$ divides the set $\mathcal{D}$ into two connected regions $\mathcal{D}_1, \mathcal{D}_3$: the points of $\mathcal{D}_1$ are the unique solutions of the corresponding intersection problem; the points of $\mathcal{D}_3$ are solutions of an intersection problem with three solutions.

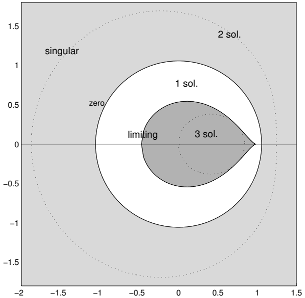
Multiple solutions: the big picture



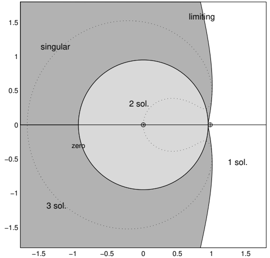
$\gamma \leq 0$



$\gamma = 1$



$0 < \gamma < 1$



$\gamma > 1$