Platonic Polyhedra, Topological Constraints and Periodic Solutions of the Classical N-Body Problem

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Propositum est mihi, Lector, hoc libello demonstrare, quod Creator Optimus Maximus, in creatione mundi huius mobilis, et dispositione coelorum, ad illa quinque regularia corpora, inde a Pythagora et Platone, ad nos utque, celebratissima respexerit, atque ad illorum naturam coelorum numerum, proportiones, et motuum rationem accommodaverit. (J. Kepler, *Myst. Cosm.* [20])

Abstract We prove the existence of a number of smooth periodic motions u_* of the classical Newtonian N–body problem which, up to a relabeling of the N particles, are invariant under the rotation group $\mathcal{R}$ of one of the five Platonic polyhedra. The number N coincides with the order $|\mathcal{R}|$ of $\mathcal{R}$ and the particles have all the same mass. Our approach is variational and u_* is a minimizer of the Lagrangean action $\mathcal{A}$ on a suitable subset $\mathcal{K}$ of the H^1 T–periodic maps $u : \mathbb{R} \to \mathbb{R}^{3N}$. The set $\mathcal{K}$ is a cone and is determined by imposing to u both topological and symmetry constraints which are defined in terms of the rotation group $\mathcal{R}$. There exist infinitely many such cones $\mathcal{K}$, all with the property that $\mathcal{A}|_{\mathcal{K}}$ is coercive. For a certain number of them, using level estimates and local deformations, we show that minimizers are free of collisions and therefore classical solutions of the N–body problem with a rich geometric–kinematic structure.

List of symbols

symbol	meaning
$\mathfrak{T}$, $\mathfrak{C}$, $\mathfrak{O}$, $\mathfrak{D}$, $\mathfrak{I}$	the five Platonic polyhedra
$\mathcal{T}, \mathcal{O}, \mathcal{I}$	symmetry groups of rotations of the Platonic polyhedra
$\mathcal{Q}_{\mathcal{T}}, \mathcal{Q}_{\mathcal{O}}, \mathcal{Q}_{\mathcal{I}}$	Archimedean polyhedra, Section 4.1
$\mathcal{X}$	configuration space
$\mathfrak{S}$	loops with collisions
Γ	set of the axes of rotation of the group $\mathcal{R} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$
$\tilde{\mathcal{R}}$	reflection group associated to $\mathcal{R}$
u	generic element of $H^1_T(\mathbb{R}, R^{3N})$ and
P_1	generating particle
τ_1	trajectory of the generating particle
u_1	generating loop for u
$\check{u}_1$, $\hat{u}_1$	variations of u_1
u_*, $u_{*,1}$	minimizer and related generating loop
$\Lambda^{(\mathsf{a})}$, $\Lambda_0^{(\mathsf{a})}$	loop spaces
$\Lambda_0^{(\mathsf{a})}/\sim$	space of equivalence classes of loops in $\Lambda_0^{(\mathsf{a})}$
$\mathcal{K}$, $\mathcal{K}_i^P$, $\tilde{\mathcal{K}}^\nu$	cones of loops
S^2	unit sphere in $\mathbb{R}^3$
e_j	direction vector of the axis $\xi_j, j = 1, 2, 3$
$\mathbb{N}$	natural numbers excluding 0
$\mathrm{u}_1^{(\sigma,n)}$	map associated to a periodic sequence (σ, n) of domains D_k, see (4.3)
$\mathrm{v}_1^{(\nu,n)}$	map associated to a periodic sequence (ν, n) of vertexes of $\mathcal{Q}_{\mathcal{R}}$, see (4.14)
D	fundamental domain
S_1, S_2, S_3	faces of D
D_k	domains obtained by applying the elements of $\tilde{\mathcal{R}}$ to D
S_k, S^k	faces of D_k
R_S	reflection with respect to the plane of the face S
$R \mapsto R^S$	bijection defined in $\mathcal{R} \setminus \{I\}$ for a given S, cfr. (4.17)

1. Introduction

In the last few years many interesting periodic motions of the classical Newtonian N–body problem have been discovered as minimizers of the *Action functional*

$$\mathcal{A} : \Lambda_G \to \mathbb{R} \cup \{+\infty\},$$

$$\mathcal{A}(u) = \int_0^T \Big(\frac{1}{2} \sum_{h=1}^{N} m_h |\dot{u}_h|^2 + \sum_{1 \leq h < k \leq N} \frac{m_h m_k}{|u_h - u_k|} \Big)\, dt \tag{1.1}$$

on loop spaces $\Lambda_G \subset H^1_T(\mathbb{R}, \mathcal{X})$, of T–periodic motions, equivariant with respect to the action of a suitably chosen group G [14],[5]. We denote by $u = (u_1, \ldots, u_N) : \mathbb{R} \to \mathcal{X}$ a typical element of Λ_G, by $u_h : \mathbb{R} \to \mathbb{R}^3$ the motion of the mass m_h and by $\mathcal{X} \subset \mathbb{R}^{3N}$ the configuration space

$$\mathcal{X} = \Big\{ x = (x_1, \ldots, x_N) \in \mathbb{R}^{3N} : \sum_{h=1}^{N} m_h x_h = 0 \Big\}.$$

$H^1_T(\mathbb{R}, \mathcal{X})$ denotes the Sobolev space of L^2 T–periodic maps $u : \mathbb{R} \to \mathcal{X}$ with L^2 first derivative.

The interest in this classical problem was revived by the discovery of the now famous Eight [22], [3]: the rather surprising fact that three equal masses can move periodically one after the other with a time shift of $T/3$ on the same fixed planar trajectory which has the shape of a symmetric eight. Another remarkable motion is the Hip–Hop [8], [12] where four equal masses, at intervals of $T/2$, coincide alternatively with the vertexes of two tetrahedra, each one symmetric of the other with respect to the center of mass.

The lack of coercivity of the action functional on the whole set $H^1_T(\mathbb{R}, \mathcal{X})$ of T–periodic motions and the fact that, on the basis of Sundman's estimates [31], [26], collisions give a bounded contribution to the action are the main mathematical obstructions in the search for T–periodic motions of the N–body problem as minimizers of the action. One of the main results of the research effort developed in the last ten years is a quite systematic way to deal with these obstructions. A basic observation concerning the problem of coercivity is that the action functional (1.1) is coercive when restricted to a loop space Λ_G of motions that possess suitable symmetries. The idea to impose symmetries to obtain coercivity was introduced in [11] and used in [7], and it is considered in a general abstract context in [14], where also a necessary and sufficient condition for coercivity is given.

The original motivation for our work was aesthetical: we wondered about the existence of new periodic motions which we could compare in perfection and beauty with the Eight, the Hip–Hop and the other interesting motions that have been recently discovered, see [27], [2], [28].

Let P be one of the five Platonic polyhedra, that is $P \in \{\mathfrak{T}, \mathfrak{C}, \mathfrak{O}, \mathfrak{D}, \mathfrak{I}\}$ where $\mathfrak{T}, \mathfrak{C}, \mathfrak{O}, \mathfrak{D}, \mathfrak{I}$ stand for *Tetrahedron*, *Cube*, *Octahedron*, *Dodecahedron* and *Icosahedron*. Let $\mathcal{T}, \mathcal{O}, \mathcal{I}$ be the groups of rotations of $\mathfrak{T}$, of $\mathfrak{C}$ and $\mathfrak{O}$, of $\mathfrak{D}$ and $\mathfrak{I}$ respectively, and denote by $\mathcal{R} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$ the group of rotations of P. Let $K = 4, 6, 8, 12, 20$ be the number of the faces of P and $H = 3, 4, 3, 5, 3$ the number of the vertexes of each face of P. Let F be one of the faces of P, L one of the sides of F and M the middle point of L. Consider a right–handed orthogonal frame $O\xi_1\xi_2\xi_3$, with the origin in the center of P, the axis ξ_1 oriented from O to the center of F and the axes ξ_2, ξ_3 such that M lies in the half–plane $\{\xi_3 = 0, \xi_2 > 0\}$. We let V be the vertex of L in the half–space $\xi_3 > 0$ (see Figure 3). We denote by e_j the direction vector of $\xi_j, j = 1, 2, 3$. The question that was at the origin of the present work is the following:

(Q) *Does it exist a T–periodic motion of the classical Newtonian N–body problem with $N = HK = 12, 24, 60$ equal masses that satisfies conditions* (a)*,* (b)*,* (c) *below?*

(a) If $u_1 : \mathbb{R} \to \mathbb{R}^3$, $u_1(t + T) = u_1(t)\ \forall t \in \mathbb{R}$, is the motion of one of the N particles, called the *generating particle*, the motion of the N particles is determined by a bijection:

$$\{2, \dots, N\} \ni j \to R_j \in \mathcal{R} \setminus \{I\} : u_j = R_j u_1 \ . \qquad (1.2)$$

(b) Associated to each face of P there are H particles that move one after the other on the same trajectory with a time shift of T/H.

(c) The motion of the generating particle satisfies

$$u_1(t) = S_3 u_1(-t) \qquad \forall t \in \mathbb{R}\,,$$

where S_3 is the reflection with respect to the plane $\xi_3 = 0$.

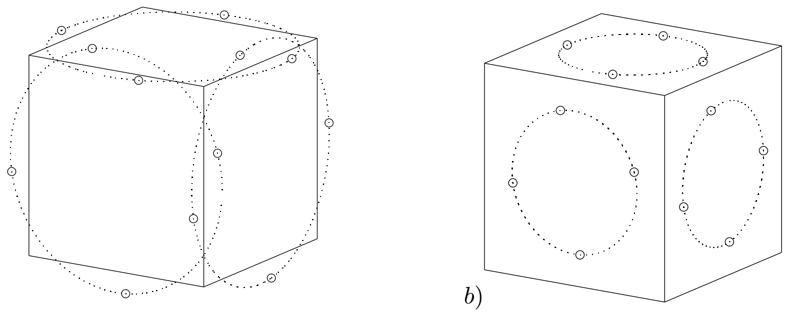
We remark that (a), (b), (c) imply in particular that the trajectory

$$\tau_1 = \{x \in \mathbb{R}^3 : \exists\, t \text{ with } x = u_1(t)\}$$

has all the symmetries of F.

We denote by $\Lambda^P \subset H^1_T(\mathbb{R}, \mathcal{X})$ the subset of T–periodic maps that satisfy conditions (a), (b), (c). Figure 1 visualizes possible structures of motions that satisfy (a), (b), (c) for the case $P = \mathfrak{C}$. One checks immediately that the action functional (with $m_j = 1$, $j = 1 \dots N$)

$$\mathcal{A}(u) = \frac{N}{2} \int_0^T \left(|\dot{u}_1|^2 + \sum_{R_j \in \mathcal{R} \setminus \{I\}} \frac{1}{|(R_j - I)u_1|} \right) dt \qquad (1.3)$$



a) *b)*

Figure 1. Loops in the space Λ^P for $P = \mathfrak{C}$.

is not coercive on the loop space Λ^P. Indeed, if we set $u_1^\lambda = u_1 + \lambda \mathsf{e}_1$, $\lambda > 0$, $\mathsf{e}_1 = (1,0,0)$, we have

$$\mathcal{A}(u) - \mathcal{A}(u^\lambda) = \frac{N}{2}\int_0^T \sum_{R_j \in \mathcal{R}\setminus\{I\}} \left(\frac{1}{|(R_j - I)u_1|} - \frac{1}{|(R_j - I)u_1 + \lambda(R_j - I)\mathsf{e}_1|} \right) dt$$

and therefore $\mathcal{A}(u) - \mathcal{A}(u^\lambda) > 0$ for λ large enough. This is also a consequence of the abstract coercivity condition formulated in [14]. In [14] a motion $u : \mathbb{R} \to \mathcal{X}$ is said to be *equivariant* with respect to the action of a finite group G if

$$\rho(g)\ u_{\sigma(g^{-1})(i)}(t) = u_i(\tau(g)(t)) \qquad \forall\, g \in G,\ \forall\, t \in \mathbb{R},\ \forall\, i \in \{1 \ldots N\}\ . \tag{1.4}$$

Here $\rho : G \to O(3)$, $\tau : G \to O(2)$ are orthogonal representations of G and $\sigma : G \to S_N$ is a homomorphism of G to S_N, the group of permutations of $\{1 \ldots N\}$.[1] If we let $\Lambda_G \subset H^1_T(\mathbb{R}, \mathcal{X})$ be the set of loops that fulfill the equivariance condition (1.4), then the coercivity condition formulated and proven in [14] is the following:

Theorem 1.1. *The action functional $\mathcal{A}$ is coercive on Λ_G if and only if*

$$\mathcal{X}^G = \{0\} \tag{1.5}$$

where $\mathcal{X}^G \subset \mathbb{R}^{3N}$ is the subset of the configuration space invariant under G, that is

$$\mathcal{X}^G = \left\{ x \in \mathcal{X} : \rho(g) x_{\sigma(g^{-1})(i)} = x_i, \forall\, g \in G,\ \forall\, i \in \{1 \ldots N\} \right\}\ .$$

For the loop space $\Lambda_G = \Lambda^P$, defined by (a), (b), (c), condition (1.5) is not satisfied. Indeed the nonzero vector

$$x = (x_1, \ldots, x_N), \qquad x_1 = \mathsf{e}_1\,, x_j = R_j \mathsf{e}_1,\ j = 2, \ldots, N$$

[1] If the masses of the N particles are not all equal it is required that $\sigma(g)(i) = j \Rightarrow m_i = m_j$.

belongs to $\mathcal{X}^G$.

The fact that for $\Lambda_G = \Lambda^P$ condition (1.5) is violated does not exclude *a priori* a positive answer to question (Q). Actually, in spite of the non–coercivity of $\mathcal{A}$ on Λ^P, motions defined by (a), (b), (c) may correspond to local minimizers, that may exist even though (1.5) is not satisfied. We also remark that for a loop $u(t)$ that satisfies (1.5) we necessarily have

$$\bar{u} = \frac{1}{T}\int_0^T u(t)\,dt = 0\ . \tag{1.6}$$

This follows from (1.4), that implies

$$\int_0^T u_i(t)\,dt = \int_0^T u_i(\tau(g)(t))\,dt = \rho(g)\int_0^T u_{\sigma(g^{-1})(i)}(t)\,dt\,,$$

which is equivalent to $\bar{u}_i = \rho(g)\bar{u}_{\sigma(g^{-1})(i)}$. Clearly (1.6) poses strong geometric restrictions on the motion and we may expect that many complex and interesting motions with a rich geometric–kinematic structure correspond to local minimizers that, as the loops defined by (a), (b), (c), do not need to satisfy condition (1.6). Is it possible to detect some of these local minimizers? Let $\mathfrak{S} \subset \Lambda_G$ be the subset of the loops that present collisions:

$$\mathfrak{S} = \{u \in \Lambda_G : \exists\, t_c \in \mathbb{R}, h \neq k \in \{1 \dots N\} : u_h(t_c) = u_k(t_c)\}\ .$$

This is a well defined subset of Λ_G, closed in the C^0 topology. We focus on *open cones* $\mathcal{K} \subset \Lambda_G$ with the property

$$\partial\mathcal{K} \subset \mathfrak{S} \tag{1.7}$$

where $\partial\mathcal{K}$ is the C^0 boundary of $\mathcal{K}$. The idea is that $\mathcal{A}$ can be coercive on $\mathcal{K}$ even though (1.5) is not satisfied; if we are able to prove that a minimizer u_* of $\mathcal{A}|_{\overline{\mathcal{K}}}$ exists and is collision free, then automatically we have $u_* \in \mathcal{K}$ and therefore a genuine solution of the N–body problem. In the following we discuss non–trivial situations where the above ideas can be successfully applied. Indeed we show the existence of new T–periodic solutions of the classical N–body problem with a rather rich and complex structure. In particular we give a positive answer to question (Q). These results are precisely stated in Theorems 2.1, 3.1, 4.1 below.
We remark that restricting the action to a cone $\mathcal{K} \subset \Lambda_G$ that satisfies (1.7) corresponds to the introduction of *topological* constraints beside the *symmetry* constraints imposed by the equivariance condition (1.4). The idea to obtain coercivity by introducing topological constraints, that restrict the action to subsets $\mathcal{K}$ satisfying condition (1.7), goes back to Poincaré [25] and has been exploited in [18], [29], [28]. In the proof that minimizers u_* of $\mathcal{A}|_{\overline{\mathcal{K}}}$, for the considered

cones $\mathcal{K}$, are free of collisions we take advantage of ideas of various authors [21], [5], [30]. Moreover we need to overcome the extra difficulty of dealing with a topological constraint, that does not allow general perturbations of $u_* \in \partial\mathcal{K}$, but only those that move u_* inside $\mathcal{K}$. For this reason Marchal's idea of averaging the action on a sphere or the average on a suitable circle, leading to the definition of the rotating circle property in [14], can not be applied in our context. The paper is organized as follows: in Sections 2, 3 and 4 we define different kinds of cones $\mathcal{K}$ that satisfy (1.7), and we prove the coercivity of $\mathcal{A}|_{\mathcal{K}}$. In Section 5 we prove Theorems 2.1, 3.1, 4.1 by showing that, for certain cones $\mathcal{K}$, minimizers $u_* \in \mathcal{K}$ are collision free. In Section 5.1 we exclude total collisions by means of level estimates. In Section 5.2 we exclude partial collisions via local perturbations. In Section 6 we present conjectures and numerical experiments, and prove the existence of T–periodic motions that violate (1.6).

2. An example of cone $\mathcal{K}$

We consider the set of T-periodic loops of $N = 4$ unit masses that satisfy

$$\begin{cases} u_1(t) = S_3 u_1(-t) \\ u_1(\frac{T}{4} + t) = S_2 u_1(\frac{T}{4} - t) \end{cases} \tag{2.1}$$

$$\begin{cases} u_2(t) = R_3 u_1(t) \\ u_3(t) = R_1 u_1(t) \\ u_4(t) = R_2 u_1(t) \end{cases} \tag{2.2}$$

where S_j is the reflection with respect to the plane $\xi_j = 0$, $j = 1, 2, 3$, and R_j is the rotation of π around the axis ξ_j, $j = 1, 2, 3$ (cfr. Figure 2). The loop space defined by (2.1) and (2.2) does not satisfy condition (1.5). Indeed the vector $x \in \mathcal{X} \subset \mathbb{R}^{12}$ defined by

$$x_1 = x_3 = \mathsf{e}_1 \,, \quad x_2 = x_4 = -\mathsf{e}_1$$

is in $\mathcal{X}^G$ and therefore, by Theorem 1.1, $\mathcal{A}$ is not coercive on Λ_G as one can also directly verify. Following the approach outlined above, we regain coercivity by restricting the set of allowed loops to the cone

$$\mathcal{K}_4 = \{u \in \Lambda_G : u_{11}(0)u_{11}(T/4) < 0\} \,. \tag{2.3}$$

We denote with (u_{11}, u_{12}, u_{13}) the components of u_1.

Proposition 2.1. *The cone $\mathcal{K}_4$ satisfies condition (1.7) and $\mathcal{A}|_{\mathcal{K}_4}$ is coercive.*

Proof. From (2.3) it follows $[u_{11}(0)-u_{11}(T/4)]^2 > (u_{11}(0))^2+(u_{11}(T/4))^2$. This and (2.1), that implies $u_{13}(0) = u_{12}(T/4) = 0$, yield

$$\left(|u_1(0)|^2 + |u_1(T/4)|^2\right)^{\frac{1}{2}} < |u_1(0) - u_1(T/4)| \,. \qquad (2.4)$$

Assume u_1^k and t^k, $k \in \mathbb{N}$, are sequences such that $\|u_1^k\|_{C^0} = |u_1^k(t^k)| \to +\infty$ as $k \to +\infty$. Then, if the sequence $u_1^k(0)$ is bounded, we have $\lim_{k\to+\infty} |u_1^k(0) - u_1^k(t^k)| = +\infty$. On the other hand, if the sequence $u_1^k(0)$ is unbounded, (2.4) implies that $\lim_{k\to+\infty} |u_1^k(0) - u_1^k(T/4)| = +\infty$. This proves coercivity.

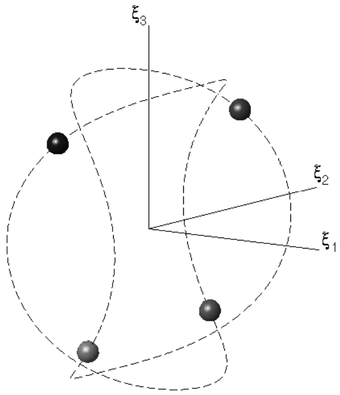


Figure 2. Structure of a motion (the map u_* in Theorem 2.1) that satisfies (2.1), (2.2).

To complete the proof we observe that u belongs to $\partial\mathcal{K}_4$ if and only if one of the following is true: $u_{11}(0) = 0$; $u_{11}(T/4) = 0$. As already remarked $u_{13}(0) = 0$. Therefore $u_{11}(0) = 0 \Rightarrow u_1(0) = u_{12}(0)\mathsf{e}_2 = u_4(0)$ by $(2.2)_3$. The other case is analogous.

□

Proposition 2.1 is the first step in the proof of the following Theorem, that we establish in Section 5:

Theorem 2.1. *There exists a T-periodic solution $u_* \in \mathcal{K}_4$ of the classical Newtonian 4-body problem.*

In Figure 2 we show the geometry of the solution u_* computed numerically.

3. Cones $\mathcal{K}$ and Platonic Polyhedra I.

The space Λ^P characterized by $(\mathsf{a}), (\mathsf{b}), (\mathsf{c})$ includes maps $u : \mathbb{R} \to \mathbb{R}^{3N}$, such that the map $u_1 : \mathbb{R} \to \mathbb{R}^3$ of the generating particle

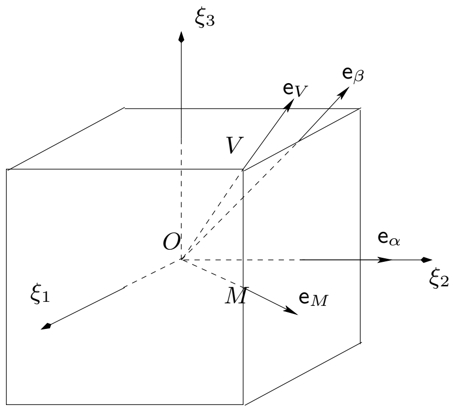


Figure 3. The unit vectors $\mathsf{e}_M, \mathsf{e}_V, \mathsf{e}_\alpha, \mathsf{e}_\beta$ for $P = \mathfrak{C}$.

satisfies

$$u_{11}(t_m) = \min_{t\in\mathbb{R}} u_{11}(t) > 0\,, \tag{3.1}$$

in contrast with (1.6). From (3.1) it follows that, if u is a real motion of the N–body problem, then, at time t_m, the N masses of the system are distributed on both sides of the plane $\xi_1 = u_{11}(t_m)$. Indeed (3.1) and (a) imply that some of the masses lie in the half–space $\xi_1 < u_{11}(t_m)$. From this and (3.1), that implies $\ddot{u}_{11}(t_m) \geq 0$, it follows $u_{j1}(t_m) > u_{11}(t_m)$ for some $j \neq 1$. This suggests that in the search of local minimizers of $\mathcal{A}|_{\Lambda^P}$ one should concentrate on situations of the type shown in Figure 1a and disregard the ones sketched in Figure 1b. Moreover it is clear from the picture that a continuous deformation from a loop $u \in \Lambda^P$ of the type in Figure 1a into a loop of the type of Figure 1b and vice-versa can not be done without collisions. It is also clear that if the H^1–norm of a map $u \in \Lambda^P$, corresponding to Figure 1a, tends to infinity, also the diameter of the orbit of each particle tends to infinity and we have coercivity in spite of the fact that (1.5) is violated. Based on these observations we now define cones $\mathcal{K} \subset \Lambda^P$ that satisfy (1.7). Given unit vectors $\mathsf{e}_r, \mathsf{e}_s$, $\mathsf{e}_r \neq \pm\mathsf{e}_s$, we let $\widehat{\mathsf{e}_r\mathsf{e}_s}$ be the angle

$$\widehat{\mathsf{e}_r\mathsf{e}_s} = \{x \in \mathbb{R}^3 : x = a\mathsf{e}_r + b\mathsf{e}_s, a, b > 0\}\ .$$

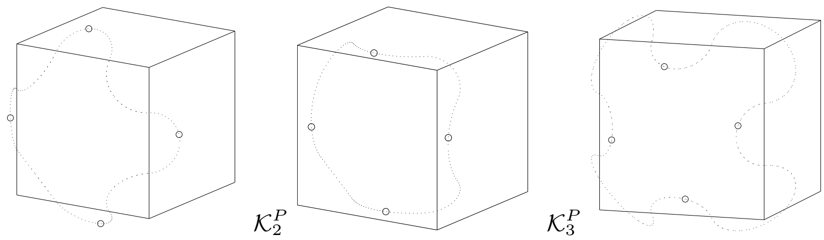
Let r_M (r_V) be the line through OM (OV) and let e_M (e_V) be the unit vectors directed as OM (OV). Observe that, beside ξ_1 and r_M (ξ_1 and r_V), on the plane $\xi_1 r_M$ ($\xi_1 r_V$) there is at least another axis r_α (r_β) of some rotations in $\mathcal{R}\setminus\{I\}$. We choose $r_\alpha \notin \{\xi_1, r_M\}$ ($r_\beta \notin \{\xi_1, r_V\}$) and its direction vector e_α (e_β) such that $M \in \widehat{\mathsf{e}_1\mathsf{e}_\alpha}$ ($V \in \widehat{\mathsf{e}_1\mathsf{e}_\beta}$) and the measure ψ_α (ψ_β) of the angle $\widehat{\mathsf{e}_1\mathsf{e}_\alpha}$ ($\widehat{\mathsf{e}_1\mathsf{e}_\beta}$) is minimum. We also let ϕ_M (ϕ_V) be the measure of $\widehat{\mathsf{e}_1\mathsf{e}_M}$ ($\widehat{\mathsf{e}_1\mathsf{e}_V}$).

We define

$$\mathcal{K}_1^P = \left\{ u \in \Lambda^P : u_1(0) \in \widehat{\mathsf{e}_M\mathsf{e}_\alpha}, \quad u_1(\tfrac{T}{2H}) \in \widehat{\mathsf{e}_1\mathsf{e}_V} \right\}, \tag{3.2}$$

$$\mathcal{K}_2^P = \left\{ u \in \Lambda^P : u_1(0) \in \widehat{\mathsf{e}_M\mathsf{e}_\alpha}, \quad u_1(\tfrac{T}{2H}) \in \widehat{\mathsf{e}_V\mathsf{e}_\beta} \right\}, \tag{3.3}$$

$$\mathcal{K}_3^P = \left\{ u \in \Lambda^P : u_1(0) \in \widehat{\mathsf{e}_1\mathsf{e}_M}, \quad u_1(\tfrac{T}{2H}) \in \widehat{\mathsf{e}_V\mathsf{e}_\beta} \right\}. \tag{3.4}$$



$\mathcal{K}_1^P$ $\mathcal{K}_2^P$ $\mathcal{K}_3^P$

Figure 4. The trajectory τ_1 for typical loops in the cones $\mathcal{K}_1^P, \mathcal{K}_2^P, \mathcal{K}_3^P$ respectively, in the case $P = \mathfrak{C}$.

The definition of $\mathcal{K}_1^P$ is illustrated in Figure 4 for the case $P = \mathfrak{C}$. We remark that

$$\mathcal{K}_j^P \cap \mathcal{K}_h^P = \emptyset \quad \text{for } j \neq h \,. \tag{3.5}$$

Proposition 3.1. *The cone* $\mathcal{K}_i^P \subset \Lambda^P$, $i = 1, 2, 3$, *satisfies (1.7) and* $\mathcal{A}|_{\mathcal{K}_i^P}$ *is coercive.*

Proof. Let $R \in \mathcal{R}$ be the rotation of angle $2\pi/H$ around ξ_1. Then (b) implies $Ru_1(t - \frac{T}{H}) = u_1(t)$ for all t, and in particular $Ru_1(-\frac{T}{2H}) = u_1(\frac{T}{2H})$. Hence from (c) it follows $RS_3u_1(\frac{T}{2H}) = u_1(\frac{T}{2H})$. Since RS_3 coincides with the reflection with respect to the plane $\xi_1 r_V$ we conclude that, for $u \in \Lambda^P$, $u_1(\frac{T}{2H})$ lies on the plane $\xi_1 r_V$. Moreover by (c), $u_1(0)$ lies in the plane $\xi_1 r_M$. Therefore $u \in \partial\mathcal{K}_i^P$ implies that either $u_1(0)$ or $u_1(\frac{T}{2H})$ belongs to the boundary of one of the angles in the definition of $\mathcal{K}_i^P$, that is to the axis r of some rotation in $\mathcal{R} \setminus \{I\}$. Thus by (a) we have a collision of the generating particle with all the other particles associated to the maximal cyclic group of the rotations with axis r. This proves (1.7).
To show coercivity of $\mathcal{A}|_{\mathcal{K}_i^P}$ we observe that from the definition of $\mathcal{K}_i^P$ and (b) it follows the existence of constants $c_i^P > 0$, depending only on P and $i = 1, 2, 3$, such that at least one of the two following inequalities holds true:

$$\left| u_1(\frac{T}{H}) - u_1(0) \right| = |Ru_1(0) - u_1(0)| \geq c_1^P |u_1(0)|\,, \tag{3.6}$$

$$\Big|u_1(\frac{T}{2H}) - u_1(-\frac{T}{2H})\Big| = \Big|Ru_1(-\frac{T}{2H}) - u_1(-\frac{T}{2H})\Big| \geq c_i^P \Big|u_1(-\frac{T}{2H})\Big|, \tag{3.7}$$

for all $u \in \mathcal{K}_i^P$. If $\bar{t} \in (0,T)$ is such that $|u_1(\bar{t})| = \|u_1\|_{C^0}$, then from (3.6), (3.7) we have either

$$\begin{aligned} 2T^{1/2}\Big(\int_0^T |\dot{u}_1|^2\Big)^{1/2} &\geq \big|u_1(\tfrac{T}{H}) - u_1(0)\big| + |u_1(\bar{t}) - u_1(0)| \\ &\geq c_i^P |u_1(0)| + \big|\|u_1\|_{C^0} - |u_1(0)|\big| \end{aligned}$$

or

$$\begin{aligned} 2T^{1/2}\Big(\int_0^T |\dot{u}_1|^2\Big)^{1/2} &\geq |u_1(\tfrac{T}{2H}) - u_1(-\tfrac{T}{2H})| + \big|u_1(\bar{t}) - u_1(-\tfrac{T}{2H})\big| \\ &\geq c_i^P \,\big|u_1(-\tfrac{T}{2H})\big| + \big|\|u_1\|_{C^0} - |u_1(-\tfrac{T}{2H})|\big| \end{aligned}$$

and coercivity follows.

□

On the basis of Proposition 3.1 we shall prove the following theorem (Section 5):

Theorem 3.1. *Given $P \in \{\mathfrak{T}, \mathfrak{C}, \mathfrak{O}, \mathfrak{D}, \mathfrak{I}\}$ let $\mathcal{K}_i^P \subset \Lambda^P$, $i = 1,2,3$, be the cones defined in (3.2), (3.3), (3.4):*

(i) there exists a minimizer $u_*^{P,i} \in \mathcal{K}_i^P$ of $\mathcal{A}|_{\mathcal{K}_i^P}$, $i = 1,2,3$, and $u_*^{P,i}$ is a smooth T–periodic solution of the classical Newtonian N–body problem ($N = 12$ for $P = \mathfrak{T}$; $N = 24$ for $P = \mathfrak{C}, \mathfrak{O}$; $N = 60$ for $P = \mathfrak{D}, \mathfrak{I}$).

(ii) $u_*^{P,i} \neq u_*^{P,j}$, $i \neq j$.

T–periodic solutions of the classical Newtonian N–body problem for $N = |\mathcal{R}|$ particles, $\mathcal{R} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$, satisfying condition (a) and the coercivity condition (1.6) have already appeared in [13] and [23]. We stress that we do not require condition (1.6), see also Theorem 6.1 below. Figure 5 shows numerical simulations of the trajectory of the particles for some of the minimizers $u_*^{P,i}$ in Theorem 3.1.

4. Cones $\mathcal{K}$ and Platonic Polyhedra II.

Definitions (3.2), (3.3), (3.4) give only three examples of cones $\mathcal{K}$ satisfying (1.7) and ensuring coercivity of $\mathcal{A}|_{\mathcal{K}}$. Actually for each choice of $\mathcal{R} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$ there are infinitely many cones $\mathcal{K}$ with these properties. For each $R \in \mathcal{R} \setminus \{I\}$, let $r(R)$ be the axis of rotation of R and define

$$\Gamma = \bigcup_{R \in \mathcal{R} \setminus \{I\}} r(R) \subset \mathbb{R}^3 .$$

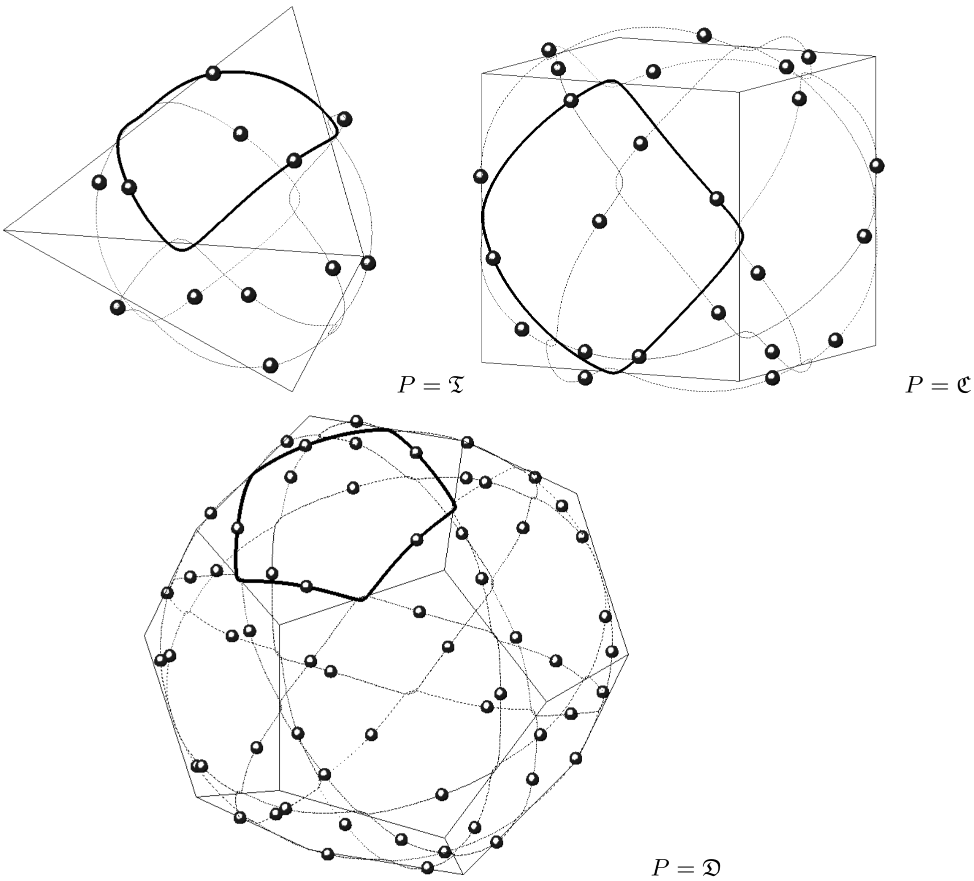


Figure 5. Geometry of minimizers $u_*^{P,1}$ for the cones $\mathcal{K}_1^P$, with $P = \mathfrak{T}, \mathfrak{C}, \mathfrak{D}$.

We denote by $\Lambda^{(\mathsf{a})} \subset H^1_T(\mathbb{R}, \mathcal{X})$ the space of T–periodic maps $u : \mathbb{R} \to \mathcal{X}$ defined by (a) when the map $u_1 \in H^1_T(\mathbb{R}, \mathbb{R}^3)$, representing the motion of the generating particle, satisfies the condition

$$u_1(\mathbb{R}) \cap \Gamma = \emptyset \ . \tag{4.1}$$

We introduce a notion of equivalence on $\Lambda^{(\mathsf{a})}$.

Definition 4.1. *We say that u and $v \in \Lambda^{(\mathsf{a})}$ are equivalent and we write $u \sim v$ if the corresponding maps of the generating particle u_1 and v_1 are homotopic in $\mathbb{R}^3 \setminus \Gamma$, that is if there exists a continuous map $h : \mathbb{R} \times [0,1] \to \mathbb{R}^3$, T-periodic in the first variable, and such that*

(i) $h(\mathbb{R}, s) \cap \Gamma = \emptyset, \ \forall s \in [0,1]$,
(ii) $h(\cdot, 0) = u_1, \ h(\cdot, 1) = v_1$.

Let $\Lambda_0^{(\mathsf{a})} \subset \Lambda^{(\mathsf{a})}$ be the subset of all the maps u satisfying the following condition:

(C) *u_1 is not homotopic to any map $v_1 \in H^1_T(\mathbb{R}, \mathbb{R}^3)$ of the form:*

$$v_1(t) = \mathsf{e}'_1 + \delta \left[\cos\left(2\pi k \frac{t}{T}\right) \mathsf{e}'_2 + \sin\left(2\pi k \frac{t}{T}\right) \mathsf{e}'_3\right]$$

where e'_j, $j = 1, 2, 3$ is an orthonormal basis, with e'_1 parallel to one of the axes in Γ, $0 < \delta << 1$, $k \in \mathbb{N} \cup \{0\}$.

Let $\mathcal{K}(u) \in \Lambda_0^{(\mathsf{a})}/\sim$ denote the equivalence class of $u \in \Lambda_0^{(\mathsf{a})}$. We have:

Proposition 4.1. *Each cone $\mathcal{K} \in \Lambda_0^{(\mathsf{a})}/\sim$ satisfies (1.7) and $\mathcal{A}|_{\mathcal{K}}$ is coercive.*

Proof. $u \in \partial\mathcal{K}$ if and only if there exists a time $\bar{t}$ and $R \in \mathcal{R} \setminus \{I\}$ such that $u_1(\bar{t}) \in r(R)$. Then from (1.2), writing $R \to j_R$ for the inverse of $j \to R_j$, we have: $u_1(\bar{t}) = u_{j_R}(\bar{t}) = Ru_1(\bar{t})$ and therefore a collision. This establishes (1.7). To show coercivity we observe that condition (C) above implies the existence of a constant $c_{\mathcal{K}}$ such that

$$\max_{t_1, t_2 \in [0,T]} |u_1(t_1) - u_1(t_2)| \geq c_{\mathcal{K}} \min_{t \in [0,T]} |u_1(t)|, \ \forall u \in \mathcal{K} \ .$$

Therefore, if t_m satisfies $|u_1(t_m)| = \min_{t \in [0,T]} |u_1(t)|$, we have

$$|u_1(t)| \leq |u_1(t_m)| + |u_1(t) - u_1(t_m)| \leq (1/c_{\mathcal{K}} + 1) \max_{t_1, t_2 \in [0,T]} |u_1(t_1) - u_1(t_2)| \ .$$

This concludes the proof.

□

On the basis of Proposition 4.1 we shall show (see Theorem 4.1) that for several $\mathcal{K} \in \Lambda_0^{(\mathsf{a})}/\sim$ there exists $u_* \in \mathcal{K}$ corresponding to a smooth periodic motion of the Newtonian N–body problem. For the precise statement of Theorem 4.1 and for the detailed analysis of the existence of collisions we make use of two different ways of characterizing the topology of the maps in a given $\mathcal{K} \in \Lambda_0^{(\mathsf{a})}/\sim$. Indeed we associate to $\mathcal{K}$ two different topological invariants in the form of periodic sequences of integers. The following Subsection is devoted to the definition of these invariants.

4.1. Characterizing the topology of $\mathcal{K}$

For $\mathcal{R} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$ we denote with $\tilde{\mathcal{R}} \in \{\tilde{\mathcal{T}}, \tilde{\mathcal{O}}, \tilde{\mathcal{I}}\}$ the associated reflection group. A fundamental domain [16] for $\tilde{\mathcal{R}}$ can be identified with the open convex cone $\mathrm{D} \subset \mathbb{R}^3$ generated by the positive ξ_1 semiaxis, and the semiaxes determined by OM and by OV.[2] Then

$$\tilde{R}\mathrm{D} \cap \mathrm{D} = \emptyset \quad \forall \tilde{R} \in \tilde{\mathcal{R}} \setminus \{I\}\,, \qquad \bigcup_{\tilde{R}\in\tilde{\mathcal{R}}} \tilde{R}\overline{\mathrm{D}} = \mathbb{R}^3\ .$$

This means that $\mathbb{R}^3$ is divided into exactly $|\tilde{\mathcal{R}}|$ non–overlapping *chambers* ($|\tilde{\mathcal{R}}| = 2|\mathcal{R}| = 24, 48, 120$ for $\mathcal{R} = \mathcal{T}, \mathcal{O}, \mathcal{I}$) each of which is an isometric copy of D. Let $\mathcal{D} = \{D \subset \mathbb{R}^3 : D = \tilde{R}\mathrm{D}, \tilde{R} \in \tilde{\mathcal{R}}\}$. We also let $\mathrm{S}_i, i = 1, 2, 3$ be the (open) faces of D and let $\mathcal{S} = \{S \subset \mathbb{R}^3 : S = \tilde{R}\mathrm{S}_i, \tilde{R} \in \tilde{\mathcal{R}}, i = 1, 2, 3\}$ be the set of the faces of all the elements of $\mathcal{D}$. For each $S \in \mathcal{S}$ we define $\tilde{R}_S \in \tilde{\mathcal{R}}$ as the reflection with respect to the plane of S. We also note that each $S \in \mathcal{S}$ uniquely determines a pair $D_S^i \in \mathcal{D}, i = 1, 2$ such that $\overline{S} = \overline{D_S^1} \cap \overline{D_S^2}$ and $D_S^2 = \tilde{R}_S D_S^1$.

We consider the set of sequences $\sigma = \{D_k\}_{k\in\mathbb{Z}} \subset \mathcal{D}$ that satisfy

(I) σ is periodic: $\exists\, K \in \mathbb{N}$ such that $D_{k+K} = D_k, k \in \mathbb{Z}$;
(II) D_{k+1} is the mirror image of D_k with respect to one of the faces of D_k and $D_{k+1} \neq D_{k-1}$;
(III) $\bigcap_{k\in\mathbb{Z}} \overline{D_k} = \{0\}$.

We identify sequences that coincide up to translations, that is $\sigma = \{D_k\}, \sigma' = \{D'_k\}$ are identified whenever there exists $K \in \mathbb{N}$ such that

$$D_k = D'_{k+K},\ \ k \in \mathbb{Z}\ . \tag{4.2}$$

Each sequence σ satisfying (I), (II), (III) can be regarded as periodic of period nK_σ, with $n \in \mathbb{N}$ and K_σ the minimal period. We write (σ, n) to indicate that we are considering σ with the particular period nK_σ and regard (σ, n_1), (σ, n_2), with $n_1 \neq n_2$, as different objects.

Our first algebraic characterization of the topology of maps in $\mathcal{K}$ is described in the following Proposition:

Proposition 4.2. *Each pair* (σ, n) *with* $\sigma = \{D_k\}_{k\in\mathbb{Z}}$ *satisfying* (I), (II), (III) *and* $n \in \mathbb{N}$ *uniquely determines a cone* $\mathcal{K} \in \Lambda_0^{(\mathsf{a})}/\!\sim$*. Viceversa each* $\mathcal{K} \in \Lambda_0^{(\mathsf{a})}/\!\sim$ *uniquely determines* $n \in \mathbb{N}$ *and (up to translation) a sequence* σ *satisfying* (I), (II), (III).

Proof. To each pair (σ, n) we associate in a canonical way a map $\mathrm{u}^{(\sigma,n)} \in \Lambda_0^{(\mathsf{a})}$ as follows. Let $\tau_k, k \in \mathbb{Z}$ be the spherical triangle, intersection of D_k with the unit sphere $\mathsf{S}^2 \subset \mathbb{R}^3$, and let $c_k \in \tau_k$ be the

[2] The axes $\xi_j, j = 1, 2, 3$, are defined as in Section 1. If $\mathcal{R} = \mathcal{O}$ ($\mathcal{R} = \mathcal{I}$), the definition can be based indifferently on $\mathfrak{C}$ or $\mathfrak{O}$ (on $\mathfrak{D}$ or $\mathfrak{I}$).

center of τ_k. Set $\ell_k = |c_{k+1} - c_k|$, $\ell_0 = 0$, $L = \sum_{j=1}^{nK_\sigma} \ell_j$ and define $\mathrm{u}^{(\sigma,n)}$ by

$$\begin{gathered}\mathrm{u}_1^{(\sigma,n)}(t) = \Big(\sum_{j=0}^{k} \ell_j - \frac{t}{T}L\Big)\frac{c_k}{\ell_k} + \Big(\frac{t}{T}L - \sum_{j=0}^{k-1}\ell_j\Big)\frac{c_{k+1}}{\ell_k}\,, \\ \sum_{j=0}^{k-1}\frac{\ell_j}{L} \leq \frac{t}{T} \leq \sum_{j=0}^{k}\frac{\ell_j}{L}\,,\quad k = 1,\ldots,nK_\sigma\,.\end{gathered} \tag{4.3}$$

By construction $\mathrm{u}_1^{(\sigma,n)}$ satisfies (4.1) and condition (C) by (III). It follows that $\mathrm{u}^{(\sigma,n)} \in \Lambda_0^{(\mathsf{a})}$ and therefore each pair (σ, n) uniquely determines an element $\mathcal{K}(\mathrm{u}^{(\sigma,n)}) \in \Lambda_0^{(\mathsf{a})}/\!\sim$. To prove that the viceversa is also true we shall show that for each $u \in \Lambda_0^{(\mathsf{a})}$ there is a unique pair (σ_u, n_u) such that

$$u \sim \mathrm{u}^{(\sigma_u,n_u)}\,.$$

By definition of $\Lambda_0^{(\mathsf{a})}$, for each $u \in \Lambda_0^{(\mathsf{a})}$ there is $d_u > 0$ such that

$$d(u_1(\mathbb{R}), \Gamma) = d_u\,. \tag{4.4}$$

Therefore the set $\Theta = \{t \in \mathbb{R} : u_1(t) \in S, S \in \mathcal{S}\}$ is closed. We note in addition that (4.4) yields

$$\mathcal{A}(u) < +\infty\,, \qquad \forall\, u \in \Lambda_0^{(\mathsf{a})}\,.$$

From this and (4.4) we conclude that, if (t_1, t_2) is a connected component of the open set $\mathbb{R} \setminus \Theta$ with the property that

$$u_1(t_1) \in S, u_1(t_2) \in S', \text{ for some } S, S' \in \mathcal{S}\,, \text{ with } S \neq S', \tag{4.5}$$

then $t_2 - t_1 > \delta_u$ for some $\delta_u > 0$. This inequality shows that in any time interval of size T there are only a finite number, say M, of connected components of $\mathbb{R} \setminus \Theta$ that satisfy (4.5).

Let J be the set of the connected components of $\mathbb{R} \setminus \Theta$ that satisfy (4.5) and consider two consecutive intervals (t_1, t_2), $(t_1', t_2') \in J$. The assumption that between t_2 and t_1' there is no other interval belonging to J implies the existence of $S \in \mathcal{S}$ such that

$$u_1(t_2), u_1(t_1') \in S\,, \qquad u_1([t_2, t_1']) \subset D_S^- \cup S \cup D_S^+\,, \tag{4.6}$$

where $D_S^\pm \in \{D_S^1, D_S^2\}$ are determined by the conditions

$$u_1((t_1, t_2)) \subset D_S^-\,, \qquad D_S^- \neq D_S^+\,.$$

Define $\check{u}_1$ by setting, for each interval $(t_1, t_2) \in J$,

$$\check{u}_1(t) = \begin{cases} \tilde{R}_S u_1(t) & t \in (t_2, t_1'), u_1(t) \in D_S^+\,, \\ u_1(t) & \text{otherwise}\,, \end{cases} \tag{4.7}$$

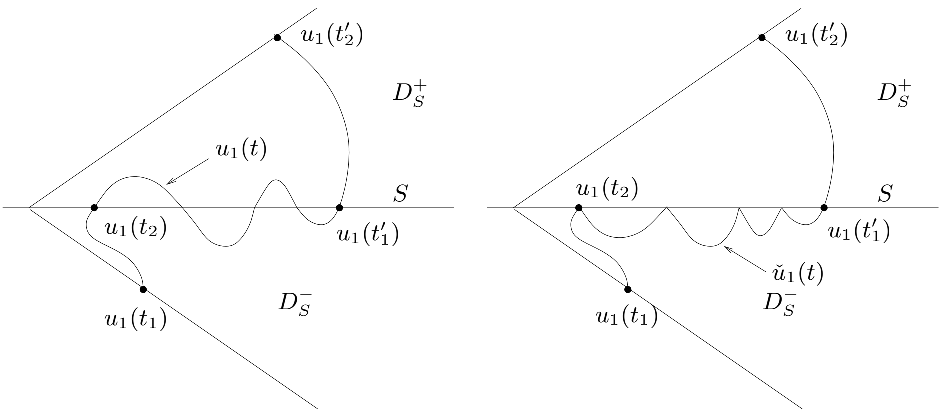


Figure 6. Reflecting u_1 into $\breve{u}_1$.

as in the example in Figure 6. The convexity of $D_S^- \cup S \cup D_S^+$ implies that the map

$$h(t,s) = (1-s)u_1(t) + s\breve{u}_1(t)\,, \quad t \in \mathbb{R}, s \in [0,1]$$

satisfies conditions (i), (ii) of Definition 4.1, and therefore $\breve{u} \sim u$. Write J in the form $J = \{(t_1^k, t_2^k),\ t_1^k < t_1^{k+1},\ k \in \mathbb{Z}\}$, let $S_{k+1} \in \mathcal{S}$ be the face associated to the interval (t_2^k, t_1^{k+1}), as in (4.6), and let $D_k \in \mathcal{D}$ be the set D_S^- corresponding to $S = S_{k+1}$, that is

$$u_1\big((t_1^k, t_2^k)\big) \subset D_k\ . \tag{4.8}$$

From (4.7) and (4.8) it follows that

$$\breve{u}_1\big([t_1^k, t_1^{k+1}]\big) \subset \overline{D_k} \setminus \varGamma\,,\ k \in \mathbb{Z} \tag{4.9}$$

and moreover that

$$\breve{u}_1\big((t_1^k, t_1^k + \delta_u)\big) \subset D_k\,,\ k \in \mathbb{Z}\ . \tag{4.10}$$

From now on we shall drop the subscript 1 and write simply t^k instead of t_1^k. The periodicity of $\breve{u}_1$ implies

$$[t^k + jT, t^{k+1} + jT] = [t^{k+jM}, t^{k+1+jM}], \quad D_k = D_{k+jM}, \quad j \in \mathbb{Z}\ .$$

We can assume that

$$D_k \neq D_{k+1}\ . \tag{4.11}$$

Indeed if $D_h = D_{h+1}$ for some $h \in \mathbb{Z}$, then (4.9) implies

$$\breve{u}_1\big([t^h + jT, t^{h+2} + jT]\big) \subset \overline{D_{h+jM}} \setminus \varGamma$$

and therefore, if we erase the subsequences $\{t^{h+1+jM}\}_{j\in\mathbb{Z}}$, $\{S_{h+1+jM}\}_{j\in\mathbb{Z}}$, $\{D_{h+1+jM}\}_{j\in\mathbb{Z}}$ from the sequences $\{t^k\}_{k\in\mathbb{Z}}$, $\{S_k\}_{k\in\mathbb{Z}}$, $\{D_k\}_{k\in\mathbb{Z}}$, then, after relabeling, we still have that (4.9), (4.10) hold. A finite number of steps of this kind establishes (4.11).

By homotopy we can transform $\check{u}_1$ into a map $\hat{u}_1$ that satisfies (4.9), (4.10), 4.11) and moreover has the property that

$$D_{k-1} \neq D_{k+1} \ . \tag{4.12}$$

To see this we observe that $D_{h-1} = D_{h+1}$ together with (4.9), (4.11) imply $S_h = S_{h+1}$ and

$$\check{u}_1(t^h), \check{u}_1(t^{h+1}) \in S_h \,, \qquad \check{u}_1\big([t^h, t^{h+1}]\big) \subset \overline{D_h} \setminus \Gamma \ . \tag{4.13}$$

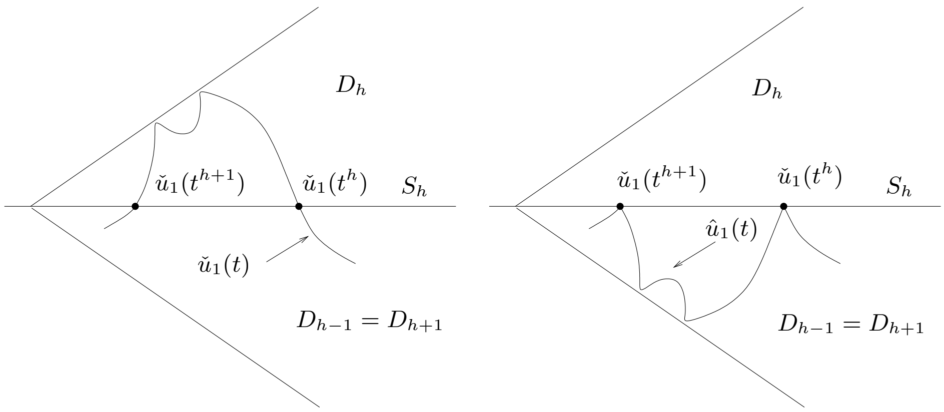


Figure 7. Reflecting $\check{u}_1$ into $\hat{u}_1$.

From (4.13) and the same argument used above to define $\check{u}$ it follows that the map defined by

$$\hat{u}_1(t) = \begin{cases} \tilde{R}_{S_h}\check{u}_1(t)\,, & t \in (t^{h+jM}, t^{h+1+jM})\,,\ j \in \mathbb{Z}\,, \\ \check{u}_1(t)\,, & \text{otherwise} \end{cases}$$

(see Figure 7) is homotopic to $\check{u}_1$ and it has the property that

$$\hat{u}_1\big([t^{h-1}, t^{h+2}]\big) \subset \overline{D_{h-1}} \setminus \Gamma = \overline{D_{h+1}} \setminus \Gamma \ .$$

This shows that, after erasing t^{h+jM} and t^{h+1+jM}, $j \in \mathbb{Z}$, from $\{t^k\}_{k\in\mathbb{Z}}$ ($S_{h+jM}, S_{h+1+jM}, j \in \mathbb{Z}$, from $\{S_k\}_{k\in\mathbb{Z}}$, $D_{h+jM}, D_{h+1+jM}, j \in \mathbb{Z}$, from $\{D_k\}_{k\in\mathbb{Z}}$), and after relabeling, the relations (4.9), (4.10), (4.11) still hold and M is reduced by 2 units. Therefore after a finite number of steps we obtain a map $\hat{u}_1$ homotopic to u_1 and such that the corresponding sequences $\{t^k\}_{k\in\mathbb{Z}}$, $\{S_k\}_{k\in\mathbb{Z}}$, $\{D_k\}_{k\in\mathbb{Z}}$ satisfy (4.9)– (4.12). In particular the sequence $\sigma_u = \{D_k\}_{k\in\mathbb{Z}}$ satisfies (I) and (II) and we claim that $\hat{u}_1$ and therefore u_1 is homotopic to the map $\mathrm{u}_1^{(\sigma_u,n_u)}$ where n_u is defined by $n_u = M/K_u$, K_u being the minimal period of σ_u. To prove the claim it suffices to observe that the transformation $s \to h(\cdot, s)$, defined by

$$h(\lambda, s) = (1-s)\hat{u}_1(t^k+\delta_u+\lambda(t^{k+1}-t^k))+s(c_k+\lambda(c_{k+1}-c_k)), \;\; \lambda, s \in [0,1]\,,$$

continuously deforms, without touching Γ, each arc $\hat{u}_1([t^k+\delta_u, t^{k+1}+\delta_u]) \subset (\overline{D_k} \cup \overline{D_{k+1}}) \setminus \Gamma$ into the segment $[c_k, c_{k+1}]$ joining the centers of the spherical triangles $\tau_k = D_k \cap \mathsf{S}^2$, $\tau_{k+1} = D_{k+1} \cap \mathsf{S}^2$. To conclude the proof we only need to show that σ_u satisfies (III). This follows from the fact that, should (III) be violated, then $\mathrm{u}_1^{(\sigma_u,n_u)}$, and in turn u_1, would not satisfy condition (C).

□

The above discussion establishes a one to one correspondence between $\Lambda_0^{(\mathsf{a})}/\!\sim$ and the set of the pairs (σ, n) with σ satisfying (I), (II), (III) (with the identification (4.2)).

Definition 4.2. *A pair* (σ, n) *is said to be* simple *if* σ *does not contain a string* $D_k, \ldots, D_{k+H}$ *such that*

a) $\bigcap_{j=0}^{H} \overline{D_{k+j}} = r(R)$*, for some* $R \in \mathcal{R} \setminus \{I\}$*;*
b) $H = 2|\mathcal{C}|$*, where* $\mathcal{C} \subset \mathcal{R}$ *is the maximal cyclic group of the rotations around* $r(R)$*.*

We say that $u \in \Lambda_0^{(\mathsf{a})}$ is simple if the corresponding (σ_u, n_u) is simple. On the basis of Definition 4.2, simple u are the ones such that u_1 does not coil around any of the axes of the rotations in $\mathcal{R}$. We remark that the subset of $\Lambda_0^{(\mathsf{a})}/\!\sim$ of the cones corresponding to simple u is infinite.

Next we introduce our second algebraic characterization of the topology of $\mathcal{K} \in \Lambda_0^{(\mathsf{a})}/\!\sim$. We identify the faces S_1 and S_2 of D, the fundamental domain introduced above, with the faces generated by ξ_1, OM and OM, OV respectively. Let $\Pi_i, i = 1, 2, 3$ be the planes of S_i and let $\tilde{R}_i \in \tilde{\mathcal{R}}$ be the reflection with respect to Π_i. Since Π_1 and Π_2 are orthogonal there is a unique $q \in \mathrm{S}_3 \cap \mathsf{S}^2$ such that $q, \tilde{R}_1 q, \tilde{R}_2 q, \tilde{R}_2\tilde{R}_1 q$ are the vertexes of a square. Since $q = \tilde{R}_3 q$, the orbit of q under $\tilde{\mathcal{R}}$ contains only $|\tilde{\mathcal{R}}|/2 = |\mathcal{R}|$ distinct points and coincides with the orbit $\{Rq\}_{R\in\mathcal{R}}$ of q under $\mathcal{R}$. The convex hull $\mathcal{Q}_{\mathcal{R}}$ of $\{Rq\}_{R\in\mathcal{R}}$ is an Archimedean polyhedron [9] which is naturally associated to the rotation group $\mathcal{R}$. Since the action of $\mathcal{R}$ on the square defined by $q, \tilde{R}_1 q, \tilde{R}_2 q, \tilde{R}_3 q$ generates $|\tilde{\mathcal{R}}|/4$ distinct squares, $\mathcal{Q}_{\mathcal{R}}$ has exactly $|\tilde{\mathcal{R}}|$ equal sides and therefore all its faces $\mathcal{F}$ are regular polygons with axes coinciding with $r(R)$ for some $R \in \mathcal{R} \setminus \{I\}$.

In Figure 8 we show the three Archimedean Polyhedra $\mathcal{Q}_{\mathcal{T}}, \mathcal{Q}_{\mathcal{O}}, \mathcal{Q}_{\mathcal{I}}$ corresponding to $\mathcal{R} = \mathcal{T}, \mathcal{O}, \mathcal{I}$ respectively. The vertex configurations

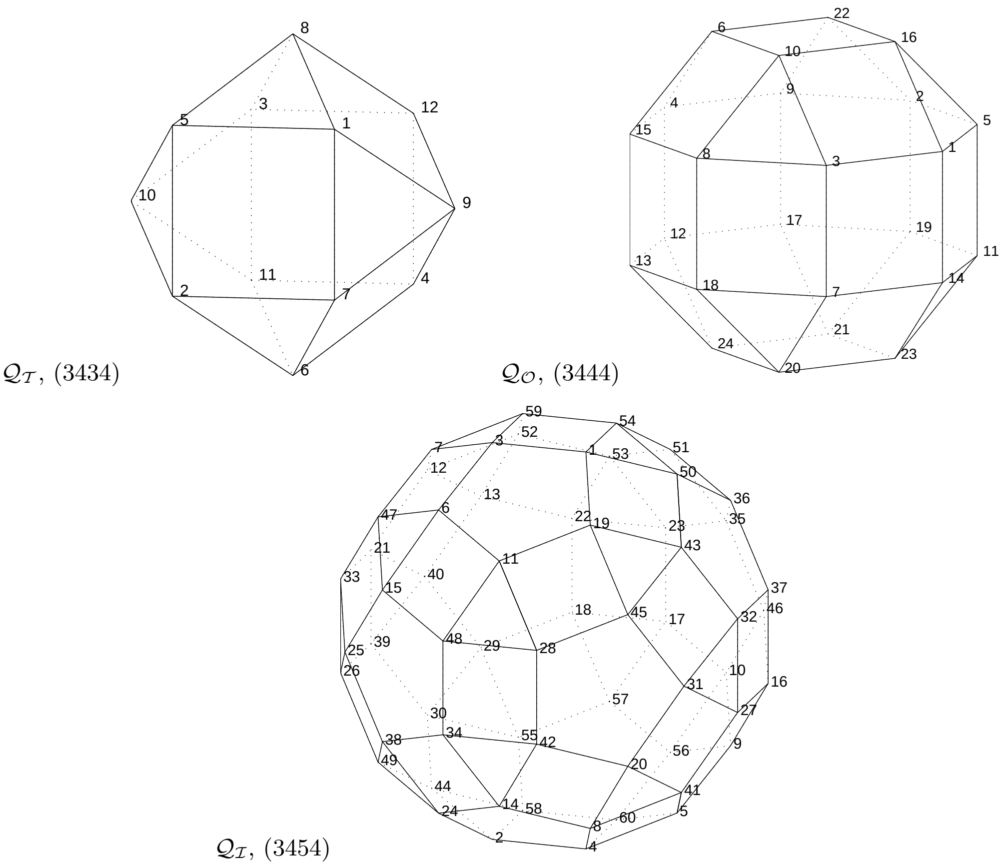


Figure 8. The three Archimedean polyhedra $\mathcal{Q}_{\mathcal{T}}, \mathcal{Q}_{\mathcal{O}}, \mathcal{Q}_{\mathcal{I}}$; the vertexes have been numbered for later reference.

[9] of $\mathcal{Q}_{\mathcal{T}}, \mathcal{Q}_{\mathcal{O}}, \mathcal{Q}_{\mathcal{I}}$ are $(3434), (3444), (3454)$. By construction $\mathcal{L}_{\mathcal{R}}$, the union of the edges of $\mathcal{Q}_{\mathcal{R}}$, avoids Γ. This property allows us to introduce another algebraic characterization of $\mathcal{K} \in \Lambda_0^{(a)}/\!\sim$.

Proposition 4.3. *Each* $\mathcal{K} \in \Lambda_0^{(a)}/\!\sim$ *uniquely determines a number* $n \in \mathbb{N}$ *and (up to translations) a periodic sequence* $\nu = \{\nu_k\}_{k\in\mathbb{Z}}$ *of vertexes of* $\mathcal{Q}_{\mathcal{R}}$ *such that*

- [i] *for each* $k \in \mathbb{Z}$ *the segment* $[\nu_k, \nu_{k+1}]$ *coincides with one of the edges of* $\mathcal{Q}_{\mathcal{R}}$*;*
- [ii] $\nu \not\subset \overline{\mathcal{F}}$*, for all the faces* $\mathcal{F}$ *of* $\mathcal{Q}_{\mathcal{R}}$*.*

Viceversa each pair (ν, n)*,* ν *a periodic sequence of vertexes of* $\mathcal{Q}_{\mathcal{R}}$ *that satisfies* [i], [ii]*, uniquely determines a cone* $\mathcal{K} \in \Lambda_0^{(a)}/\!\sim$*.*

Proof. Let c be the center of the spherical triangle $\tau = \mathrm{D} \cap \mathsf{S}^2$ and set

$$c_{\tilde{R}}(s) = (1-s)\tilde{R}c + s\tilde{R}q,\ \ s \in [0,1],\ \ \tilde{R} \in \tilde{\mathcal{R}}\ .$$

Given $u \in \mathcal{K}$ let (σ_u, n_u), $\sigma_u = \{D_k\}_{k\in\mathbb{Z}}$, $n_u \in \mathbb{N}$, be the pair associated to u by Proposition 4.2 and let $c_k(s)$ be determined by the condition

$$c_k(s) = c_{\tilde{R}}(s)\,; \quad \tilde{R}\mathrm{D} = D_k\ .$$

Moreover let $\mathrm{u}_1^{(\sigma_u,n_u,s)}$ be the map defined by (4.3) when c_k, c_{k+1} are replaced by $c_k(s), c_{k+1}(s)$ and ℓ_k by $\ell_k(s) = |c_{k+1}(s) - c_k(s)|$. The map $h(t,s) = \mathrm{u}_1^{(\sigma_u,n_u,s)}(t), (t,s) \in \mathbb{R}\times[0,1]$ defines a homotopy that transforms $\mathrm{u}_1^{(\sigma_u,n_u)} = \mathrm{u}_1^{(\sigma_u,n_u,0)}$ into $\mathrm{u}_1^{(\sigma_u,n_u,1)}$. By definition $\mathrm{u}_1^{(\sigma_u,n_u,1)}$ ranges in $\mathcal{L}_{\mathcal{R}}$ and describes with constant speed a closed path on $\mathcal{L}_{\mathcal{R}}$. This determines a sequence $\nu = \{\nu_j\}_{j\in\mathbb{Z}}$ consisting of the vertexes of $\mathcal{Q}_{\mathcal{R}}$ visited one after the other by $\mathrm{u}_1^{(\sigma_u,n_u,1)}$. If $\nu_j = \mathrm{u}_1^{(\sigma_u,n_u,1)}(t_j)$ we have

$$\nu_j = \mathrm{u}_1^{(\sigma_u,n_u,1)}(t_j + T) = \nu_{j+K}\,,$$

for some $K \in \mathbb{N}$. The integer n is determined by $n = K/K_\nu$ with K_ν the minimal period of ν. The sequence ν satisfies [i], and also [ii] since otherwise $\mathrm{u}^{(\sigma_u,n_u,1)}$ and therefore u will not satisfy condition (C).

To prove the last statement of the Proposition, given (ν, n), ν a periodic sequence of vertexes of $\mathcal{Q}_{\mathcal{R}}$ satisfying [i], [ii], we define the T-periodic map $\mathrm{v}^{(\nu,n)}$ by setting

$$\begin{cases} \gamma^{(\nu,n)} = \displaystyle\prod_{j=1}^{nK_\nu} \gamma_{\nu,j}\,, \quad \gamma_{\nu,j}(s) = (1-s)\nu_j + s\nu_{j+1}\,, \quad s \in [0,1] \\ \mathrm{v}_1^{(\nu,n)}(t) = \gamma^{(\nu,n)}(t/T)\ . \end{cases} \tag{4.14}$$

By definition the map $\mathrm{v}^{(\nu,n)}$ is T–periodic and $\mathrm{v}_1^{(\nu,n)}(\mathbb{R}) \cap \mathit{\Gamma} = \emptyset$. Moreover $\mathrm{v}_1^{(\nu,n)}$ satisfies condition (C) by [ii]. Therefore we have $\mathrm{v}^{(\nu,n)} \in \mathit{\Lambda}_0^{(\mathsf{a})}$.

□

4.2. Results on the existence and the geometric structure of periodic motions in $\mathcal{K}$

Based on the algebraic characterization of $\mathcal{K} \in \mathit{\Lambda}_0^{(\mathsf{a})}/\!\sim$ discussed in the previous Section we can now state the following

Theorem 4.1. *For each one of the sequences* ν *listed below there exists a* T*–periodic solution of the classical Newtonian* N*–body problem* *which is equivalent to* $\mathrm{v}^{(\nu,1)}$ *in the sense of Definition 4.1. (The sequences are given with reference to the enumeration of the vertexes of* $\mathcal{Q}_{\mathcal{R}}$ *in Figure 8).*

$\mathcal{R} = \mathcal{T}$

$$\begin{aligned}
\nu^1 &= [1, 8, 3, 12, 9, 7, 1]\,,\\
\nu^2 &= [1, 8, 3, 11, 6, 7, 1]\,,\\
\nu^3 &= [9, 1, 8, 12, 4, 6, 7, 9, 12, 3, 11, 4, 9]\,,
\end{aligned}$$

$\mathcal{R} = \mathcal{O}$

$$\begin{aligned}
\nu^1 &= [16, 5, 11, 14, 23, 20, 18, 8, 3, 10, 16]\,,\\
\nu^2 &= [1, 16, 10, 3, 7, 20, 23, 14, 1]\,,\\
\nu^3 &= [1, 14, 23, 11, 5, 16, 1]\,,\\
\nu^4 &= [1, 5, 16, 1, 3, 7, 18, 20, 7, 14, 1]\,,\\
\nu^5 &= [1, 16, 10, 8, 18, 7, 3, 10, 6, 15, 8, 3, 1]\,,\\
\nu^6 &= [11, 5, 2, 22, 16, 10, 6, 15, 8, 18, 13, 24, 20, 23, 21, 19, 11]\,,
\end{aligned}$$

$\mathcal{R} = \mathcal{I}$

$$\begin{aligned}
\nu^1 &= [11, 48, 34, 14, 42, 28, 11]\,,\\
\nu^2 &= [11, 48, 34, 42, 28, 11, 6, 15, 48, 28, 45, 19, 11]\,,\\
\nu^3 &= [15, 48, 34, 42, 28, 45, 31, 32, 43, 50, 36, 51, 54, 59, 52, 12, 7, 47, 33, 25, 15]\,.
\end{aligned}$$

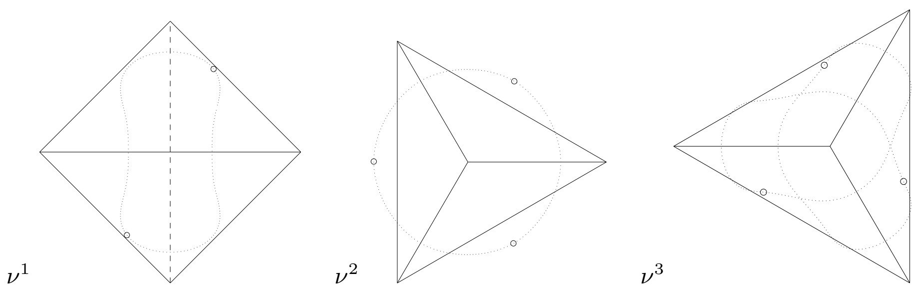


Figure 9. Typical loops in $\mathcal{K}^\nu$ corresponding to ν^1, ν^2, ν^3 for $\mathcal{R} = \mathcal{T}$.

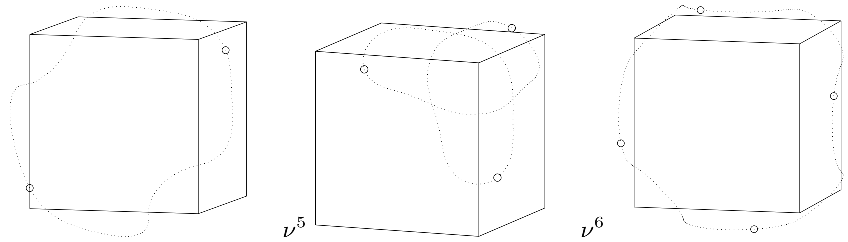


Figure 10. Typical loops in $\mathcal{K}^\nu$ corresponding to ν^1, ν^5, ν^6 for $\mathcal{R} = \mathcal{O}$.

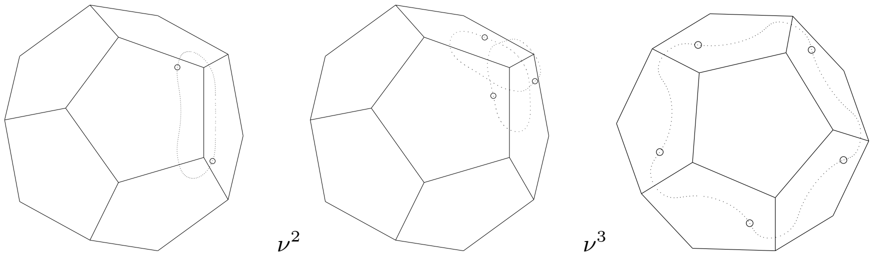


Figure 11. Typical loops in $\mathcal{K}^\nu$ corresponding to ν^1, ν^2, ν^3 for $\mathcal{R} = \mathcal{I}$.

For later reference we observe that if ν is one of the particular sequences listed in Theorem 4.1, then we can associate to the corresponding cone

$$\mathcal{K}^\nu = \mathcal{K}(\mathrm{v}^{(\nu,1)})$$

(where $\mathrm{v}^{(\nu,1)}$ is the map defined in (4.14)) a plane Π with the associated reflection $\tilde{R}_\Pi \in \tilde{\mathcal{R}} \backslash \{I\}$, a number $M \in \{2,3,4,5\}$ and a rotation $R \in \mathcal{R} \setminus \{I\}$ of angle $2\pi/M$ such that the symmetry conditions

$$\begin{cases} u_1(t) = \tilde{R}_\Pi u_1(-t) \\ u_1(t + T/M) = R u_1(t) \end{cases} \tag{4.15}$$

are compatible with membership in $\mathcal{K}^\nu$. This is a straightforward consequence of the fact that the map $\mathrm{v}_1^{(\nu,1)}$ itself satisfies these conditions for suitable $\tilde{R}_\Pi, M, R$. For each ν the corresponding value of M is given in column 3 of Table 3. We denote by $\tilde{\mathcal{K}}^\nu \subset \mathcal{K}^\nu$ the subset of $\mathcal{K}^\nu$ of the maps that satisfy (4.15). Figures 9, 10, 11 illustrate the structure of the map u_1 for a typical element $u \in \tilde{\mathcal{K}}^\nu$.
In preparation for the proof of Theorem 4.1 we note the following

Proposition 4.4. *Given* $u \in \Lambda_0^{(\mathsf{a})}$ *there exist* $\hat{u} \sim u$, $\delta_u > 0$ *and sequences* $\{t^k\}, \{S_k\}, \{D_k\}$, $k \in \mathbb{Z}$ *such that*

(i) $\hat{u}_1(t^k) \in S_k\,,$
(ii) $\hat{u}_1((t^k, t^k + \delta_u)) \subset D_k\,,\ \ \hat{u}_1([t^k, t^{k+1}]) \subset \overline{D_k} \setminus \Gamma\,,$

(iii)

$$S_{k+1} \neq S_k \,, \tag{4.16}$$

(iv) $\mathcal{A}(\hat{u}) \leq \mathcal{A}(u)$.

Proof. We identify $\hat{u}$ with the map constructed in the proof of Proposition 4.2. Then (i), (ii), (iii) hold trivially. To prove (iv) we observe that, given $S \in \{\mathcal{S}\}$ there is a bijection $\mathcal{R} \setminus \{I\} \ni R \to R^S \in \mathcal{R} \setminus \{I\}$ such that (see Figure 12)

$$\tilde{R}_S R x = R^S \tilde{R}_S x, \;\; x \in \mathbb{R}^3 \;. \tag{4.17}$$

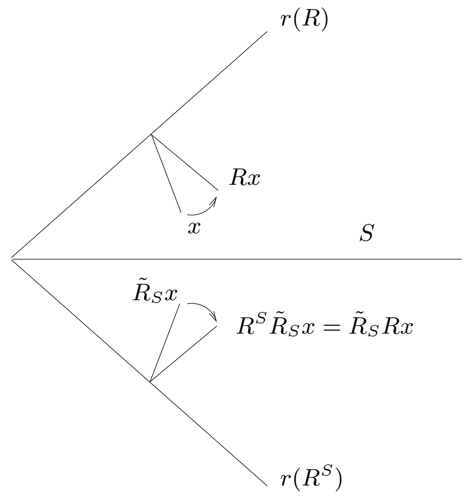


Figure 12. Skematic illustration of R, $\tilde{R}_S$, R^S and equation (4.17)

From this it follows

$$|R^S \tilde{R}_S x - \tilde{R}_S x| = |\tilde{R}_S (Rx - x)| = |Rx - x|$$

and therefore we have, for every $t \in \mathbb{R}$ and for each $S \in \mathcal{S}$,

$$\sum_{R \in \mathcal{R} \setminus \{I\}} \frac{1}{|R\tilde{R}_S u_1(t) - \tilde{R}_S u_1(t)|} = \sum_{R \in \mathcal{R} \setminus \{I\}} \frac{1}{|R^S \tilde{R}_S u_1(t) - \tilde{R}_S u_1(t)|} =$$

$$= \sum_{R \in \mathcal{R} \setminus \{I\}} \frac{1}{|Ru_1(t) - u_1(t)|} \;.$$

This shows that the reflections used in the proof of Proposition 4.2 to construct $\check{u}_1$ and then $\hat{u}_1$ do not change the potential term of the action. This and the fact that also the kinetic part of the action is unchanged by a reflection proves (iv).

□

Before proceeding we observe that, on the basis of the coercivity of $\mathcal{A}|_{\mathcal{K}}$ proved in Propositions 2.1, 3.1, 4.1, standard arguments from Calculus of Variations [10], [15], [30] yield the existence of a minimizer $u_* \in H^1_T(\mathbb{R},\mathcal{X}) \cap \overline{\mathcal{K}}$, $\overline{\mathcal{K}}$ the closure of $\mathcal{K}$ in the C^0–topology. We also remark that, in all cases where it can be proved that $u_* \notin \partial\mathcal{K}$, we can invoke the *principle of symmetric criticality* [24] to deduce that u_* is actually a critical point of the unconstrained action functional.

An interesting consequence of Proposition 4.4 is the following

Theorem 4.2. *Given* $u \in \Lambda_0^{(\mathsf{a})}$*, assume* $u_* \in \mathcal{K}(u)$ *is a collision free minimizer of* $\mathcal{A}|_{\mathcal{K}(u)}$*. Then* $\Theta_* := \{t \in [0,T) : u_{*,1}(t) \in S, S \in \mathcal{S}\}$ *is a finite set and*

(i) $\#\Theta_* = n_u K_u$ *(*n_u, K_u *as in Proposition 4.2) and therefore* $\#\Theta_*$ *is the minimum compatible with the topological structure of* u*;*
(ii) if $u_{*,1}(\bar{t}) \in S$ *for some* $S \in \mathcal{S}$ *and some* $\bar{t} \in [0,T)$*, then* $\dot{u}_{*,1}(\bar{t})$ *is transversal to* S*.*

Proof. Since u_* is a collision free minimizer, by elliptic regularity it is a smooth function, therefore, if $\bar{t} \in [0,T)$ is such that $u_{*,1}(\bar{t}) \in S$ and $\dot{u}_{*,1}(\bar{t})$ is not transversal to S, then necessarily $\dot{u}_{*,1}(\bar{t})$ is parallel to S. This implies

$$\begin{cases} \tilde{R}_S R u_{*,1}(\bar{t}) = R^S \tilde{R}_S u_{*,1}(\bar{t}) = R^S u_{*,1}(\bar{t}) \\ \tilde{R}_S R \dot{u}_{*,1}(\bar{t}) = R^S \tilde{R}_S \dot{u}_{*,1}(\bar{t}) = R^S \dot{u}_{*,1}(\bar{t}) \end{cases}, \qquad R \in \mathcal{R}\,. \tag{4.18}$$

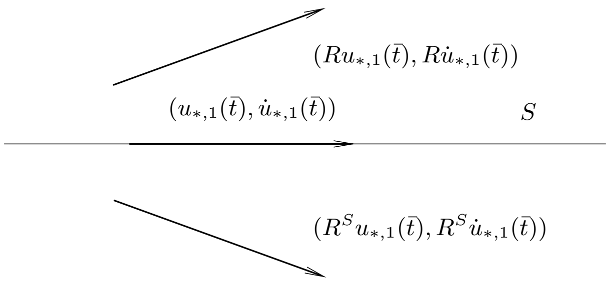


Figure 13. The reflection $\tilde{R}_S$ changes the set $\{(Ru_{*,1}(\bar{t}), R\dot{u}_{*,1}(\bar{t})), R \in \mathcal{R}\}$ into itself.

Equations (4.18) say that at time $t = \bar{t}$ the set of positions and velocities of the $N = |\mathcal{R}|$ particles of the system is changed into itself by the reflection $\tilde{R}_S$. From the symmetry of the equations of motion it follows that the same is true for all t. In particular this implies that $u_{*,1}(t)$ belongs to the plane of S for all t. This is clearly incompatible with membership in $\mathcal{K}(u)$. Therefore (ii) is established. Let $\hat{u}_*$ be the map associated to u_* as in Proposition 4.4, then $\hat{u}_*$ is a minimizer and if (i) does not hold then $\hat{u}_* \neq u_*$. In particular there is a $\bar{t} \in \mathbb{R}$ and $S \in \mathcal{S}$ such that $\hat{u}_{*,1}(\bar{t}) = u_{*,1}(\bar{t}) \in S$ and

$$\dot{\hat{u}}_{*,1}(\bar{t}^-) = \dot{u}_{*,1}(\bar{t}) \neq \tilde{R}_S \dot{u}_{*,1}(\bar{t}) = \dot{\hat{u}}_{*,1}(\bar{t}^+)$$

since by (ii) $\dot{u}_{*,1}(\bar{t})$ is transversal to S. This is in contradiction with (ii).

□

Remark 4.1. Let $\mathcal{K} \in \Lambda_0^{(a)}/\sim$ and (σ, n) be the corresponding pair. An important consequence of Theorem 4.2 is that in the search of classical $T-$ periodic solutions, minimizing on $\mathcal{K}$ is equivalent to minimize on the subset of $\mathcal{K}$ of the loops u such that the map $t \to u_1(t)$ visits periodically one after the other all the chambers D_k in the sequence $\{D_1, \ldots, D_{nK_\sigma}\}$ entering D_k from S_k and exiting D_k from S_{k+1} without touching the third face $S^k \notin \{S_k, S_{k+1}\}$ of D_k.

Remark 4.1 has important consequences on the kind of partial collisions that have to be excluded in the proof that a minimizer $u_* \in \overline{\mathcal{K}}$ is collision free. First of all, if a minimizer u_* has a partial collision at time t_c, then there is k such that $u_{*,1}(t_c) \neq 0$ belongs to one of the semiaxes on the boundary of D^k. This largely reduces the set of partial collisions that $u_* \in \mathcal{K}$ may exhibit. Since each time we add one of the sets $\overline{D_k}$ to the previous one $\overline{D_{k-1}}$ we introduce the semiaxis $(\overline{S_{k+1}} \cap \overline{S^k}) \setminus \{0\}$; the semiaxes on the boundary of D_k, $k \in \mathbb{Z}$ all appear in the sequence $k \to r_k := (\overline{S_{k+1}} \cap \overline{S^k}) \setminus \{0\}$. Therefore the sequence $k \to r_k$ characterizes the semiaxes where partial collisions can occur for the particular cone $\mathcal{K}$ under consideration (see Figure 14). Other consequences of the preceding discussion on the possible partial collisions and on their geometric structure will be considered in Section 5 below.

Remark 4.2. Theorem 4.2 and Remark 4.1 apply also to minimizers $u_* \in \tilde{\mathcal{K}}^\nu$ of $\mathcal{A}|_{\tilde{\mathcal{K}}^\nu}$, with $\tilde{\mathcal{K}}^\nu \subset \mathcal{K}^\nu$ the cone introduced above (cfr. (4.15).

5. Collisions

In this Section we always denote by $\mathcal{K}$ either $\mathcal{K}_4$, or $\mathcal{K}_i^P$, or $\tilde{\mathcal{K}}^\nu$ with ν as in Theorem 4.1. We show that minimizers $u_* \in \mathcal{K}$ are collision free. We start by excluding total collisions. In Section 5.2 we shall discuss the case of partial collisions.

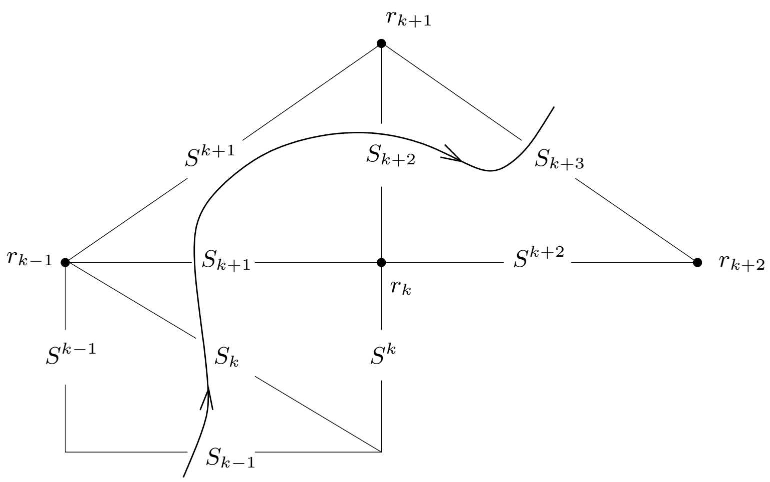


Figure 14. The sequence $k \to r_k = (\overline{S_{k+1}} \cap \overline{S^k}) \setminus \{0\}$.

5.1. Total collisions

Our strategy to show that actually in all cases a minimizer $u_* \in \overline{\mathcal{K}}$ does not have total collisions is based on *level estimates*, that is we show

(a) the assumption that u_* has a total collision implies a bound of the form

$$\mathcal{A}(u_*) \geq a > 0\,;$$

(b) there exists $v \in \mathcal{K}$ such that

$$\mathcal{A}(v) < a\ .$$

This approach is quite natural in our context. Indeed a total collision implies $u_* \in \partial\mathcal{K}$ and therefore any attempt to perturb u_* into a competing function v such that $\mathcal{A}(v) < \mathcal{A}(u_*)$ to show that u_* is free of total collisions runs against the difficulty of respecting the topological constraints that characterize membership in $\mathcal{K}$.

We begin with the cone $\mathcal{K}_4$.

Proposition 5.1. *Assume $u \in \overline{\mathcal{K}_4}$ has a total collision. Then*

$$\mathcal{A}(u) \geq a_4 = \frac{18}{2^{\frac{1}{3}}}\pi^{\frac{2}{3}}T^{\frac{1}{3}}$$

Proof. From the definition of $\mathcal{K}_4$ it follows that, if $u_1 = (u_{11}, u_{12}, u_{13})$,

$$\begin{aligned}
\mathcal{A}(u) &= \int_0^T \left(2|\dot{u}_1|^2 + \frac{1}{\sqrt{u_{11}^2 + u_{12}^2}} + \frac{1}{\sqrt{u_{11}^2 + u_{13}^2}} + \frac{1}{\sqrt{u_{12}^2 + u_{13}^2}} \right) dt \\
&= \int_0^T \left(\frac{1}{2}[2(\dot{u}_{11}^2 + \dot{u}_{12}^2)] + \frac{1}{\sqrt{u_{11}^2 + u_{12}^2}} \right) dt + \int_0^T \left(\frac{1}{2}[2(\dot{u}_{12}^2 + \dot{u}_{13}^2)] + \right. \\
&+ \left. \frac{1}{\sqrt{u_{12}^2 + u_{13}^2}} \right) dt + \int_0^T \left(\frac{1}{2}[2(\dot{u}_{11}^2 + \dot{u}_{13}^2)] + \frac{1}{\sqrt{u_{11}^2 + u_{13}^2}} \right) dt \\
&\stackrel{def}{=} a(u_{11}, u_{12}, T) + a(u_{12}, u_{13}, T) + a(u_{11}, u_{13}, T) \ . \qquad (5.1)
\end{aligned}$$

If m_1, m_2, K are positive constants such that

$$\frac{m_1 m_2}{m_1 + m_2} = 2 \, , \qquad K m_1 m_2 = 1 \, ,$$

and $\xi : \mathbb{R} \to \mathbb{R}^2$ is a periodic map of period $\mathtt{T}$, we can write

$$a(\xi_1, \xi_2, \mathtt{T}) = \int_0^{\mathtt{T}} \left(\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{\xi}|^2 + \frac{K m_1 m_2}{|\xi|} \right) dt \ .$$

Gordon's Theorem [18] implies that if $\xi(t)$ vanishes at some $t \in [0, \mathtt{T})$, then

$$a(\xi_1, \xi_2, \mathtt{T}) \geq 3 \left(\frac{K^2 \pi^2}{2(m_1 + m_2)} \right)^{\frac{1}{3}} m_1 m_2 \mathtt{T}^{\frac{1}{3}} = 3 \pi^{\frac{2}{3}} \mathtt{T}^{\frac{1}{3}} \ . \qquad (5.2)$$

If $u \in \overline{\mathcal{K}_4}$ has a total collision then $u(t)$ vanishes at least at two times t_1, t_2, with $t_2 - t_1 = T/2$, in the interval $[0, T)$. From this, (5.1) and (5.2) we get, for $\mathtt{T} = T/2$,

$$\mathcal{A}(u) \geq 3 \cdot 2 \cdot 3 \pi^{\frac{2}{3}} \mathtt{T}^{\frac{1}{3}} = \frac{18}{2^{\frac{1}{3}}} \pi^{\frac{2}{3}} T^{\frac{1}{3}} \ .$$

□

Proposition 5.2. *There exists $v \in \mathcal{K}_4$ such that*

$$\mathcal{A}(v) < \frac{18}{3^{\frac{1}{3}}} \pi^{\frac{2}{3}} T^{\frac{1}{3}} \ .$$

Proof. We define v by describing the motion v_1 of P_1, the generating particle. P_1 moves with constant speed on a closed curve which is the union of four half circumferences $C_1^{\pm}, C_2^{\pm}$ of radius $\rho > 0$. $C_1^{\pm}$ has center on the axis ξ_3 and lies on the plane $\xi_3 = \pm\rho$. $C_2^{\pm}$ has center on the axis ξ_2 and lies on the plane $\xi_2 = \pm\rho$. The kinetic part $A_K(v)$ of $\mathcal{A}(v)$ is given by

$$A_K(v) = 4 \cdot \frac{T}{2}\left(\frac{4\pi\rho}{T}\right)^2 = 32\frac{\pi^2\rho^2}{T} \ .$$

From the definition of v it follows that $|v_i - v_j| \geq 2\rho$ whenever $i \neq j$. Then the potential part $A_U(v)$ of $\mathcal{A}(v)$ satisfies

$$A_U(v) = \int_0^T \sum_{i<j} \frac{1}{|v_i - v_j|} < 6\frac{T}{2\rho} \, ,$$

therefore we have

$$\mathcal{A}(v) < 32\frac{\pi^2\rho^2}{T} + 3\frac{T}{\rho} \, , \qquad \forall \rho > 0 \ . \tag{5.3}$$

If we choose $\rho = \left(\frac{3T^2}{64\pi^2}\right)^{\frac{1}{3}}$, to minimize the r.h.s. of (5.3), we obtain

$$\mathcal{A}(v) < \frac{18}{3^{\frac{1}{3}}}\pi^{\frac{2}{3}}T^{\frac{1}{3}} \ .$$

□

Propositions (5.1) and (5.2) imply that a minimizer $u_* \in \overline{\mathcal{K}}$ can not have total collisions. We now consider the cases $\mathcal{K} = \mathcal{K}_i^P$ or $\mathcal{K} = \tilde{\mathcal{K}}^\nu$.

Proposition 5.3. *Let $\Omega \subset \mathbb{R}^s$ be an open connected set, $L_k : \Omega \times \mathbb{R}^s \to \mathbb{R}$, $0 \leq k \leq M$ be smooth Lagrangian functions and $Q_k \subset \{u : u \in H_T^1(\mathbb{R}, \mathbb{R}^s), u(\mathbb{R}) \subset \Omega\}$. Let $\mathcal{A}_k : Q_k \to \mathbb{R}$ be the action functional*

$$\mathcal{A}_k(q) = \int_0^T L_k(q, \dot{q}) \, dt \, , \qquad 0 \leq k \leq M \ .$$

Assume

(i) $\sum_{k=1}^M L_k(x,y) \leq L_0(x,y), \ \ \forall (x,y) \in \Omega \times \mathbb{R}^s$,
(ii) $Q_0 \subset Q_k, \ \ 1 \leq k \leq M$.

Then

$$\sum_{k=1}^M \inf_{q \in Q_k} \mathcal{A}_k(q) \leq \inf_{q \in Q_0} \mathcal{A}_0(q).$$

Proof. (i) and (ii) imply

$$\sum_{k=1}^M \inf_{q \in Q_k} \mathcal{A}_k(q) \leq \sum_{k=1}^M \inf_{q \in Q_0} \mathcal{A}_k(q) \leq \inf_{q \in Q_0} \sum_{k=1}^M \mathcal{A}_k(q) \leq \inf_{q \in Q_0} \mathcal{A}_0(q) \ .$$

□

Proposition 5.4. *Assume $u \in \Lambda_0^{(\mathsf{a})}$ has a total collision. Then*

$$\mathcal{A}(u) \geq 3N \left(\frac{\pi^2 (N-1)^2}{32}\right)^{1/3} T^{1/3}\,, \qquad \textit{with } N = |\mathcal{R}|\ . \tag{5.4}$$

Proof. To prove (5.4) we apply Proposition 5.3 with $M = 1$ and

$$L_0(x,y) = \frac{N}{2}\left(|y|^2 + \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)x|}\right),$$

$$Q_0 = \{u_1 \in H^1_T(\mathbb{R}, \mathbb{R}^3 \setminus \Gamma) : \lim_{t\to\{0,T\}} u_1(t) = 0\},$$

$$L_1(x,y) = \frac{N}{2}\left(\frac{(y\cdot x)^2}{x\cdot x} + \frac{(N-1)}{2|x|}\right),$$

$$Q_1 = \{u_1 \in H^1_T(\mathbb{R}, \mathbb{R}^3 \setminus \{0\}) : \lim_{t\to\{0,T\}} u_1(t) = 0\}\ .$$

Set $\rho = |u_1|$. Then we have

$$\mathcal{A}_1(u) = \frac{N}{2}\int_0^T \left(\dot\rho^2 + \frac{N-1}{2\rho}\right) dt \stackrel{def}{=} N a_1(\rho)\ . \tag{5.5}$$

Therefore minimizing $\mathcal{A}_1$ on Q_1 is equivalent to the minimization of a_1 on $H^1_T(\mathbb{R}, (0,+\infty))$. From (5.5) a_1 is the action of a system of two masses $m_1 = m_2 = 1/2$ interacting with Newtonian potential of gravitational constant $2(N-1)$. From this and Gordon's Theorem it follows

$$a_1(\rho) \geq 3\left(\frac{\pi^2(N-1)^2}{32}\right)^{1/3} T^{1/3}\ .$$

This and (5.5) imply (5.4).

□

The estimate (5.4) in Proposition 5.4 is based on the assumption, fulfilled by any $u \in \Lambda_0^{(\mathsf{a})}$, that at each time all the particles have the same distance ρ from the origin O, and on the obvious bound $|u_i - u_j| \leq 2\rho$, which implies

$$\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u_1|} \geq (N-1)\frac{1}{2\rho}\ .$$

Simple geometric observations allow for sharper estimates. For instance for $\mathcal{R} = \mathcal{O}$ one can observe that at each time t the generating particle P_1 is the vertex of 3 squares and 4 equilateral triangles, the other vertexes of which are all particles distinct from each other and from P_1. Since the maximum of the side of a square with vertexes on a sphere with radius ρ is $\sqrt{2}\rho$ and the maximum of the side of an equilateral triangle with vertexes on a sphere with radius ρ is $\sqrt{3}\rho$, we conclude that at each time t there are 6 particles at distance $\leq \sqrt{2}\rho$ from P_1 and 8 particles at distance $\leq \sqrt{3}\rho$ from P_1. From these observations it follows that in the case $\mathcal{R} = \mathcal{O}$ we have

$$\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u_1|} \geq \left(\frac{9}{2} + \frac{6}{\sqrt{2}} + \frac{8}{\sqrt{3}}\right)\frac{1}{\rho} > 1.161 \cdot \frac{23}{2\rho} \,. \tag{5.6}$$

Similarly, if $\mathcal{R} = \mathcal{T}$, P_1 is the vertex of 4 equilateral triangles, therefore we have

$$\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u_1|} \geq \left(\frac{3}{2} + \frac{8}{\sqrt{3}}\right)\frac{1}{\rho} > 1.112 \cdot \frac{11}{2\rho} \,. \tag{5.7}$$

Finally, if $\mathcal{R} = \mathcal{I}$, P_1 is a vertex of 6 regular pentagons and 10 equilateral triangles. For each pentagon there are 2 particles whose distance from P_1 is equal to the side of the pentagon and 2 particles at distance equal to $2\cos(\pi/10)$ times the radius of the circumcircle. Therefore we have the estimate

$$\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u_1|} \geq \left(\frac{15}{2} + \frac{12}{2\sin(\pi/5)} + \frac{12}{2\cos(\pi/10)} + \frac{20}{\sqrt{3}}\right)\frac{1}{\rho} > 1.205 \cdot \frac{59}{2\rho} \,. \tag{5.8}$$

We denote by $a_\mathcal{R}$ the right–hand side of (5.4) and by $a'_\mathcal{R}$ the analogous lower bound obtained by using the improved estimates (5.6), (5.7), (5.8). In Table 1 we list the values of $a_\mathcal{R}, a'_\mathcal{R}$ (approximated by truncation). The lower bounds given in Table 1 applies to any $u \in \overline{\mathcal{K}}$,

Table 1. Lower bounds for the action in case of total collision.

$\mathcal{R}$	$a_\mathcal{R}/T^{1/3}$	$a'_\mathcal{R}/T^{1/3}$
$\mathcal{T}$	120.3042	129.1665
$\mathcal{C}$	393.4301	434.8151
$\mathcal{I}$	1843.1348	2087.7547

which is known to have one total collision per period. If u has $M > 1$ total collisions per period and the time intervals between subsequent collisions are all equal, then from (5.4) we derive

$$\mathcal{A}(u) \geq M a_\mathcal{R} \frac{1}{M^{1/3}} = M^{2/3} a_\mathcal{R} \tag{5.9}$$

and the same is true with $a'_\mathcal{R}$ in place of $a_\mathcal{R}$. This observation applies in particular to the cones $\mathcal{K}_i^P$. For $u \in \mathcal{K}_i^P$, the definition of $\mathcal{K}_i^P$ (cfr. (a), (b), (c)) implies

$$|u(t + T/H)| = |u(t)|, \quad \forall t \in \mathbb{R}\,,$$

therefore if u has a total collision at time t_c it also has a total collision at time $t_c + T/H$ and we can apply (5.9) with $M = H$. In Table 2 (lines 2, 3) we list the values of $H^{2/3}a_\mathcal{R}/T^{1/3}, H^{2/3}a'_\mathcal{R}/T^{1/3}$.

Table 2. Values of the action for $u \in \mathcal{K}_i^P$.

	$\mathfrak{T}$	$\mathfrak{C}$	$\mathfrak{O}$	$\mathfrak{D}$	$\mathfrak{I}$
H	3	4	3	5	3
$H^{2/3}a_{\mathcal{R}}/T^{1/3}$	250.2428	991.3818	818.3676	5389.3588	3833.8749
$H^{2/3}a'_{\mathcal{R}}/T^{1/3}$	268.6772	1095.6654	904.4519	6104.6318	4342.7048
$\mathcal{A}(v)/T^{1/3}, i = 1$	220.2007	734.9502	589.9526	2866.6116	2027.2544
$\mathcal{A}(v)/T^{1/3}, i = 2$	168.0446	553.1633	589.9526	2181.2066	2452.2053
$\mathcal{A}(v)/T^{1/3}, i = 3$	266.7542	896.4157	819.8050	3477.7486	3208.5266

Table 3. Values of the action for $u \in \tilde{\mathcal{K}}^\nu$.

$\mathcal{R}$	ν	M	$M^{2/3}a_{\mathcal{R}}/T^{1/3}$	$M^{2/3}a'_{\mathcal{R}}/T^{1/3}$	$\mathcal{A}(v)/T^{1/3}$
$\mathcal{T}$	1	2	190.9710	205.0391	168.0445
	2	3	250.2428	268.6772	168.0445
	3	3	250.2428	268.6772	266.7542
$\mathcal{O}$	1	2	624.5314	690.2260	647.2635
	2	2	624.5314	690.2260	553.1632
	3	2	624.5314	690.2260	462.9895
	4	2	624.5314	690.2260	647.2635
	5	3	818.3676	904.4519	724.8489
	6	4	991.3818	1095.6654	859.5748
$\mathcal{I}$	1	2	2925.7941	3314.1040	1556.2362
	2	3	3833.8749	4342.7048	2463.1128
	3	5	5389.3588	6104.6318	3447.1168

The inequality (5.9) can also be applied to $u \in \tilde{\mathcal{K}}^\nu$. In Table 3, for each ν considered in Theorem 4.1 we list the corresponding values of $M \in \{2, 3, 4, 5\}$ and the lower bounds given by (5.9) .

To complete the proof that minimizers $u_* \in \overline{\mathcal{K}}, \mathcal{K} = \mathcal{K}_i^P$ or $\mathcal{K} = \tilde{\mathcal{K}}^\nu$, ν as in Theorem 4.1, are free of total collisions we now show that in all these cases there exists $v \in \mathcal{K}$ which is collision free and has a value $\mathcal{A}(v)$ of the action below the lower bounds discussed above. For $\mathcal{K} = \tilde{\mathcal{K}}^\nu$ we choose $v = \lambda \mathrm{v}^{(\nu,1)}$, where $\mathrm{v}^{(\nu,1)}$ is defined by (4.14) and $\lambda > 0$ will be chosen later. For the cones $\mathcal{K}_i^P$ the sequence ν is not uniquely determined. On the other hand, if $u \in \mathcal{K}_i^P$ then the map u_1 must necessarily visit, and in a well determined order, certain domains $D \in \mathcal{D}$. This determines a minimal sequence ν compatible with membership in $\mathcal{K}_i^P$. This minimal sequence ν is the one we use for defining our test function $v = \lambda \mathrm{v}^{(\nu,1)} \in \mathcal{K}_i^P$. With reference to the

Table 4. Minimal sequences for the cones $\mathcal{K}_i^P$ considered in Theorem 3.1.

P	$\mathcal{K}_i^P$	ν
$\mathfrak{T}$	1	[2, 7, 1, 9, 12, 8, 3, 10, 5, 2]
	2	[2, 10, 3, 12, 9, 7, 2]
	3	[2, 10, 5, 8, 3, 12, 8, 1, 9, 7, 1, 5, 2]
$\mathfrak{C}$	1	[5, 1, 16, 10, 3, 8, 18, 7, 20, 23, 14, 11, 5]
	2	[5, 16, 10, 8, 18, 20, 23, 11, 5]
	3	[1, 5, 16, 1, 3, 10, 8, 3, 7, 18, 20, 7, 14, 23, 11, 14, 1]
$\mathfrak{O}$	1	[3, 1, 16, 10, 6, 15, 8, 18, 7, 3]
	2	[1, 16, 22, 6, 15, 13, 18, 7, 14, 1]
	3	[3, 10, 16, 22, 6, 10, 8, 15, 13, 18, 8, 3, 7, 14, 1, 3]
$\mathfrak{D}$	1	[1, 54, 59, 3, 7, 47, 6, 15, 48, 11, 28, 45, 19, 43, 50, 1]
	2	[54, 59, 7, 47, 15, 48, 28, 45, 43, 50, 54]
	3	[54, 1, 3, 59, 7, 3, 6, 47, 15, 6, 11, 48, 28, 11, 19, 45, 43, 19, 1, 50, 54]
$\mathfrak{I}$	1	[28, 45, 19, 11, 6, 15, 48, 34, 42, 28]
	2	[45, 19, 1, 3, 6, 15, 25, 38, 34, 42, 20, 31, 45]
	3	[45, 28, 11, 19, 1, 3, 6, 11, 48, 15, 25, 38, 34, 48, 28, 42, 20, 31, 45]

numbering of the vertexes of $\mathcal{Q}_{\mathcal{R}}$ in Figure 8, we list in Table 4 the minimal ν corresponding to each $\mathcal{K}_i^P$.

Let $A_K = A_K^{(\nu,n)}, A_U = A_U^{(\nu,n)}$, the kinetic and the potential part of the action:

$$A_K = \frac{N}{2}\int_0^T |\dot{\mathrm{v}}_1^{(\nu,n)}|^2\, dt\,, \qquad A_U = \frac{N}{2}\int_0^T \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)\mathrm{v}_1^{(\nu,n)}|}\, dt\ . \tag{5.10}$$

We choose $\lambda = (\frac{A_U}{2A_K})^{1/3}$, that gives to $\mathcal{A}(\lambda\mathrm{v}^{(\nu,n)}) = \lambda^2 A_K + \frac{1}{\lambda}A_U$ its minimum value

$$\mathcal{A}(v) = 3\Big(\frac{A_K A_U^2}{4}\Big)^{1/3}\ . \tag{5.11}$$

The main reason for considering test functions of the form $v = \lambda\mathrm{v}^{(\nu,n)}$ is that the piecewise affine character of $\mathrm{v}^{(\nu,n)}$ implies that A_U is the sum of $N-1$ elementary integrals and by consequence $\mathcal{A}(v)$ has an explicit analytic expression. Moreover, on the basis of simple observations, the computation of A_U, and in turn the computation of $\mathcal{A}(v)$, can be reduced to a purely algebraic fact. Going back to the construction of the polyhedron $\mathcal{Q}_{\mathcal{R}}$ outlined in Section 4 we set $q_i = \tilde{R}_i q, i = 1,2$ (cfr. Section 4 for the definition of q and $\tilde{R}_i$) and we deduce from the discussion in Section 4 that the set $\mathcal{L}_{\mathcal{R}}$ of the sides of $\mathcal{Q}_{\mathcal{R}}$ is the union of the two orbits $\{R[q,q_i]\}_{R\in\mathcal{R}}, i = 1,2$, of the segments $[q,q_i], i = 1,2$. It follows that we can associate to each $j \in \mathbb{Z}$ a uniquely determined pair $(R_j, i_j) \in \mathcal{R}\times\{1,2\}$ such that

$[\nu_{j-1},\nu_j]=R_j[q,q_{i_j}]$. For each given $R'\in\mathcal{R}$ set

$$\upsilon_i(R')=\int_0^1\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{ds}{\big|(R-I)R'[(1-s)q+sq_i]\big|}\,,\qquad i=1,2\ .$$

Since

$$\big|(R-I)R'[(1-s)q+sq_i]\big|=\big|((R')^{-1}RR'-I)[(1-s)q+sq_i]\big|$$

and the map $R\mapsto(R')^{-1}RR'$ is an isomorphism of $\mathcal{R}$ onto itself, we have

$$\upsilon_i(R')=\upsilon_i(I)\stackrel{def}{=}\upsilon_i\ .$$

From this and the fact that $v_1^{(\nu,n)}$ travels each side $[\nu_{j-1},\nu_j]$ in a time interval of size $T/(nK_\nu)$, it follows

$$\begin{aligned}A_U&=\frac{N}{2}\frac{T}{nK_\nu}\sum_{j=1}^{nK_\nu}\int_0^1\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{ds}{\big|(R-I)R_j[(1-s)q+sq_{i_j}]\big|}=\\&=\frac{N}{2}\frac{T}{nK_\nu}(N_1\upsilon_1+N_2\upsilon_2)\,,\end{aligned}\tag{5.12}$$

where N_i is the number of sides $[\nu_{j-1},\nu_j]$ in the orbit of $[q,q_i]$.

There is a simple geometric criterium to decide whether $[\nu_{j-1},\nu_j]$ is in the orbit of $[q,q_i]$. Besides the $|\tilde{\mathcal{R}}|/4$ squares, the other faces of $\mathcal{Q}_{\mathcal{R}}$ are the images under some $R\in\mathcal{R}$ of the polygons $\mathcal{F}_i$ with vertexes $\{Rq\}_{R\in C_i},i=1,2$, where C_1,C_2 are the cyclic groups of the rotations with axis ξ_1 and OV respectively. The side $[\nu_{j-1},\nu_j]$ is in the orbit of $[q,q_i]$ if it is one of the sides of $R\mathcal{F}_i$ for some $R\in\mathcal{R}$.

In the case $\mathcal{R}=\mathcal{T}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are both equilateral triangles and it is straightforward to check that we have $\upsilon_1=\upsilon_2\stackrel{def}{=}\upsilon$, and (5.12) becomes [3]

$$A_U=6T\upsilon\ .$$

From (5.11), (5.12) and

$$A_K=\frac{N}{2}\frac{\ell^2n^2K_\nu^2}{T}\,,$$

[3] We write the explicit expression of υ for the case $\mathcal{R}=\mathcal{T}$:

$$\begin{aligned}\upsilon=&-\ln\left(\sqrt{2}-1\right)\sqrt{2}-2\,\ln\left(2-\sqrt{3}\right)-2\,\ln(3)+\\&+2/3\sqrt{3}\ln(3)+2\,\ln\left(2+\sqrt{3}\right)-2/3\,\ln\left(2-\sqrt{3}\right)\sqrt{3}+\\&+2/3\,\ln\left(2+\sqrt{3}\right)\sqrt{3}-\ln\left(\sqrt{2}-1\right)\ .\end{aligned}$$

We omit the analogous, but longer, expressions of υ_1,υ_2 for $\mathcal{R}=\mathcal{O},\mathcal{I}$.

where ℓ is the length of a side of $\mathcal{Q}_{\mathcal{R}}$, we finally obtain

$$\mathcal{A}(v) = \frac{3}{2 \cdot 4^{1/3}} N \ell^{2/3} (N_1 v_1 + N_2 v_2)^{2/3} T^{1/3} \ . \tag{5.13}$$

In the last column of Table 3 we list the values of $\mathcal{A}(v)$ given by (5.13) for the cases considered in Theorem 4.1. We recall that the function $\mathrm{v}_1^{(\nu,n)}$, defined in (4.14), will automatically satisfy condition (4.15). That is $\mathrm{v}^{(\nu,n)} \in \tilde{\mathcal{K}}^\nu$. This and the fact that the values of $\mathcal{A}(v)$ given in the last column of Table 3 are smaller than the corresponding values given in lines 3 and 4 proves that a minimizer $u_* \in \tilde{\mathcal{K}}^\nu$ is free of total collisions.

In the last 3 lines of Table 2, for each (P, i), we list the values of $\mathcal{A}(v)$ corresponding to the minimal sequences in Table 4. Again we remark that v enjoys all the symmetries and the topological constraints required for membership in $\mathcal{K}_i^P$. From this and the fact that the values $\mathcal{A}(v)$ in the line corresponding to the pair (P, i) are strictly less than the values in lines 2 or 3 shows that minimizers $u_* \in \mathcal{K}_i^P$ are free of total collisions. This concludes our analysis of total collisions. It remains to exclude the occurrence of partial collisions: this is done in the following Subsection.

5.2. Partial collisions

Our strategy to exclude partial collision consists of two steps. Let us assume that a minimizer $u_* \in \overline{\mathcal{K}}$ has a partial collision: first we show that the collision is isolated; then we prove that we can construct a local perturbation v with $\mathcal{A}(v) < \mathcal{A}(u_*)$. In this construction we can not rely on techniques of the type used in [21], [14], based on Marchal's idea of averaging the action on a set of perturbations v_θ, depending on a parameter θ. In fact the condition $v_\theta \in \overline{\mathcal{K}}$ is a kind of unilateral constraint and may be violated for some value of θ. Besides in certain cases, for instance for the cones $\mathcal{K}_i^P$, the use of this technique is not allowed due to the presence of reflection symmetries. We base our discussion of partial collisions on the fact that, as we discuss below, all of them can be regarded as binary collisions and we can take advantage of the knowledge of the geometric–kinematic structure of such collisions.

Lemma 5.1. *Let $u_* \in \overline{\mathcal{K}}$ be a minimizer of the action. Assume that u_* has a partial collision at time t_c, then the collision is isolated.*

Proof. 1. In [21], [4], [14] it is shown that, if $M \leq N$ particles of the system all collide together at time t_c and there is $\delta > 0$ such that for $t \in (t_c - \delta, t_c + \delta)$ there is no collision involving only a proper subset

of the M particles colliding at time t_c, then t_c is not an accumulation point of collisions of the M particles.
2. Let G be the group of order $|G|=4$ generated by the rotations R_j of π around the axes $\xi_j, j=1,2,3$, if $\mathcal{K}=\mathcal{K}_4$, or $G=\mathcal{R}\in\{\mathcal{T},\mathcal{O},\mathcal{I}\}$ otherwise. As we have already observed in Proposition 3.1 at the collision time t_c the generating particle $u_{*,1}(t_c)$ lies on one of the axes, say r, of some rotation $R\in G\setminus\{I\}$. Since we deal with a partial collision we have that $u_{*,1}(t_c)\in r\setminus\{0\}$ and the collision involves all the $|C|$ particles associated to the maximal subgroup $C\subset G$ of rotations of axis r; there are $|G|/|C|$ contemporary partial collisions of clusters of $|C|$ bodies. Assume that there is another partial collision at time t_c' such that $u_{*,1}(t_c')\in r'\setminus\{0\}$ for $r'\neq r$ the axis of another rotation in G. Then

$$|t_c'-t_c|\geq\frac{N}{2\mathcal{A}(u_*)}d^2(u_{*,1}(t_c),r')\stackrel{def}{=}\delta\,;$$

indeed we have that

$$d(u_{*,1}(t_c),r')\leq|u_{*,1}(t_c')-u_{*,1}(t_c)|\leq|t_c'-t_c|^{1/2}\left[\int_{t_c}^{t_c'}|\dot{u}_{*,1}(t)|^2\,dt\right]^{1/2}\leq$$

$$\leq|t_c'-t_c|^{1/2}\left(\frac{2\mathcal{A}(u_*)}{N}\right)^{1/2}.$$

It follows that all collisions of $u_{*,1}$ on the interval $(t_c-\delta,t_c+\delta)$ take place on the axis r and involve exactly the $|C|$ particles associated to C. From this and 1. the Lemma follows.

□

If $u_*\in\overline{\mathcal{K}}$ is a minimizer and (t_1,t_2) is an interval of regularity, then $u_{*,1}$ is a solution of Newton's equation

$$\ddot{w}=\sum_{R\in G\setminus\{I\}}\frac{(R-I)w}{|(R-I)w|^3},\qquad t\in(t_1,t_2).\tag{5.14}$$

If r is the axis of some rotation in G (G as in Lemma 5.1) and $C\subset G\setminus\{I\}$ is the maximal subgroup of the rotations with axis r we can rewrite (5.14) in the form

$$\ddot{w}=\sum_{R\in C\setminus\{I\}}\frac{(R-I)w}{|(R-I)w|^3}+\sum_{R\in G\setminus C}\frac{(R-I)w}{|(R-I)w|^3}\,.\tag{5.15}$$

If we call R_π the rotation of π around r and set

$$\alpha=\sum_{j=1}^{|C|-1}\frac{1}{\sin(\frac{j\pi}{|C|})}$$

then we have

$$\sum_{R\in C\setminus\{I\}} \frac{(R-I)w}{|(R-I)w|^3} = \alpha\frac{(R_\pi - I)w}{|(R_\pi - I)w|^3}\,,$$

that shows (5.15) is of the general form

$$\ddot{w} = \alpha\frac{(R_\pi - I)w}{|(R_\pi - I)w|^3} + V_1(w)\,, \tag{5.16}$$

where $V_1(w)$ is a smooth function defined in an open set $\Omega \subset \mathbb{R}^3$ that contains $r\setminus\{0\}$. The form (5.16) of Newton's equation is well suited for the analysis of partial collisions occurring on r and implies that all partial collisions a minimizer $u_* \in \overline{\mathcal{K}}$ may present can be regarded as binary collisions.

By a similar computation the first integral of energy can be written in the form

$$|\dot{w}|^2 - \alpha\frac{1}{|(R_\pi - I)w|} - V(w) = h\ . \tag{5.17}$$

For the case at hand

$$V_1(w) = \sum_{R\in G\setminus C} \frac{(R-I)w}{|(R-I)w|^3}, \qquad V(w) = \sum_{R\in G\setminus C} \frac{1}{|(R-I)w|}\ .$$

From these expressions it follows that if $r' \neq r$ is the axis of some rotation in $G\setminus\{I\}$ and $\tilde{R}$ is the reflection with respect to the plane determined by r, r', then V_1, V satisfy the symmetry conditions

$$V_1(\tilde{R}w) = \tilde{R}V_1(w)\,, \qquad V(\tilde{R}w) = V(w)\ . \tag{5.18}$$

In the following Proposition we list a number of properties of *ejection solutions* to (5.16), that is solutions such that

$$\lim_{t\to t_c^+} w(t) = w(t_c) \in r\setminus\{0\}\ .$$

By shifting the origin of the coordinates and of the time we can assume $w(t_c) = 0$, $t_c = 0$. We denote by e_r a unit vector parallel to r.

Proposition 5.5. *Let $w : (0,\bar{t}) \to \mathbb{R}^3$ be a maximal solution of (5.16). Assume that*

$$\lim_{t\to 0^+} w(t) = 0\ . \tag{5.19}$$

Then
(i) there exists $b \in \mathbb{R}$ and a unit vector n, orthogonal to r, such that

$$\lim_{t\to 0^+} \frac{\dot{w}(t) + R_\pi\dot{w}(t)}{2} = b\mathsf{e}_r, \tag{5.20}$$

$$\lim_{t\to 0^+}\frac{w(t)-R_\pi w(t)}{|w(t)-R_\pi w(t)|}=\lim_{t\to 0^+}\frac{w(t)}{|w(t)|}=\mathsf{n}\ . \tag{5.21}$$

(ii) The rescaled function $w^\lambda:[0,1]\to\mathbb{R}^3$ *defined by* $w^\lambda(0)=0, w^\lambda(\tau)=\lambda^{2/3}w(\tau/\lambda), \lambda>1/\bar{t}$, *satisfies*

$$\begin{array}{l}\lim_{\lambda\to+\infty}|w^\lambda(\tau)-s^\alpha(\tau)\mathsf{n}|=0 \textit{ uniformly in } [0,1]\,,\\ \lim_{\lambda\to+\infty}|\dot{w}^\lambda(\tau)-\dot{s}^\alpha(\tau)\mathsf{n}|=0 \textit{ uniformly in } [\delta,1], 0<\delta<1\,,\end{array} \tag{5.22}$$

where

$$s^\alpha(\tau)=\frac{3^{2/3}}{2}\alpha^{1/3}\tau^{2/3}\,,\quad \tau\in[0,+\infty)$$

is the parabolic ejection motion, that is the solution of $\dot{s}=(\alpha/2)^{1/2}s^{-1/2}$ *that satisfies* $\lim_{\tau\to 0^+}s(\tau)=0$.

Proof. The change of variables

$$p=\frac{w+R_\pi w}{2},\qquad q=\frac{w-R_\pi w}{2}$$

transforms (5.16), (5.17) into

$$\begin{cases}\ddot{p}=\frac{1}{2}(I+R_\pi)V_1(p+q)\\ \ddot{q}=-\dfrac{\alpha}{4}\dfrac{q}{|q|^3}+\dfrac{1}{2}(I-R_\pi)V_1(p+q)\end{cases}, \tag{5.23}$$

$$|\dot{p}|^2+|\dot{q}|^2=\frac{\alpha}{2|q|}+V(p+q)+h\ . \tag{5.24}$$

Fix a number $d>0$ such that $B_d=\{|w|<d\}\subset\subset\Omega$ and let $(0,t_d)$ be the maximal interval in which the solution $(p(t),q(t))$ of (5.23) remains in B_d. For $t\in(0,t_d)$ the boundedness of V_1 and $(5.23)_1$ implies $|\ddot{p}|\le C_1$, for some constant $C_1>0$. This, the assumption (5.19) and the fact that, by definition, p is parallel to r yield the existence of $b\in\mathbb{R}$ such that

$$\lim_{t\to 0^+}\dot{p}(t)=b\mathsf{e}_r\,, \tag{5.25}$$

which proves (5.20) and implies

$$|\dot{p}(t)|\le C_2\,,\qquad t\in(0,\min\{1,t_d\}) \tag{5.26}$$

where C_2 is a positive constant that depends only on C_1 and b. If we set $\rho=|q|$ and $\mathsf{e}_q=\frac{q}{|q|}$, $(5.23)_2$ and (5.24) become

$$\ddot{\rho}\mathsf{e}_q+\frac{1}{\rho}\frac{d(\rho^2\dot{\mathsf{e}}_q)}{dt}=-\frac{\alpha}{4\rho^2}\mathsf{e}_q+\frac{1}{2}(I-R_\pi)V_1, \tag{5.27}$$

$$\dot{\rho}^2 + \rho^2 |\dot{\mathsf{e}}_q|^2 = \frac{\alpha}{2\rho} + V + h - |\dot{p}|^2. \tag{5.28}$$

In the remaining part of the proof and in the following Propositions 5.6 and 5.9 $C_\diamond$ will denote a positive constant that may depends only on b and h. The value of $C_\diamond$ can change from line to line.

By projecting (5.27) on e_q and on its orthogonal complement we get

$$\begin{cases} \ddot{\rho} = \rho |\dot{\mathsf{e}}_q|^2 - \frac{\alpha}{4\rho^2} + \frac{1}{2}((I - R_\pi)V_1) \cdot \mathsf{e}_q \\ \frac{d(\rho^2 \dot{\mathsf{e}}_q)}{dt} + \rho^2 |\dot{\mathsf{e}}_q|^2 \mathsf{e}_q = \rho \frac{1}{2}((I - R_\pi)V_1)^\perp \end{cases} \tag{5.29}$$

where the suffix $\perp$ denotes the projection on the plane orthogonal to e_q. We claim that there is a right neighborhood of $t = 0$ where

$$\dot{\rho} > 0 \ . \tag{5.30}$$

If this is not the case, in any neighborhood of $t = 0$ there is t_0 such that $\dot{\rho}(t_0) = 0$. For $t = t_0$ (5.26) and (5.28) imply

$$\rho |\dot{\mathsf{e}}_q|^2 = \frac{\alpha}{2\rho^2} + \frac{1}{\rho}(V + h - |\dot{p}|^2) \geq \frac{\alpha}{2\rho^2} - \frac{C_\diamond}{\rho}, \quad (t = t_0) \ . \tag{5.31}$$

This inequality and $(5.29)_1$ imply

$$\ddot{\rho} \geq \frac{\alpha}{4\rho^2} - C_\diamond(1 + \frac{1}{\rho}), \quad (t = t_0) \ . \tag{5.32}$$

Since the assumption (5.19) implies $\lim_{t \to 0^+} \rho(t) = 0$, from (5.32) we obtain that all points t_0 in a small neighborhood of $t = 0$ where $\dot{\rho}(t_0) = 0$ are relative minima of ρ. This is clearly impossible and (5.30) is established. Next we show

$$\lim_{t \to 0^+} \rho |\dot{\mathsf{e}}_q| = 0 \tag{5.33}$$

and therefore that by (5.25), (5.28)

$$\lim_{t \to 0^+} \dot{\rho}^2 - \frac{\alpha}{2\rho} = V(0) + h - b^2 = C_\diamond \ . \tag{5.34}$$

To show (5.33) we first prove the weaker statement

$$\lim_{t \to 0^+} \rho^2 |\dot{\mathsf{e}}_q| = 0 \ . \tag{5.35}$$

Suppose that, on the contrary, there is $\delta > 0$ and a sequence $\{t_j\}$, $t_j \to 0^+$ such that $\rho^2 |\dot{\mathsf{e}}_q| \geq \delta$ along this sequence. Then for $t = t_j$ (5.28) implies $(\frac{\delta}{\rho})^2 \leq \frac{\alpha}{2\rho} + V + h - |\dot{p}|^2$, which is impossible for large j, and

(5.35) is established. If we take the vector product of $(5.29)_2$ by e_q, integrate on $(0,t)$ and use (5.35), we get

$$\rho^2\dot{\mathsf{e}}_q \times \mathsf{e}_q = \frac{1}{2}\int_0^t \rho((I-R_\pi)V_1)^\perp \times \mathsf{e}_q dt'. \tag{5.36}$$

From this it follows that, provided t is restricted to a small neighborhood of $t=0$ so that (5.30) holds, we have

$$\rho|\dot{\mathsf{e}}_q| \leq \frac{1}{2}\int_0^t |((I-R_\pi)V_1)^\perp|dt' \leq C_\diamond t\,, \tag{5.37}$$

and (5.33) is established. From $(5.29)_1$ we have, for t in a neighborhood of $t=0$,

$$\ddot{\rho} + \frac{\alpha}{4\rho^2} \geq -C_\diamond \Rightarrow \frac{d}{dt}(\dot{\rho}^2 - \frac{\alpha}{2\rho}) \geq -2C_\diamond\dot{\rho}\,, \tag{5.38}$$

where we have also used (5.30). This and (5.34) implies that for t in a neighborhood of $t=0$

$$\dot{\rho}^2 \geq \frac{\alpha}{2\rho} - C_\diamond(1+\rho)\ . \tag{5.39}$$

On the other hand (5.28) implies

$$\dot{\rho}^2 \leq \frac{\alpha}{2\rho} + C_\diamond\ . \tag{5.40}$$

The inequalities (5.39), (5.40) imply that there exists t_0 that depends only on h, b such that (5.30) holds for $t \in (0,t_0)$ and moreover

$$\begin{cases} C_\diamond t^{\frac{2}{3}} \leq \rho(t)\,, & \rho(t) \leq C_\diamond t^{\frac{2}{3}} \\ C_\diamond t^{-\frac{1}{3}} \leq \dot{\rho}(t)\,, & \dot{\rho}(t) \leq C_\diamond t^{-\frac{1}{3}},\ t \in (0,t_0] \end{cases}. \tag{5.41}$$

For later reference we also observe that (5.39), (5.40) imply the asymptotic formulas

$$\rho(t) \propto s^\alpha(t) = \frac{3^{2/3}}{2}\alpha^{\frac{1}{3}}t^{\frac{2}{3}}\,, \qquad \dot{\rho}(t) \propto \dot{s}^\alpha(t) = 3^{-1/3}\alpha^{\frac{1}{3}}t^{-\frac{1}{3}}\ . \tag{5.42}$$

The inequality $(5.41)_1$ and (5.36) yield

$$\rho^2|\dot{\mathsf{e}}_q| \leq \frac{1}{2}\int_0^t \rho|(I-R_\pi)V_1|dt' \leq \frac{3}{10}C_\diamond t^{\frac{5}{3}},\ t \in (0,t_0] \tag{5.43}$$

which together with $(5.41)_1$ imply

$$|\dot{\mathsf{e}}_q| \leq C_\diamond t^{\frac{1}{3}},\ t \in (0,t_0]\ . \tag{5.44}$$

Therefore we deduce from (5.41) that there exists ρ_0 depending only on h, b such that

$$\Big|\frac{d\mathsf{e}_q}{d\rho}\Big| = \frac{|\dot{\mathsf{e}}_q|}{|\dot{\rho}|} \leq C_\diamond \rho,\ \rho \in (0, \rho_0]\ . \tag{5.45}$$

From the estimate (5.44) it follows that there exists a unit vector n such that

$$\mathsf{n} = \lim_{t \to 0^+} \mathsf{e}_q(t) = \lim_{t \to 0^+} \frac{w(t) - R_\pi w(t)}{|w(t) - R_\pi w(t)|}\ . \tag{5.46}$$

Moreover by definition $\mathsf{e}_q(t) \cdot \mathsf{e}_r = 0$ and therefore $\mathsf{n} \cdot \mathsf{e}_r = 0$. We set

$$\mathsf{e}_r \times \mathsf{n} = \mathsf{e}_\perp,$$

$$\begin{cases} q = x\mathsf{n} + z\mathsf{e}_\perp, \\ p = y\mathsf{e}_r \end{cases} \tag{5.47}$$

which imply

$$w = p + q = x\mathsf{n} + y\mathsf{e}_r + z\mathsf{e}_\perp\ . \tag{5.48}$$

From (5.41) we can take $\rho \in (0, \rho_0]$ as the independent variable. From (5.47) and $q = \rho\mathsf{e}_q$ we get

$$\frac{dx}{d\rho} = \mathsf{e}_q \cdot \mathsf{n} + \rho\frac{d\mathsf{e}_q}{d\rho} \cdot \mathsf{n} = 1 + \Big(\mathsf{e}_q - \mathsf{n} + \rho\frac{d\mathsf{e}_q}{d\rho}\Big) \cdot \mathsf{n}\ .$$

This and (5.45) imply

$$\Big|\frac{dx}{d\rho} - 1\Big| \leq C_\diamond \rho^2,\ \rho \in (0, \rho_0]\ . \tag{5.49}$$

From this we derive, using again (5.45),

$$\begin{aligned} \Big|\frac{dz}{d\rho}\Big| &= \Big|\frac{d}{d\rho}(\rho\mathsf{e}_q - x\mathsf{n}) \cdot \mathsf{e}_\perp\Big| = \Big|\Big(\mathsf{e}_q + \rho\frac{d\mathsf{e}_q}{d\rho} - \frac{dx}{d\rho}\mathsf{n}\Big) \cdot \mathsf{e}_\perp\Big| \\ &= \Big|\Big[\Big(1 - \frac{dx}{d\rho}\Big)\mathsf{n} + \mathsf{e}_q - \mathsf{n} + \rho\frac{d\mathsf{e}_q}{d\rho}\Big] \cdot \mathsf{e}_\perp\Big| \leq C_\diamond \rho^2, \quad \rho \in (0, \rho_0]\ . \end{aligned} \tag{5.50}$$

From (5.25) and (5.41) we get

$$\Big|\frac{dy}{d\rho}\Big| \leq C_\diamond \rho^{1/2},\ \rho \in (0, \rho_0]\ . \tag{5.51}$$

These estimates and the fact that by (5.49) we can take x as the independent variable in some interval $(0, x_0)$, with $x_0 > 0$ depending only on h, b, imply

$$\lim_{t \to 0^+} \frac{w(t)}{|w(t)|} = \lim_{x \to 0^+} \frac{\mathsf{n} + \frac{y}{x}\mathsf{e}_r + \frac{z}{x}\mathsf{e}_\perp}{(1 + \frac{y^2 + z^2}{x^2})^{\frac{1}{2}}} = \mathsf{n}\,, \tag{5.52}$$

which completes the proof of (5.21).

We denote by $'$ the derivative with respect to x and observe that (5.50), (5.51) imply

$$|y'| \leq C_\diamond \sqrt{x}\,, \qquad |z'| \leq C_\diamond x^2\,, \qquad x \in (0, x_0)\;. \tag{5.53}$$

Relations (5.49), (5.41) and (5.53) yield

$$|x(t) - \rho(t)| \leq C_\diamond t^2\,, \quad |y(t)| \leq C_\diamond t\,, \quad |z(t)| \leq C_\diamond t^2\;.$$

Using also $(5.42)_1$, that implies

$$1 = \lim_{t\to 0^+} \frac{\rho(t)}{s^\alpha(t)} = \lim_{\lambda\to+\infty} \frac{\rho(\tau/\lambda)}{s^\alpha(\tau/\lambda)} = \lim_{\lambda\to+\infty} \frac{\lambda^{2/3}\rho(\tau/\lambda)}{s^\alpha(\tau)}\,,$$

the proof of $(5.22)_1$ follows from the inequality

$$|w^\lambda(\tau) - s^\alpha(\tau)\mathsf{n}| \leq |\lambda^{2/3}\rho(\tau/\lambda) - s^\alpha(\tau)| + \lambda^{2/3}|x(\tau/\lambda) - \rho(\tau/\lambda)| + \\ +\lambda^{2/3}(|y(\tau/\lambda)| + |z(\tau/\lambda)|)\;.$$

The proof of $(5.22)_2$ is similar.

□

For later use, in the following Proposition we collect some of the estimates obtained in the proof of Proposition 5.5.

Proposition 5.6. *Let $w : (0,\bar{t}) \to \mathbb{R}^3$ be a maximal solution of (5.16). Assume that*

$$\lim_{t\to 0^+} w(t) = 0$$

and define b, n as in Proposition 5.5. Let x, y, z, ρ be defined by

$$w = x\mathsf{n} + y\mathsf{e}_r + z\mathsf{e}_r \times \mathsf{n}\,, \qquad \rho = \frac{1}{2}|(R_\pi - I)w|\;.$$

Then the following estimates hold for some positive constants t_0, ρ_0, x_0, $C_\diamond$, depending only on b and h:

$$\begin{cases} C_\diamond t^{\frac{2}{3}} \leq \rho(t)\,; \quad \rho(t) \leq C_\diamond t^{\frac{2}{3}}\,, \\ C_\diamond t^{-\frac{1}{3}} \leq \dot\rho(t)\,; \quad \rho(t) \leq C_\diamond t^{-\frac{1}{3}}, \quad t \in (0, t_0]\,, \\ |\frac{dx}{d\rho} - 1| \leq C_\diamond \rho^2\,, \rho \in (0, \rho_0]\,, \\ |y'| \leq C_\diamond x^{1/2}\,; \quad |z'| \leq C_\diamond x^2\,, \end{cases} \tag{5.54}$$

where by $'$ we mean differentiation with respect to x.

Propositions 5.5, 5.6 are stated and proved for ejection solutions. Analogous statements and proofs with obvious modifications apply to collision solutions of (5.16), that is solutions satisfying $\lim_{t\to 0^-} w(t) = 0$.

Given unit vectors $\mathsf{n}^\pm$ define $\omega = \omega^{\alpha,\mathsf{n}^\pm} : \mathbb{R} \to \mathbb{R}^3$ by setting

$$\omega^{\alpha,\mathsf{n}^\pm}(\pm t) = \mathsf{n}^\pm s^\alpha(t),\ t \geq 0\,, \qquad s^\alpha(t) = \frac{3^{2/3}}{2}\alpha^{1/3}t^{2/3}\ .$$

If $-1 < \mathsf{n}^+ \cdot \mathsf{n}^- < 1$ we let Θ_d, with $0 < \Theta_d < \pi$, be the angle determined by O and $\mathsf{n}^\pm$ and let Θ_i, with $\pi < \Theta_i < 2\pi$ be the complement of Θ_d. Given $t^+, t^- > 0$ there exist unique Keplerian arcs $\omega_d : [-t^-, t^+] \to \mathbb{R}^3$ and $\omega_i : [-t^-, t^+] \to \mathbb{R}^3$ that connect $\omega(-t^-)$ to $\omega(t^+)$ in the time interval $[-t^-, t^+]$ and satisfy

$$\begin{cases} \omega_d((-t^-, t^+)) \subset \Theta_d \\ \omega_i((-t^-, t^+)) \subset \Theta_i \end{cases} ;$$

ω_d and ω_i are called the *direct* and *indirect* Keplerian arc [1]. In the boundary case $\mathsf{n}^+ \cdot \mathsf{n}^- = -1$ both angles have measure π and the distinction between ω_d and ω_i does not make sense. In the other boundary case $\mathsf{n}^+ \cdot \mathsf{n}^- = 1$ the indirect arc does not exist and we can assume $\Theta_d = 0, \Theta_i = 2\pi$.

Proposition 5.7. *The following inequalities hold:*

(i) $\mathcal{A}(\omega_d) < \mathcal{A}(\omega|_{[-t^-,t^+]}),\ \forall \mathsf{n}^\pm\,,$

(ii) $\mathcal{A}(\omega_i) < \mathcal{A}(\omega|_{[-t^-,t^+]}),\ \forall \mathsf{n}^\pm$ *such that* $\mathsf{n}^+ \cdot \mathsf{n}^- < 1\,,$

where

$$\mathcal{A}(w) = \int_{-t^-}^{t^+} \Big(\frac{|\dot{w}|^2}{2} + \frac{\alpha}{4|w|}\Big)\, dt\ .$$

Proof. For the proof of this Proposition we refer to [21], [6], and to [28], where the generalization to the case of potentials of the form $1/r^\gamma, \gamma \in (0, 2)$, is also considered. Below we present a proof for the symmetric case $t^+ = t^- = \tau$ which is the one we use in the following.

(The case $t^+ = t^- = \tau$) We consider a unit mass P, moving in the plane under the attraction of a mass μ ($\mu = \frac{\alpha}{4}$) fixed in the origin O, and we let h be the energy and J the constant of angular momentum. We let (r, ϕ) be the polar coordinates of $\mathsf{P} \neq O$. We define $\rho > 0$ and $\theta \in [0, \pi]$ by setting $\mathsf{n}^\pm = (\cos\theta, \pm\sin\theta)$, $\omega(\pm\tau) = \mathsf{n}^\pm \rho$. Then τ and ρ are related by

$$\tau = \frac{2^{1/2}}{3}\frac{\rho^{3/2}}{\mu^{1/2}}\ . \tag{5.55}$$

For each $\theta \in (0, \pi)$ there is a unique Keplerian arc ω_θ that connects $\omega(-\tau)$ with $\omega(\tau)$ in the time interval $[-\tau, \tau]$ and intersects the polar axis at a point $(\rho_0, 0)$ with $\rho_0 > 0$. The arc ω_θ is therefore the direct arc if $\theta \in (0, \pi/2)$ and the indirect arc if $\theta \in (\pi/2, \pi)$. We denote by $2A$ the action of the arc ω_θ and by

$$A_0 = 2^{3/2}(\rho\mu)^{1/2} \tag{5.56}$$

the action of the parabolic ejection arc connecting O with $\omega(\tau)$. To prove the Proposition is the same as to show that the ratio $a := A/A_0$ is strictly < 1 for all $\theta \in [0, \pi)$. It is easily seen that

$$\theta \lesseqqgtr \frac{2^{1/2}}{3} \qquad \Longrightarrow \qquad \rho_0 \gtreqqless \rho$$

and moreover the eccentricity e of ω_θ satisfies

$$\begin{cases} e \in [0, 1), & 0 < \theta \leq 2^{1/2}/3 \\ e \in (0, +\infty), & 2^{1/2}/3 < \theta \leq \pi/2 \\ e \in (0, -1/\cos\theta), & \pi/2 < \theta \leq \pi \end{cases} . \tag{5.57}$$

In particular it follows that $(\rho_0, 0)$ is the apocenter if $\theta < 2^{1/2}/3$ and the pericenter if $\theta > 2^{1/2}/3$. Therefore the polar equation of ω_θ reads

$$r = \frac{J^2/\mu}{1 \mp e\cos\phi}, \quad -\theta \leq \phi \leq \theta \tag{5.58}$$

where here and in the following $\mp = -$ if $\theta < 2^{1/2}/3$ and $\mp = +$ if $\theta > 2^{1/2}/3$. The constants h and J are related to ρ_0 and e by

$$h = \frac{\mu}{2\rho_0}(-1 \mp e), \tag{5.59}$$

$$J^2 = \rho_0\mu(1 \mp e) . \tag{5.60}$$

The values of ρ_0 and e are determined by the conditions

$$\rho = \frac{J^2/\mu}{1 \mp e\cos\theta}, \tag{5.61}$$

$$\frac{1}{J}\int_0^\theta r^2\, d\phi = \frac{J^3}{\mu^2}\int_0^\theta \frac{d\phi}{(1 \mp e\cos\phi)^2} = \tau \tag{5.62}$$

which express the fact that $(r, \phi) = (\rho, \theta)$ fulfills (5.58) and that the travel time from $(\rho_0, 0)$ to (ρ, θ) along ω_θ coincides with τ. From (5.60), and (5.61) it follows

$$\frac{\rho}{\rho_0} = \frac{1 \mp e}{1 \mp e\cos\theta} .$$

Using this, (5.55) and (5.60), we obtain from (5.62) the equation

$$\int_0^\theta \frac{d\phi}{(1 \mp e\cos\phi)^2} = \frac{2^{1/2}}{3}\frac{1}{(1 \mp e\cos\theta)^{3/2}} \tag{5.63}$$

that determines $e = e(\theta)$. For the action ratio we have

$$a = \frac{1}{A_0}\Big(h\tau + 2\mu\int_0^\tau \frac{dt}{r}\Big) = \frac{1}{A_0}\Big(h\tau + 2J\int_0^\theta \frac{d\phi}{1 \mp e\cos\phi}\Big)\ . \tag{5.64}$$

If we set $I_n = \int_0^\theta \frac{d\phi}{(a+b\cos\phi)^n}$ we have the identity

$$(a^2 - b^2)I_2 = aI_1 - b\frac{\sin\theta}{a + b\cos\theta} \qquad (a^2 \neq b^2)\ .$$

From this with $a = 1, b = \mp e$ and (5.63) it follows

$$\int_0^\theta \frac{d\phi}{1 \mp e\cos\phi} = \frac{2^{1/2}}{3}\frac{1 - e^2}{(1 \mp e\cos\theta)^{3/2}} \mp \frac{e\sin\theta}{1 \mp e\cos\theta}\ .$$

If we introduce this expression of I_1 into (5.64) and use

$$\frac{h\tau}{A_0} = -\frac{1 - e^2}{12(1 \mp e\cos\theta)}\,, \qquad \frac{2J}{A_0} = \frac{1}{2^{1/2}}(1 \mp e\cos\theta)^{1/2}\,,$$

that follow from (5.55), (5.56), (5.59)–(5.61), we finally get

$$a = \frac{1}{4}\frac{(1 - e^2)}{(1 \mp e\cos\theta)} \mp \frac{1}{2^{1/2}}\frac{e\sin\theta}{(1 \mp e\cos\theta)^{1/2}}\ . \tag{5.65}$$

Note that this expression, derived under the assumption $0 < \theta$, is valid also for $\theta = 0$. To conclude the proof, instead of studying directly the function $a(\theta)$ obtained by inserting the solution $e = e(\theta)$ of (5.63) into (5.65), we show that a has a unique maximum $a_M < 1$ on each line

$$e\cos\theta = \text{const}\,, \tag{5.66}$$

for e, θ satisfying (5.57). Differentiating (5.66) with respect to e we get $\theta' = \frac{1}{e\tan\theta}$. From this and (5.65), it follows that the derivative a' of a with respect to e along the lines $e\cos\theta = \text{const}$ is given by

$$a' = \frac{1}{2^{1/2}\sin\theta(1 \mp e\cos\theta)^{1/2}}\Big(\mp 1 - \frac{e\sin\theta}{2^{1/2}(1 \mp e\cos\theta)^{1/2}}\Big)\ . \tag{5.67}$$

For $\theta \leq 2^{1/2}/3$ we have from (5.57) that $e \in [0, 1)$, and (5.67) implies $a' \leq 0$, therefore a takes its maximum $a_M = \frac{1}{4}(1 + e) < \frac{1}{2}$ for $\theta = 0$. For $\theta > 2^{1/2}/3$ we have $a' = 0$ on the curve ℓ defined by

$$1 = \frac{e\sin\theta}{2^{1/2}(1 + e\cos\theta)^{1/2}} \quad \Longleftrightarrow \quad e = e_0(\eta) \stackrel{def}{=} (1 + (1 + \eta)^2)^{1/2} > 1\,, \tag{5.68}$$

with $\eta = e\cos\theta$. From (5.67) it follows that

$$a' \lesseqqgtr 0 \iff e \gtreqqless e_0$$

and therefore a attains its maximum a_M on ℓ. Inserting (5.68) into (5.65) yields

$$a_M = 1 - \frac{1}{4}(1 + e\cos\theta) < 1 \ .$$

This concludes the proof.

□

It is exactly the possibility of choosing between ω_d and ω_i still reducing the action that, whenever u_* is assumed to have a partial collision that implies $u_* \in \partial\mathcal{K}$, allows us to perturb u_* inside $\mathcal{K}$, thus preserving the constraint of membership in $\mathcal{K}$. On the basis of Proposition 5.7 this can always be done if $\mathsf{n}^+ \cdot \mathsf{n}^- < 1$. The special case $\mathsf{n}^+ \cdot \mathsf{n}^- = 1$ is excluded since in this case the indirect Keplerian arc does not exist at all and, if the direct Keplerian arc does not allow the construction of a perturbation $v \in \mathcal{K}$, then Proposition 5.7 can not be used. The discussion of these situations is more delicate and it is based on a uniqueness result that we prove below (cfr. Proposition 5.9).

We begin our analysis of partial collisions by

Proposition 5.8. *Let $u_* \in \overline{\mathcal{K}}$ be a minimizer of the action and assume that u_* has a partial collision at time t_c. Let $\mathsf{n}^+, \mathsf{n}^-$ be the unit vectors associated to the collision of the generating particle in the sense of Proposition 5.5. Then*

$$\mathsf{n}^+ = \mathsf{n}^- \ .$$

Proof. We show that the assumption that u_* has a partial collision with $\mathsf{n}^+ \cdot \mathsf{n}^- < 1$ leads to the contradiction of the existence of a perturbation $v \in \overline{\mathcal{K}}$ of u_* such that $\mathcal{A}(v) < \mathcal{A}(u_*)$. The equivariance condition that characterizes $u \in \mathcal{K}$ implies that it suffices to define v only in a fundamental interval $I_{\mathcal{K}}$ that contains the collision time t_c. Then v is automatically extended to the whole $\mathbb{R}$ by equivariance.

We call r the axis where the collision of the generating particle takes place and $C \subset G$ the maximal subgroup of the rotations with axis r. Set $w(t) = u_{*,1}(t_c + t) - u_{*,1}(t_c)$. For every fixed $\lambda > 1$ the restriction of w to $\left[-\frac{1}{\lambda}, \frac{1}{\lambda}\right]$ is a minimizer of

$$\mathcal{A}^\lambda(\phi) = \lambda^{1/3}\frac{|G|}{2}\int_{-1/\lambda}^{1/\lambda}\left(|\dot\phi|^2 + \sum_{R\in C\setminus\{I\}}\frac{1}{|(R-I)\phi|}\right)dt \, +$$

$$\lambda^{1/3}\frac{|G|}{2}\int_{-1/\lambda}^{1/\lambda}\sum_{R\in G\setminus C}\frac{1}{|(R-I)(\phi + u_{*,1}(t_c))|}\,dt$$

on the set of functions ϕ in $H^1\left((-\frac{1}{\lambda},\frac{1}{\lambda}),\mathbb{R}^3\right)$ that satisfy

$$\phi\Big(\pm\frac{1}{\lambda}\Big)=w\Big(\pm\frac{1}{\lambda}\Big)\;.$$

The map

$$f:H^1\Big(\Big(-\frac{1}{\lambda},\frac{1}{\lambda}\Big),\mathbb{R}^3\Big)\to H^1\left((-1,1),\mathbb{R}^3\right)$$

defined by

$$\begin{cases} f(\phi)=\psi \\ \phi(t)=\lambda^{-2/3}\psi(\lambda t), \quad t\in[-1/\lambda,1/\lambda] \end{cases} \tag{5.69}$$

is a bijection and we have

$$\mathcal{A}^\lambda(\phi)=\hat{\mathcal{A}}^\lambda(\psi)\overset{def}{=}\frac{|G|}{2}\int_{-1}^{1}\Big(\Big|\frac{d\psi}{d\tau}\Big|^2+\sum_{R\in C\setminus\{I\}}\frac{1}{|(R-I)\psi|}\Big)\,d\tau\,+$$

$$+\frac{|G|}{2}\int_{-1}^{1}\sum_{R\in G\setminus C}\frac{1}{|(R-I)(\psi+\lambda^{2/3}u_{*,1}(t_c))|}\,d\tau\overset{def}{=}a(\psi)+a^\lambda(\psi)\,, \tag{5.70}$$

where $a(\psi)$ and $a^\lambda(\psi)$ denote the two terms in the definition of $\hat{\mathcal{A}}^\lambda(\psi)$. Therefore from the minimality of $w|_{[-1/\lambda,1/\lambda]}$ and (5.70) it follows that the map $w^\lambda:[-1,1]\to\mathbb{R}^3$ defined by

$$w^\lambda(\tau)=\lambda^{2/3}w\Big(\frac{\tau}{\lambda}\Big)$$

is a minimizer of $\hat{A}^\lambda(\psi)=a(\psi)+a^\lambda(\psi)$. From Proposition 5.5, for $\lambda\to+\infty$, w^λ converges uniformly in $[-1,1]$ to $w^\infty=\omega^{\alpha,\mathsf{n}^\pm}$ with $\mathsf{n}^\pm$ orthogonal to r. Assume $\mathsf{n}^+\cdot\mathsf{n}^-<1$ and let $w^\pm$ be the direct and indirect Keplerian arcs connecting $w^\infty(-1)$ to $w^\infty(+1)$ in the interval $[-1,1]$, and define

$$\hat{w}^{\lambda,\pm}=w^\pm+\tilde{w}^\lambda\,,$$

where we have set

$$\tilde{w}^\lambda(\tau)=(w^\lambda(-1)-w^\infty(-1))(\frac{1-\tau}{2})+(\frac{1+\tau}{2})(w^\lambda(1)-w^\infty(1)),\;\;\tau\in[-1,1].$$

From (5.70), the boundedness of $\hat{w}^{\lambda,\pm}$ and Lebesgue's dominate convergence theorem we have

$$\begin{cases} \lim_{\lambda\to+\infty}\hat{A}^\lambda(w^\lambda)=a(w^\infty) \\ \lim_{\lambda\to+\infty}\hat{A}^\lambda(\hat{w}^{\lambda,\pm})=a(w^\pm) \end{cases}\;. \tag{5.71}$$

This and Proposition 5.7 imply that, for $\lambda>>1$,

$$\mathcal{A}(v^{\lambda,\pm})<\mathcal{A}(u_*)\,, \tag{5.72}$$

where $v^{\lambda,\pm}$ is defined through

$$v_1{}^{\lambda,\pm}(t) = \begin{cases} u_{*,1}(t), \ t \in I_{\mathcal{K}} \setminus [t_c - 1/\lambda, t_c + 1/\lambda] \\ \lambda^{-2/3}\hat{w}^{\lambda,\pm}(\lambda(t-t_c)) + u_{*,1}(t_c), \ t \in [t_c - 1/\lambda, t_c + 1/\lambda] \end{cases}.$$

The inequality (5.72) contradicts the minimality of u_* because the definition of $\mathcal{K}$ implies that either $v^{\lambda,+}$ or $v^{\lambda,-}$ belong to $\overline{\mathcal{K}}$. In the above argument we have tacitally assumed that t_c is in the interior of $I_{\mathcal{K}}$. If t_c is one of the boundary points of $I_{\mathcal{K}}$ the same argument applies verbatim with the provision of replacing $t \in [t_c - 1/\lambda, t_c + 1/\lambda]$ with $t \in [t_c - 1/\lambda, t_c + 1/\lambda] \cap I_{\mathcal{K}}$ in the definition of $v_1^{\lambda,\pm}(t)$.

□

Remark 5.1. The idea of constructing local variations of the parabolic ejection–collision orbit obtained by blowing up the rescaled collision solution has already been used in [30], [14].

Remark 5.2. Let (σ, n) be a pair such that the sequence σ is simple in the sense of Definition 4.2. The argument in the proof of Proposition 5.8 can be applied to show that a minimizer $u_* \in \overline{\mathcal{K}^{(\sigma,n)}}$ can not have isolated partial collisions such that the unit vectors $\mathsf{n}^{\pm}$ associated to the collisions of the generating particle, as in Proposition 5.5, satisfy $\mathsf{n}^+ \cdot \mathsf{n}^- < 1$. In fact the assumption that σ is simple and the fact that the curve $\gamma = \{w^{\pm}(\tau) + u_{*,1}(t_c), \tau \in [-1, 1]\}$ is a simple closed curve linked to the axis r imply that either $v^{\lambda,+}$ or $v^{\lambda,-}$ belongs to $\overline{\mathcal{K}}$. If σ is not simple it may be very difficult, or even impossible, to use a local argument, as in Proposition 5.8, to show that a minimizer $u_* \in \overline{\mathcal{K}^{(\sigma,n)}}$ does not have partial collisions. Indeed from Gordon's Theorem [18] we know that, in the class of loops with index $\neq -1, 0, 1$ with respect to the origin, the only minimizers of the planar Kepler problem with an attracting fixed mass at the origin are collision–ejection loops.

Remark 5.3. The arguments in the proof of Proposition 5.8 are based on the possibility of choosing between $v^{\lambda,+}$ and $v^{\lambda,-}$, and in turn on the possibility of choosing between the direct and indirect Keplerian arcs used in the construction of $v^{\lambda,\pm}$. If $\mathsf{n}^+ = \mathsf{n}^-$, then the indirect arc does not exist and the construction in the proof of Proposition 5.8 yields a single perturbation v^{λ} of u_* by utilizing the direct Keplerian arc. In all cases where it results $v^{\lambda} \in \overline{\mathcal{K}}$, again one reaches the contradiction $\mathcal{A}(v^{\lambda}) < \mathcal{A}(u_*)$ and the supposed partial collision is excluded.

From Remark 5.3 it follows that a minimizer $u_* \in \overline{\mathcal{K}}$ is collision free if it can be excluded that u_* has an isolated partial collision such that

(i) $\mathsf{n}^+ = \mathsf{n}^- = \mathsf{n}$;

(ii) $v^\lambda \notin \overline{\mathcal{K}}$,

where v^λ is the perturbation constructed as in the proof of Proposition 5.8, by means of the direct Keplerian arc.
We refer to partial collisions that satisfy (i) and (ii) as collisions of type $(\rightrightarrows)$

Lemma 5.2. *Assume $u_* \in \overline{\mathcal{K}}$ has a collision of type $(\rightrightarrows)$ at $t = t_c$; let r be the axis on which the collision of the generating particle takes place and set $\mathsf{n} = \mathsf{n}^\pm$. Then*

$$\mathsf{n} \in \Pi \,,$$

where Π is a plane through r that contains two distinct axes of rotations in $G \setminus \{I\}$.

Proof. We give different proofs for the cases $\mathcal{K} = \mathcal{K}_4, \mathcal{K}_i^P$ and $\tilde{\mathcal{K}}^\nu$.

1) $\mathcal{K} = \mathcal{K}_4$. From (2.1) it follows $u_1(t + T/2) = R_1 u_1(t)$. This implies that if a minimizer $u_* \in \overline{\mathcal{K}}$ has a collision at time t_c, then it also has a collision at time $t_c + T/2$. The constraint (2.3) that defines $\mathcal{K}$ involves the values of $u_1(t)$ only for $t = 0, T/4$. Therefore if $t_c \notin \{0, T/4\} \mod T/2$, then any sufficiently small perturbation of u_* in a compact interval $[t_c - \delta, t_c + \delta]$ will remain in $\overline{\mathcal{K}}$ provided it satisfies conditions (2.1), (2.2). It follows that the collision is not of type $(\rightrightarrows)$. By consequence if $u_* \in \overline{\mathcal{K}}$ has a collision of type $(\rightrightarrows)$, then necessarily $t_c \in \{0, T/4\} \mod T/2$. From this and (2.1) it follows

$$\begin{cases} \mathsf{n}^+ = S_3 \mathsf{n}^- & \text{if } t_c = 0\,, \\ \mathsf{n}^+ = S_2 \mathsf{n}^- & \text{if } t_c = T/4\,, \end{cases}$$

which together with $\mathsf{n} = \mathsf{n}^\pm$ imply

$$\begin{cases} \mathsf{n} \in \mathrm{span}\{\mathsf{e}_1, \mathsf{e}_2\} & \text{if } t_c = 0\,, \\ \mathsf{n} \in \mathrm{span}\{\mathsf{e}_1, \mathsf{e}_3\} & \text{if } t_c = T/4\;. \end{cases}$$

2) $\mathcal{K} = \mathcal{K}_i^P$. By arguing as in 1) we conclude that if a minimizer $u_* \in \overline{\mathcal{K}}$ has a collision of type $(\rightrightarrows)$ at time t_c, then $t_c \in \{0, T/(2H)\}$ mod T/H. Then from (b), (c) it follows

$$\begin{cases} \mathsf{n}^+ = S_3 \mathsf{n}^- & \text{if } t_c = 0\,, \\ \mathsf{n}^+ = R S_3 \mathsf{n}^- & \text{if } t_c = \frac{T}{2H}\,, \end{cases} \tag{5.73}$$

where R is the rotation of $2\pi/H$ around ξ_1. The operator RS_3 coincides with the reflection with respect to the plane determined by ξ_1 and V. From this observation, from equation (5.73) and $\mathsf{n} = \mathsf{n}^\pm$ it follows

$$\begin{cases} \mathsf{n} \in \mathrm{span}\{\mathsf{e}_1, \mathsf{e}_M\} & \text{if } t_c = 0\,, \\ \mathsf{n} \in \mathrm{span}\{\mathsf{e}_1, \mathsf{e}_V\} & \text{if } t_c = T/(2H)\;. \end{cases}$$

3) $\mathcal{K} = \tilde{\mathcal{K}}^{\nu}$, ν as in Theorem 4.1. From the discussion at the end of Section 4.2, if $u_* \in \overline{\mathcal{K}}$ is a minimizer which has a partial collision, then a collision of the generating particle takes necessarily place on one of the semiaxes in the sequence $k \to r_k := (\overline{S_{k+1}} \cap \overline{S^k}) \setminus \{0\}$. For each k we let $\tilde{k} > k$ be defined by the condition that $S^{\tilde{k}}$ borders with S^k along r_k in the sense that $r_k \subset \overline{S^{\tilde{k}}} \cap \overline{S^k}$. The formal definition of $\tilde{k}$ is: $\tilde{k} = k + h$, where $h \geq 1$ is determined by the conditions (cfr. Figure 15)

$$\begin{cases} r_k \subset \overline{S^k} \cap \overline{S^{k+h}} \\ 0 < j < h \Rightarrow \overline{S^k} \cap \overline{S^{k+j}} = \emptyset \end{cases} .$$

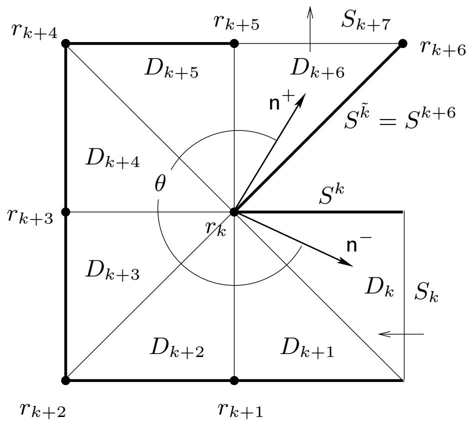


Figure 15. The definition of $S^{\tilde{k}}$. The marked segments correspond to S^j, $j = k, \ldots, k+6 = \tilde{k}$.

Let $E = \cup_{j=0}^{h}(S_{k+j} \cup D_{k+j} \cup S_{k+j+1})$. Since $u_{*,1}$ is the limit of a sequence of maps that enter E through S_k and exit E through S_{k+h+1}, the unit vectors n^- and n^+ applied at $u_{*,1}(t_c) \in r_k$ point inside $\overline{E}$. We let $0 \leq \theta \leq 2\pi$ be the angle swept by n^- when

(i) n^- is rotated around r_k until it coincides with n^+,
(ii) during the rotation n^- always points inside E.

We observe that $\theta = 2\pi$ is equivalent to

$$S^k = S^{\tilde{k}}, \qquad \mathsf{n}^- = \mathsf{n}^+ = \mathsf{n} \in S^k .$$

This concludes the proof. Indeed, if $0 \leq \theta < 2\pi$, by using either the direct or the indirect Keplerian arc, as in the proof of Proposition 5.8, the collision can be excluded. More precisely, to construct the perturbation of u_* we use the direct Keplerian arc if $0 \leq \theta \leq \pi$ and the indirect arc if instead $\pi \leq \theta < 2\pi$.

□

For the analysis of collisions of type ($\rightrightarrows$) we need a uniqueness result that we present next. Assume $u_* \in \overline{\mathcal{K}}$ has an isolated partial collision at time t_c, then, as we have discussed at the beginning of this Section, we have $u_{*,1}(t_c) \in r \setminus \{0\}$, r being the axis of one of the rotations in $G \setminus \{I\}$, and $w(t) = u_{*,1}(t_c + t) - u_{*,1}(t_c)$ is a solution of (5.16).

Proposition 5.9. *Let* $w_i : (0, \bar{t}_i) \to \mathbb{R}^3, \bar{t}_i > 0, i = 1, 2$ *be two maximal solutions of (5.16) such that*

$$\lim_{t \to 0^+} w_i(t) = 0 \ .$$

If h_i, b_i, n_i *are the corresponding values of the energy and the values of* b *and* n *given by Proposition 5.5, then*

$$\begin{cases} h_1 = h_2 \\ b_1 = b_2 \\ \mathsf{n}_1 = \mathsf{n}_2 \end{cases} \Longrightarrow \quad \begin{cases} \bar{t}_1 = \bar{t}_2 \\ w_1 = w_2 \end{cases} .$$

Proof. We project (5.16) onto $\mathsf{n}, \mathsf{e}_r, \mathsf{e}_\perp$. Recalling that $p = y\mathsf{e}_r, q = x\mathsf{n} + z\mathsf{e}_\perp$ and (5.23) we get

$$\begin{cases} \ddot{x} = -\frac{\alpha}{4x^2(1+\frac{z^2}{x^2})^{\frac{3}{2}}} + V_1 \cdot \mathsf{n} \\ \ddot{y} = V_1 \cdot \mathsf{e}_r \\ \ddot{z} = -\frac{\alpha z}{4x^3(1+\frac{z^2}{x^2})^{\frac{3}{2}}} + V_1 \cdot \mathsf{e}_\perp \end{cases} . \tag{5.74}$$

From Proposition 5.6 we can take x as the independent variable. We rewrite (5.17) in the form

$$\dot{x}^2(1 + |y'|^2 + |z'|^2) = \frac{\alpha}{2x(1 + \frac{z^2}{x^2})^{\frac{1}{2}}} + V + h \,, \tag{5.75}$$

where $'$ denotes differentiation with respect to x. From this and $(5.74)_1$ we get

$$\begin{cases} \frac{1}{\dot{x}^2} = \frac{2x}{\alpha} \frac{(1 + |y'|^2 + |z'|^2)(1 + \frac{z^2}{x^2})^{\frac{1}{2}}}{1 + 2x(1 + \frac{z^2}{x^2})^{\frac{1}{2}} \frac{V+h}{\alpha}} := \frac{2x}{\alpha}(1 + \mathcal{W}) \\ \frac{\ddot{x}}{\dot{x}^2} = -\frac{1}{2x} \frac{(1 + |y'|^2 + |z'|^2)}{(1 + \frac{z^2}{x^2})} \frac{\left(1 - 4x^2(1 + \frac{z^2}{x^2})^{\frac{3}{2}} \frac{V_1 \cdot \mathsf{n}}{\alpha}\right)}{\left(1 + 2x(1 + \frac{z^2}{x^2})^{\frac{1}{2}} \frac{V+h}{\alpha}\right)} := -\frac{1}{2x}(1 + \mathcal{U}) \end{cases} . \tag{5.76}$$

Therefore, taking into account that for any function $f(x(t))$ we have

$$\dot{f} = f'\dot{x} \,, \qquad \ddot{f} = (f'' + \frac{\ddot{x}}{\dot{x}^2} f')\dot{x}^2 \,, \tag{5.77}$$

we can rewrite system (5.74) in the form

$$\begin{cases} y'' - \frac{1}{2x}(1+\mathcal{U})y' = 2x(1+\mathcal{W})\frac{V_1\cdot \mathsf{e}_r}{\alpha} := x\mathcal{A}\,, \\ z'' - \frac{1}{2x}(1+\mathcal{U})z' = \left[-\frac{z}{2x^2(1+\frac{z^2}{x^2})^{\frac{3}{2}}} + \frac{2x}{\alpha} V_1 \cdot \mathsf{e}_\perp \right](1+\mathcal{W}) \\ \qquad := -\frac{z}{2x^2}(1+\mathcal{V}) + x\mathcal{B}\,. \end{cases} \tag{5.78}$$

We now change the independent variable and rewrite (5.78) as a first order system. We set $x = e^s$, $s \in (-\infty, s_0]$, where $s_0 < 0$ is chosen later. We introduce the new variables

$$\eta = \frac{dy}{ds}, \qquad \zeta = \frac{dz}{ds}$$

and observe that

$$y' = \frac{1}{x}\frac{dy}{ds} = \frac{\eta}{x}\,, \qquad y'' = \frac{1}{x^2}\Big(\frac{d\eta}{ds} - \eta\Big)$$

and similarly for z. If we insert these expressions into (5.78) and multiply by $x^2 = e^{2s}$, we get the first order system

$$\begin{cases} \frac{dy}{ds} = \eta\,, \quad \frac{d\eta}{ds} = \Big(\frac{3}{2} + \frac{\mathcal{U}}{2}\Big)\eta + e^{3s}\mathcal{A} \\ \frac{dz}{ds} = \zeta\,, \quad \frac{d\zeta}{ds} = \Big(\frac{3}{2} + \frac{\mathcal{U}}{2}\Big)\zeta - \Big(\frac{1}{2} + \frac{\mathcal{V}}{2}\Big)z + e^{3s}\mathcal{B} \end{cases}. \tag{5.79}$$

We rewrite (5.79) in the compact form

$$\frac{d\gamma}{ds} = M\gamma + \mathcal{N}(\gamma) \tag{5.80}$$

where $\gamma = (y, z, \eta, \zeta)^T$, $\mathcal{N}(\gamma) = (0, 0, \frac{\mathcal{U}}{2}\eta + e^{3s}\mathcal{A}, \frac{\mathcal{U}}{2}\zeta - \frac{\mathcal{V}}{2}z + e^{3s}\mathcal{B})^T$ and M is the constant matrix

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix}.$$

From the analysis in the proof of Proposition 5.5 and the equivalence between systems (5.74) and (5.80) it follows that to each solution w of (5.16) that satisfies (5.19) there corresponds a solution γ_w of (5.80) and $w \neq \tilde{w} \Rightarrow \gamma_w \neq \gamma_{\tilde{w}}$. From the estimates in Proposition 5.6 it is straightforward to check that γ_w satisfies

$$|\gamma_w(s)| \leq C_\diamond e^{\frac{3}{2}s}\,, \qquad s \in (-\infty, s_0]\,. \tag{5.81}$$

A key point in the proof of the claimed uniqueness is the fact that the constant $C_\diamond$, as discussed in the proof of Proposition 5.5, depends only on h, b and therefore is the same for all solutions of (5.16), (5.19) that can be associated to given h, b, n.

The matrix M has eigenvalues λ_i and eigenvectors ρ_i as follows

$$\begin{array}{ll} \lambda_1 = 0\,, & \rho_1 = (1,0,0,0)^T \\ \lambda_2 = \frac{1}{2}\,, & \rho_2 = (0,1,0,\frac{1}{2})^T \\ \lambda_3 = 1\,, & \rho_3 = (0,1,0,1)^T \\ \lambda_4 = \frac{3}{2}\,, & \rho_4 = (1,0,\frac{3}{2},0)^T \end{array} \quad ;$$

these properties of the matrix M imply the estimate

$$|e^{Ms}| \leq Ce^{3/2s} \tag{5.82}$$

for some constant $C > 0$ and $s \in [0, +\infty)$.

Let P be the matrix $[\rho_1, \rho_2, \rho_3, \rho_4]$. Given a constant $\delta \in \mathbb{R}$ define $\hat{\delta} \in \mathbb{R}^4$ by setting

$$P\hat{\delta} = \delta\rho_4$$

and let

$$\gamma_\delta(s) = e^{Ms}P\hat{\delta} = e^{\frac{3}{2}s}\delta\rho_4\ . \tag{5.83}$$

Clearly γ_δ is a solution of the homogeneous equation $\frac{d\gamma}{ds} = M\gamma$. Given $K > 0$ and $c \in (0, \frac{1}{2}]$, consider the set X of continuous maps $\gamma : (-\infty, s_0] \to \mathbb{R}^4$ defined by

$$X = \{\gamma : (-\infty, s_0] \to \mathbb{R}^4 : |(\gamma - \gamma_\delta)(s)| \leq Ke^{(1+c)s}\}\ . \tag{5.84}$$

The set X with the distance

$$d(\gamma, \tilde{\gamma}) = \max_{(-\infty, s_0]} |\gamma(s) - \tilde{\gamma}(s)|e^{-s} \tag{5.85}$$

is a complete metric space. Note that, for each fixed δ and provided the constant K is taken sufficiently large, the estimate (5.81) implies $\gamma_w \in X$ for all solutions w of (5.16), (5.19) corresponding to given values of h, b, n. Solutions to (5.16) corresponds through (5.23), (5.74), etc. to continuous solutions $\gamma : (-\infty, s_0] \to \mathbb{R}^4$ of the nonlinear integral equation

$$\gamma(s) = \gamma_\delta(s) + \int_{-\infty}^{s} e^{M(s-r)}\mathcal{N}(\gamma(r))dr. \tag{5.86}$$

To conclude the proof of Proposition 5.9 we shall show that

(I) If $-s_0 > 0$ is sufficiently large, then

$$(T\gamma)(s) = \gamma_\delta(s) + \int_{-\infty}^{s} e^{M(s-r)}\mathcal{N}(\gamma(r))dr \tag{5.87}$$

defines a contraction on X and therefore (5.86) has a unique solution for each $\delta \in \mathbb{R}$.

(II) the choice of δ is uniquely determined by the value of b in (5.20).

To prove (I) we need to use the estimates in Proposition 5.6 to derive corresponding estimates for the functions $\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}$ appearing in the expression of $\mathcal{N}$ in (5.80). From (5.76) and Proposition 5.6 we have

$$\begin{cases} \mathcal{U}, \mathcal{V} = O(|y'|^2 + |z'|^2 + \frac{|z|^2}{x^2} + x) = \\ \qquad = O(e^{-2s}(|\eta|^2 + |\zeta|^2 + |z|^2) + e^s) = O(e^s)\,, \\ \mathcal{A}, \mathcal{B} = O(1)\ . \end{cases} \tag{5.88}$$

Similarly we have the following estimates for the gradients:

$$\mathcal{U}_\gamma\,, \mathcal{V}_\gamma\,, \mathcal{A}_\gamma\,, \mathcal{B}_\gamma = O(e^{-\frac{s}{2}})\ . \tag{5.89}$$

We have from (5.88), (5.89) and (5.81)

$$\begin{aligned} |(\mathcal{U}\eta)(s) - (\tilde{\mathcal{U}}\tilde{\eta})(s)| &\le |(\mathcal{U}(s) - \tilde{\mathcal{U}}(s))\eta(s)| + |\tilde{\mathcal{U}}(s)(\eta(s) - \tilde{\eta}(s))| \\ &\le Ce^{-\frac{s}{2}}|\eta(s)||\gamma(s) - \tilde{\gamma}(s)| + Ce^s|\eta(s) - \tilde{\eta}(s)| \\ &\le Ce^s|\gamma(s) - \tilde{\gamma}(s)| \le Ce^{2s}d(\gamma, \tilde{\gamma}), \end{aligned} \tag{5.90}$$

$$e^{3s}|\mathcal{A}(s) - \tilde{\mathcal{A}}(s)| \le Ce^{\frac{5}{2}s}|\gamma(s) - \tilde{\gamma}(s)| \le Ce^{\frac{7}{2}s}d(\gamma, \tilde{\gamma})\,; \tag{5.91}$$

here and in the remaining part of the proof C is a generic constant that may change value from line to line.

From these and similar estimates for the other terms appearing in $\mathcal{N}$ we get

$$\begin{cases} |\mathcal{N}(\gamma(s))| \le Ce^{\frac{5}{2}s} \\ |\mathcal{N}(\gamma(s)) - \tilde{\mathcal{N}}(\tilde{\gamma}(s))| \le Ce^{2s}d(\gamma, \tilde{\gamma}) \end{cases}. \tag{5.92}$$

From (5.92), (5.82) it follows

$$|(T\gamma)(s) - \gamma_\delta(s)| \le Ce^{\frac{5}{2}s}\,, \tag{5.93}$$

$$|(T\gamma)(s) - (T\tilde{\gamma})(s)| \le Cd(\gamma, \tilde{\gamma}) \int_{-\infty}^{s} e^{2r}dr \le Ce^{2s}d(\gamma, \tilde{\gamma})\,, \tag{5.94}$$

so that

$$e^{-s}|(T\gamma)(s) - (T\tilde{\gamma})(s)| \le Ce^{s_0}d(\gamma, \tilde{\gamma}) \ \ \forall\, s \in (-\infty, s_0]\,,$$

$$\implies \ \ d(T\gamma, T\tilde{\gamma}) \le Ce^{s_0}d(\gamma, \tilde{\gamma})\ .$$

Therefore we see that, provided we take $-s_0 > 0$ sufficiently large, $T : X \to X$ is a contraction. This concludes the proof of (I).

To prove (II) we first observe that if $\overline{\gamma}$ is the fixed point of T, the estimate (5.93) implies that

$$\lim_{s\to-\infty}|\overline{\gamma}(s)-\gamma_\delta(s)|e^{-\frac{3}{2}s}=0\ . \tag{5.95}$$

We also observe that from (5.42), (5.54) and $x=e^s$ it follows

$$\begin{cases} t \propto \frac{2}{3}\sqrt{\frac{2}{\alpha}}x^{\frac{3}{2}}=\frac{2}{3}\sqrt{\frac{2}{\alpha}}e^{\frac{3}{2}s}, \\ \frac{dt}{dx}\propto\sqrt{\frac{2}{\alpha}}x^{\frac{1}{2}}=\sqrt{\frac{2}{\alpha}}e^{\frac{s}{2}} \end{cases}. \tag{5.96}$$

From these asymptotic formulas and the definition of η it follows

$$\eta=y'x=\dot{y}\frac{x}{\dot{x}}\propto\dot{y}\sqrt{\frac{2}{\alpha}}e^{\frac{3}{2}s}\propto\dot{y}\frac{3}{2}t\ . \tag{5.97}$$

From (5.83) and $(5.96)_1$ we obtain

$$\gamma_\delta(t)\propto\frac{3}{2}\sqrt{\frac{\alpha}{2}}\delta\rho_4 t\ . \tag{5.98}$$

Inserting (5.97), (5.98) into (5.95) yields

$$\begin{cases} \lim_{t\to0^+}\frac{y(t)-\frac{3}{2}\sqrt{\frac{\alpha}{2}}t\delta}{t}=0, \\ \lim_{t\to0^+}(\dot{y}(t)-\frac{3}{2}\sqrt{\frac{\alpha}{2}}\delta)=0 \end{cases}.$$

If we take $\delta=\frac{2}{3}\sqrt{\frac{2}{\alpha}}b$, then these equations show that the unique solution determined by the fixed point $\overline{\gamma}$ of T satisfies (5.20). This concludes the proof of Proposition 5.9.

□

Proposition 5.9 is formulated for *ejection* solutions. An analogous uniqueness result applies to *collision* solutions and can be derived from Proposition 5.9 by the variable change $t\to-t$.

Corollary 5.1. *Let $w:(0,\bar{t})\to\mathbb{R}^3$ $((-\bar{t},0)\to\mathbb{R}^3)$ be a maximal ejection (collision) solution to (5.16) and let $\mathsf{n}=\lim_{t\to0^+}\frac{w(t)}{|w(t)|}$ $(=\lim_{t\to0^-}\frac{w(t)}{|w(t)|})$ be the unit vector in Proposition 5.5. Assume there exists an axis $r'\neq r$ of some rotation in $G\setminus\{I\}$ such that n is parallel to the plane rr'. Then*

$$w(t)\in\text{span}\{\mathsf{e}_r,\mathsf{n}\},\ \forall t\in(0,\bar{t})\ \ (\forall t\in(-\bar{t},0))\ . \tag{5.99}$$

Proof. From (5.18) it follows that if w is a solution to (5.16), then $\tilde{R}w$ is also a solution to (5.16) with the same values of b and h. Then Proposition 5.5 implies $\tilde{R}w = w$ and (5.99) follows.

□

The following consequence of Proposition 5.9 is not used in the rest of the paper, but it is of independent interest:

Corollary 5.2. *Let* $w : (-\bar{t}^-, 0) \cup (0, \bar{t}^+) \to \mathbb{R}^3$ *be a maximal ejection-collision solution of (5.16),(5.17). Assume that there exists a unit vector* n *such that*

(i) $\lim_{t\to 0^+} \frac{w(t)}{|w(t)|} = \lim_{t\to 0^-} \frac{w(t)}{|w(t)|} = \mathsf{n}$,
(ii) $\lim_{t\to 0^\pm} (\dot{w}(t) + R\dot{w}(t)) = 0$,
(iii) the energy constant h^- *in the interval* $(-\bar{t}^-, 0)$ *coincides with the energy constant* h^+ *in the interval* $(0, \bar{t}^+)$.

Then $\bar{t}^- = \bar{t}^+ = \bar{t}$ *and* $w(t) = w(-t)$, $t \in (-\bar{t}, \bar{t})$.

We are now in the position of proving

Proposition 5.10. *Let* $u_* \in \overline{\mathcal{K}}$ *be a minimizer of the action. Then* $u_* \in \mathcal{K}$ *and it is collision free.*

Proof. By the discussion in Section 5.1 u_* does not have total collisions. From Proposition 5.8 and Remark 5.3 u_* can not have partial collisions apart from collisions of type ($\rightrightarrows$). From Lemma 5.2 and Corollary 5.1 it follows that

$$u_{*,1}(t) \in \Pi\,, \qquad \forall t \in (t_c - \bar{t}, t_c + \bar{t})\ .$$

Since the only possible collisions for u_* are partial collisions of type ($\rightrightarrows$) at times $t = t_c - \bar{t}^-, t_c + \bar{t}^+$, u_* has collisions of this kind. Therefore, invoking again Corollary 5.1, we obtain

$$u_{*,1}(t) \in \Pi\,, \qquad \forall t \in \mathbb{R}\ .$$

This contradicts membership in $\overline{\mathcal{K}}$ and concludes the proof.

□

Remark 5.4. As we have seen there are infinitely many pairwise disjoint cones $\mathcal{K}(u) \subset \Lambda_0^{(\mathsf{a})}$ corresponding to simple u_1. Moreover the subset of such cones that satisfy the condition

$$S^{\tilde{k}} \neq S^k\,, \qquad k \in \mathbb{Z} \tag{5.100}$$

is infinite. This is trivially seen by observing that if $u \in \Lambda_0^{(\mathsf{a})}$ is such that the sequence σ_u satisfies condition (5.100). it follows that all the cones $\mathcal{K}_n, n = 1, ..$ corresponding to the pairs $(\sigma_u, n), n = 1, ...$ satisfy condition (5.100).

It is easily seen that besides these rather trivial cases there are many other situations where (5.100) holds.

Are all these minimizers genuine periodic solutions of the classical Newtonian N–body problem? Our conjecture is that when the linking of the trajectory of the generating particle τ_1 with Γ becomes more and more complex, from the point of view of minimization of the action, it may be preferable to *crash* some of the complications into a total collision. If this conjecture is correct then only a finite number of minimizers corresponding to simple linking of τ_1 with Γ are genuine periodic solutions of the N-body problem.

6. Complements, conjectures and numerical results

Theorem 4.2 yields valuable information on the geometric structure of the periodic motion $u_* \in \mathcal{K}$ considered in Theorem 4.1. In particular we know that the orbit τ_1 of the generating particle is contained in the cone $C_{\mathcal{K}} \subset \mathbb{R}^3$, $C_{\mathcal{K}} = \cup_{k\in\mathbb{Z}} S_k \cup D_k$ with $\{D_k\}_{k\in\mathbb{Z}}$ the sequence corresponding to $\mathcal{K}$. Another consequence of Theorem 4.2 is the existence of motions that violate the coercivity condition (1.6) (cfr. Theorem 1.1). We have indeed

Theorem 6.1. *Assume $P \in \{\mathfrak{C}, \mathfrak{I}, \mathfrak{D}\}$. Then there exists a smooth T–periodic solution $v_*^{P,i} \in \mathcal{K}_i^P$ of the classical N–body problem such that the trajectory τ_1 of the generating particle is contained in the half space $\xi_1 > 0$. In particular we have*

$$\int_0^T v_{*,1}^{P,i}(t)\, dt > 0\ .$$

Proof. Let ν be the minimal sequence associated to $\mathcal{K}_i^P$ in Section 5 (cfr. Table 4). Let $v_*^{P,i}$ be the minimizer of the action restricted to the cone $\mathcal{K}(\mathrm{v}^{(\nu,1)}) \subset \mathcal{K}_i^P$. All the arguments developed in Section 5 to prove that a minimizer $u_*^{P,i}$ of $\mathcal{A}|_{\mathcal{K}_i^P}$ is free of collisions apply verbatim to show that also $v_*^{P,i}$ is free of collisions. Therefore $v_*^{P,i}$ is a classical solution of the N–body problem. To conclude the proof we observe that if $P \in \{\mathfrak{C}, \mathfrak{I}, \mathfrak{D}\}$ and $u = v_*^{P,i}$, the cone $C_{\mathcal{K}(u)} \subset \mathbb{R}^3, C_{\mathcal{K}(u)} = \cup_{k\in\mathbb{Z}} S_k \cup D_k$ with $\{D_k\}_{k\in\mathbb{Z}}$ the sequence corresponding to $\mathcal{K}(u)$, is actually contained in the half space $\xi_1 > 0$

□

The motion $u_*^{P,i}$ is a minimizer of $\mathcal{A}$ on $\mathcal{K}_i^P$, which properly contains $\mathcal{K}(\mathrm{v}^{(\nu,1)})$; therefore $u_*^{P,i}$ and $v_*^{P,i}$ may well be different solutions of the N–body problem in $\mathcal{K}_i^P$. However we conjecture that $v_*^{P,i} = u_*^{P,i}$, and this conjecture is supported by numerical experiments.

Given a pair (σ, n), with σ satisfying (I), (II), (III) in Section 4.1 and $n \in \mathbb{N}$, the associated $\mathrm{u}_1^{(\sigma,n)}$ defined in (4.3) possesses all the space and time symmetries compatible with its topological structure. We can minimize the action on $\mathcal{K}(\mathrm{u}^{(\sigma,n)})$ or on the subset $\mathcal{K}^S(\mathrm{u}^{(\sigma,n)})$ of the maps u with the property that u_1 has the same symmetries as $\mathrm{u}_1^{(\sigma,n)}$. All numerical experiments we have made confirm the conjecture that a global minimizer $u_* \in \mathcal{K}(\mathrm{u}^{(\sigma,n)})$ is actually in $\mathcal{K}^S(\mathrm{u}^{(\sigma,n)})$.

Assume $u_* \in \mathcal{K}(\mathrm{u}^{(\sigma,1)})$ is a minimizer of $\mathcal{A}|_{\mathcal{K}(\mathrm{u}^{(\sigma,1)})}$. Then the map $u_*^n \in \mathcal{K}(\mathrm{u}^{(\sigma,n)})$ defined by

$$u_*^n(t) := n^{-2/3} u_*(nt), \;\; t \in [0, T)$$

is a critical point of $\mathcal{A}|_{\mathcal{K}(\mathrm{u}^{(\sigma,n)})}$. This follows from the fact that the map $h_n : \mathcal{K}(\mathrm{u}^{(\sigma,1)}) \to \mathcal{K}(\mathrm{u}^{(\sigma,n)})$ defined by $(h_n u)(t) = u^n(t) = n^{-2/3} u(nt)$ satisfies

$$\mathcal{A}(u^n) = n^{2/3} \mathcal{A}(u) \; . \tag{6.1}$$

However we do not expect u_*^n to be a minimizer of $\mathcal{A}|_{\mathcal{K}(\mathrm{u}^{(\sigma,n)})}$ for every $n > 1$. Indeed we have

Proposition 6.1. *Given $u \in \mathcal{K}(\mathrm{u}^{(\sigma,1)})$, if $n > n_0$, for some $n_0 > 1$, there exists $\widehat{u^n} \in \mathcal{K}(\mathrm{u}^{(\sigma,n)})$ such that*

$$\mathcal{A}(u^n) - \mathcal{A}(\widehat{u^n}) > 0 \; .$$

Proof. We consider the case of n even. The other case is analogous. Fix a number $\tau \in (0, 1)$ and set

$$\begin{aligned} T_1 &= \Big(\frac{1}{2} - \frac{1}{n}\Big)(1+\tau)T \, , \\ T_2 &= T_1 + \frac{T}{n} \, , \\ T_3 &= T_2 + \Big(\frac{1}{2} - \frac{1}{n}\Big)(1-\tau)T \; . \end{aligned}$$

Define $\widehat{u^n}$ by setting

$$\widehat{u_1^n}(t) = \begin{cases} \Big(\dfrac{1}{1+\tau}\Big)^{-2/3} u_1^n\Big(\dfrac{t}{1+\tau}\Big) \, , & 0 \le t \le T_1 \\ n\left[\dfrac{T_2 - t}{T}\Big(\dfrac{1}{1+\tau}\Big)^{-2/3} + \dfrac{t - T_1}{T}\Big(\dfrac{1}{1-\tau}\Big)^{-2/3}\right] u_1^n(t - T_1) \, , & T_1 \le t \le T_2 \\ \Big(\dfrac{1}{1-\tau}\Big)^{-2/3} u_1^n\Big(\dfrac{t - T_2}{1-\tau}\Big) \, , & T_2 \le t \le T_3 \\ n\left[\dfrac{T - t}{T}\Big(\dfrac{1}{1-\tau}\Big)^{-2/3} + \dfrac{t - T_3}{T}\Big(\dfrac{1}{1+\tau}\Big)^{-2/3}\right] u_1^n(t - T_3) \, , & T_3 \le t \le T \end{cases} \; . \tag{6.2}$$

Then we have, with obvious notation, recalling also (6.1),

$$\begin{cases} \mathcal{A}(\widehat{u^n},(0,T_1)) = \dfrac{n-2}{2}\Big(\dfrac{1+\tau}{n}\Big)^{1/3}\mathcal{A}(u) = \dfrac{n-2}{2n}(1+\tau)^{1/3}\mathcal{A}(u^n) \leq \\ \qquad\qquad \leq \Big[\dfrac{1}{2}(1+\tau)^{1/3} + \dfrac{C}{n}\Big]\mathcal{A}(u^n) \\ \mathcal{A}(\widehat{u^n},(T_2,T_3)) = \dfrac{n-2}{2}\Big(\dfrac{1-\tau}{n}\Big)^{1/3}\mathcal{A}(u) = \dfrac{n-2}{2n}(1-\tau)^{1/3}\mathcal{A}(u^n) \leq \\ \qquad\qquad \leq \Big[\dfrac{1}{2}(1-\tau)^{1/3} + \dfrac{C}{n}\Big]\mathcal{A}(u^n) \\ \mathcal{A}(\widehat{u^n},(T_1,T_2)\cup(T_3,T)) \leq \dfrac{C}{n}\mathcal{A}(u^n) \end{cases} \tag{6.3}$$

where $C > 0$ is a constant that does not depend on n. It follows

$$\mathcal{A}(u^n) - \mathcal{A}(\widehat{u^n}) \geq \left[\frac{1}{2}(2-(1+\tau)^{1/3}-(1-\tau)^{1/3}) - 3\frac{C}{n}\right]\mathcal{A}(u^n)\ .$$

This inequality and $2 > (1+\tau)^{1/3} + (1-\tau)^{1/3}$ conclude the proof.

□

By analogy with the fact that, in the Kepler problem, if the index of the orbit with respect to the center of attraction is different from $-1, 0, 1$, the only minimizer is the collision–ejection motion [18], we don't expect that, if $u \in \Lambda_0^{(\mathsf{a})}$ is not simple in the sense of Definition 4.2, then a minimizer $u_* \in \mathcal{K}(u)$ is collision free. In Section 5, Remark 5.4, we have conjectured that, in spite of the fact that there are infinitely many pairwise disjoint cones $\mathcal{K}(u) \subset \Lambda_0^{(\mathsf{a})}$ corresponding to a simple u, only for a finite number of them the corresponding minimizers are indeed genuine T–periodic solutions of the Newtonian N–body problem. To put this conjecture in perspective we consider the N-body problem with generalized potential $\frac{1}{r^\alpha}$, $\alpha > 0$, and the corresponding action functional

$$\mathcal{A}^\alpha(u) = \frac{N}{2}\int_0^T \Big(|\dot{u}_1(t)|^2 + \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u_1|^\alpha}\Big)\ . \tag{6.4}$$

It is well known that on the basis of Sundman's estimates in the case of strong forces ($\alpha \geq 2$) $\mathcal{A}^\alpha(u) < +\infty$ implies that u is collision free. Therefore, if $\mathcal{N}^\alpha$ is the number of the cones $\mathcal{K} \in \Lambda_0^{(\mathsf{a})}/\!\sim$ such that a minimizer $u_*^\alpha \in \mathcal{K}$ of $\mathcal{A}^\alpha|_{\mathcal{K}}$ is collision free, we have $\mathcal{N}^\alpha = +\infty$ for $\alpha \in [2,+\infty)$. We conjecture that, on the other hand, $\mathcal{N}^\alpha < +\infty$ for $\alpha \in (0,2)$ and that $\mathcal{N}^\alpha$ is a non-decreasing function of α. It is also natural to conjecture that for each (σ, n) there is a critical value

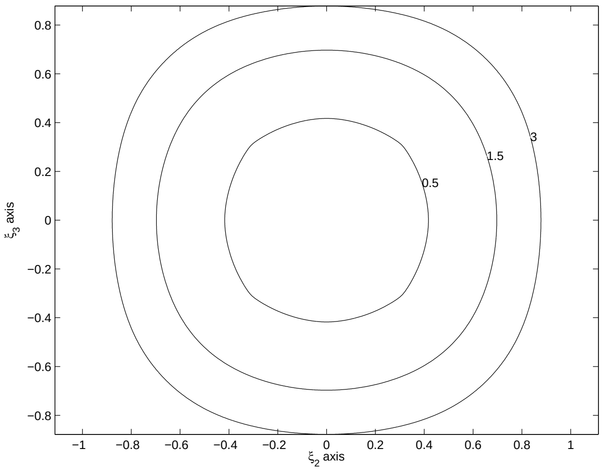


Figure 16. Shrinking of the trajectory of the periodic orbits found for the cone $\mathcal{K}_2^P$, with $P = \mathfrak{C}$, as α is decreasing. In the figure we use $\alpha = 3, 1.5, 0.5$

$\alpha^{(\sigma,n)} > 0$ such that a minimizer of $\mathcal{A}^\alpha|_{\mathcal{K}(\mathrm{u}^{(\sigma,n)})}$ is collision free for $\alpha > \alpha^{(\sigma,n)}$, while it has collisions for $\alpha \in (0, \alpha^{(\sigma,n)}]$.

Natural questions to ask are: what are the transformations a minimizer $u_*^\alpha \in \mathcal{K}(\mathrm{u}^{(\sigma,n)})$ undergoes when α crosses the critical value $\alpha_{(\sigma,n)}$ and decreases to 0^+? What are the asymptotic properties of a minimizer $u_*^\alpha \in \mathcal{K}(\mathrm{u}^{(\sigma,n)})$ when $\alpha \to +\infty$?

When $\alpha \to 0^+$ the radius of action of the attractive forces between particles converges to zero. This implies that, at each time t, the generating particle P_1 has to be near some of the other $N-1$ particles. Indeed, otherwise, P_1 could not accelerate to get around the axes of the rotations in $\mathcal{R}$ as required by the topological constraint of membership in $\mathcal{K}(\mathrm{u}^{(\sigma,n)})$. This leads to conjecture that the diameter of the orbit of the generating particle P_1 converges to zero as $\alpha \to 0^+$. Then it is also to be expected that for travelling a shorter and shorter trajectory in a fixed time T also the average speed of P_1 tends to zero:

$$\begin{cases} \lim_{\alpha\to 0^+} ||u_{*,1}^\alpha||_{L^\infty} = 0 \\ \lim_{\alpha\to 0^+} ||\dot{u}_{*,1}^\alpha||_{L^1} = 0 \end{cases} . \qquad (6.5)$$

In Figure 16 we show the trajectory τ_1^α of P_1 for various values of α. The shrinking of τ_1^α for decreasing α is clearly visible.

If $\alpha >> 1$ the force of attraction between particles is very small when the interparticle distance $|x_i - x_j|$ is larger than 1 and very large when it is smaller than 1 (recall that the gravitational constant is normalized to 1). This observation suggests:

1) the limit behavior of minimizers for $\alpha \to +\infty$ is constrained to the sub-region $\mathcal{Y} \subset \mathcal{X}$ of the configuration space defined by

$$\mathcal{Y} = \{x \in \mathcal{X} : |x_i - x_j| \geq 1, \forall i \neq j\} \ .$$

Indeed violating this condition generates a large contribution of the potential term to the action integral;
2) in the limit $\alpha \to +\infty$ the trajectory τ_1^α of P_1, corresponding to a minimizer $u_*^\alpha \in \mathcal{K}(\mathrm{u}^{(\sigma,n)})$, is the shortest possible compatible with the condition $|(R - I)x_1| \geq 1$, $\forall R \in \mathcal{R} \setminus \{I\}$ and with the topological constraint that $u_{*,1}^\alpha$ is homotopic to $\mathrm{u}_1^{(\sigma,n)}$.

The rationale behind this is that, in the limit $\alpha \to +\infty$, the interparticle attraction should act as a 'perfect holonomic constraint' and the limit motion should be a kind of 'geodesic motion' with constant kinetic energy. Then minimizing the action should be equivalent to the minimization of the length of the trajectory of the generating particle. In conclusion we advance the following conjecture. To formulate it we first observe that the characterization of the set $\mathcal{Y} \subset \mathcal{X}$ is equivalent to the characterization of the set $\mathcal{Y}_1 \subset \mathbb{R}^3$, $\mathcal{Y}_1 = \{x_1 : |Rx_1 - x_1| \geq 1, \forall R \in \mathcal{R} \setminus \{I\}\}$. Let $\mathcal{P} = \Gamma \cap \{x_1 \in \mathbb{R}^3 : |x_1| = 1\}$, the set of poles of $\mathcal{R}$. For each $p \in \mathcal{P}$ let $Cyl_p \subset \mathbb{R}^3$ be the open cylinder with axis the line through O and P and radius $r_p = \frac{1}{2\sin(\pi/|C_p|)}$, where $C_p \subset \mathcal{R}$ is the maximal cyclic group corresponding to p. Then we have

$$\mathcal{Y}_1 = \mathbb{R}^3 \setminus \bigcup_{p \in \mathcal{P}} Cyl_p \ .$$

Conjecture 6.1. Let $u_*^\alpha \in \mathcal{K}(\mathrm{u}^{(\sigma,n)})$ be a minimizer of $\mathcal{A}^\alpha|_{\mathcal{K}(\mathrm{u}^{(\sigma,n)})}$. Then, in the limit $\alpha \to +\infty$, $u_{*,1}^\alpha$ converges (possibly up to subsequences) in the C^1 topology to a minimizer of the problem

$$\min_{u_1 \in \mathcal{U}} \frac{1}{2} \int_0^T |\dot{u}_1(t)|^2 \, dt \,,$$

where $\mathcal{U}$ is the subset of H^1 T–periodic maps $u_1 : \mathbb{R} \to \mathcal{Y}_1$ such that u_1 is homotopic to $\mathrm{u}_1^{(\sigma,n)}$.

In other words, in the limit $\alpha \to +\infty$, the speed of the generating particle is constant and the trajectory coincides with the path of a wire stretched on the boundary of $\mathcal{Y}_1$ and homotopic to $\mathrm{u}_1^{(\sigma,n)}$. Since $\mathcal{Y}_1$ is a unilateral constraint the limit trajectory is expected to be the union of 'geodesic arcs' lying on the $\partial\mathcal{Y}_1$ and segments in $\mathbb{R}^3 \setminus \mathcal{Y}_1$ touching $\partial\mathcal{Y}_1$ at the extrema. If we apply Conjecture 6.1 to the case of four bodies considered in Section 2 we get that the limit trajectory of the generating particle is the union of four circular arcs of angle π and radius 1/2 and that the constant limit speed $|\dot{u}^{\infty}_{*,1}|$ and corresponding action $\mathcal{A}^{\infty}$ are given, for $T = 1$, by

$$\begin{cases} |\dot{u}^{\infty}_{*,1}| = 2\pi \\ \mathcal{A}^{\infty} = 8\pi \end{cases} . \tag{6.6}$$

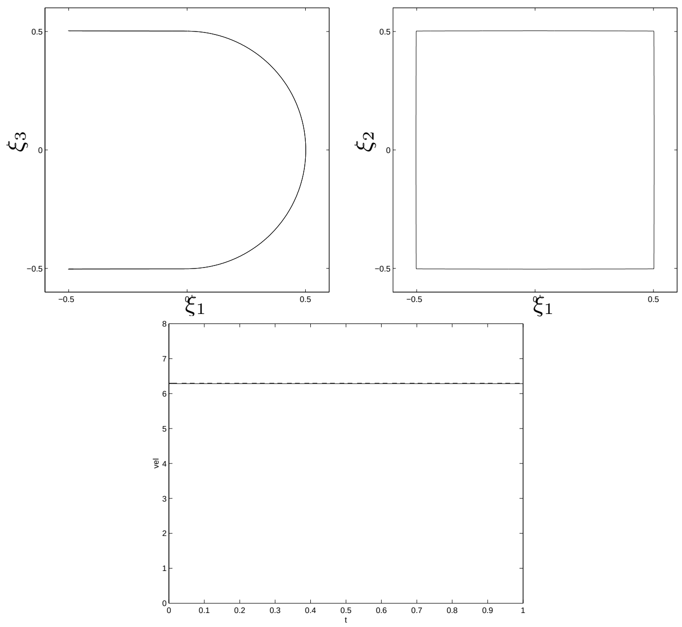


Figure 17. Projections of τ_1^{1000} and graph of $t \to |\dot{u}^{1000}_{*,1}(t)|$. The dashed line corresponds to the conjectured value for $|\dot{u}^{\infty}_{*,1}(t)|$.

In Figure 17 we show orthogonal projections of τ_1^{α} and the graph of the map $t \to |\dot{u}^{\alpha}_{*,1}(t)|$ computed numerically for $\alpha = 1000$. The value of $|\dot{u}^{1000}_{*,1}(t)|$ is practically coincident with the limit value 2π given by (6.6). Also the numerical value of $\mathcal{A}^{1000}$ is practically equal to 8π.

We found good agreement with this conjecture in several other situations. For instance, for the cone $\mathcal{K}_2^P$ defined by (3.3) with $P = \mathfrak{C}$, a simple computation (cfr. Figure 18) based on Conjecture 6.1 shows that the limit motion of the generating particle should satisfy

$$\begin{cases} u_{*,1}{}^{\infty}(0) = (\frac{1}{\sqrt{2}}, \sqrt{2}, 0) \\ u_{*,1}{}^{\infty}(\frac{T}{8}) = (\frac{1}{2}, \frac{1}{\sqrt{2}}(1 + \frac{1}{\sqrt{2}}), \frac{1}{\sqrt{2}}(1 + \frac{1}{\sqrt{2}})) \end{cases} \tag{6.7}$$

In Figure 19 we plot as functions of α the components of $u^{\alpha}_{*,1}(0)$ and $u^{\alpha}_{*,1}(T/8)$ and the theoretical values given by (6.7). We also plot

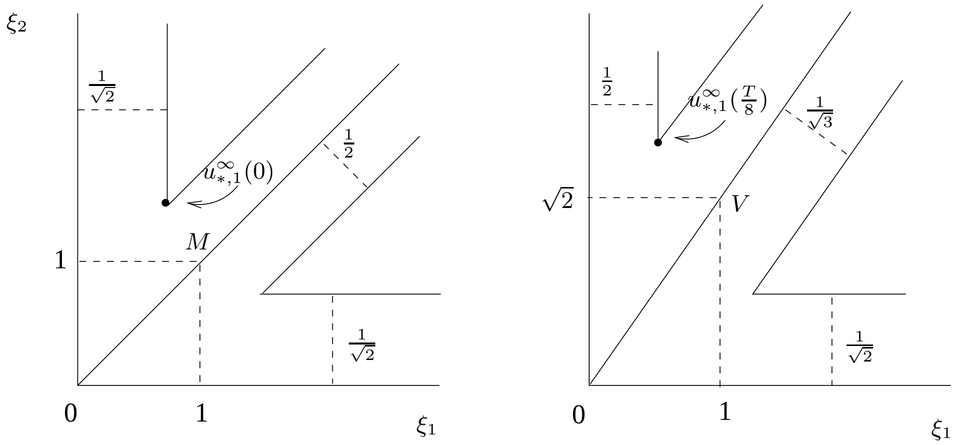


Figure 18. The geometry of $\mathcal{Y}_1$ and $u^\infty_{*,1}(0), u^\infty_{*,1}(T/8)$ for $\mathcal{K}_2^P$, $P = \mathfrak{C}$. We plot the section of $\mathcal{Y}_1$ with the plane $\xi_3 = 0$ (left) and the one with the plane determined by ξ_1 and V (right).

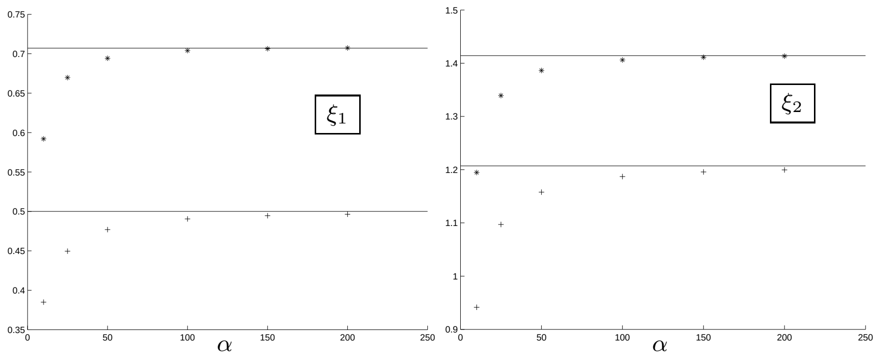


Figure 19. Numerical simulations of $u^\alpha_{*,1}(0) \cdot \mathsf{e}_j$ and $u^\alpha_{*,1}(\frac{T}{8}) \cdot \mathsf{e}_j$ for $j = 1, 2$ and $\alpha = 10, 25, 50, 100, 150, 200$. The asterisks correspond to $u^\alpha_{*,1}(0)$, the crosses to $u^\alpha_{*,1}(\frac{T}{8})$. The horizontal lines indicate the values in (6.7).

the speed $|\dot{u}^\alpha_{*,1}(t)|$ for several values of $\alpha >> 1$. For $\alpha = 200$ the relative error is within 1/100.

The class of cones $\mathcal{K}$ such that (1.7) holds can be largely generalized by requiring the sequence $\sigma = \{D_k\}_{k\in\mathbb{Z}}$ introduced in Subsection 4.1 to satisfy only the conditions (II), (III). As before we can associate to σ a motion $\mathrm{u}^\sigma : \mathbb{R} \to \mathbb{R}^{3N}$ as in Section 4.1 and consider the cone $\mathcal{K}^\sigma \subset H^1(\mathbb{R}, \mathbb{R}^{3N})$ of the maps homotopic to u^σ. The cones $\mathcal{K}^\sigma$ satisfy (1.7) and, even if we restrict to sequences σ that are simple in the sense of Definition 4.2, form an uncountable family. Now we can not talk about a minimizer $u_* \in \mathcal{K}^\sigma$ since the action $\mathcal{A}(u)$ of each $u \in \mathcal{K}^\sigma$ is infinite. But we can still regard as a minimizer a function $u_* \in \mathcal{K}^\sigma$ with the property that the restriction $u_*|_{[t_1,t_2]}$

to each compact interval $[t_1, t_2], t_1 < t_2$ minimizes the action on the set of maps $u : [t_1, t_2] \to \mathbb{R}^{3N}$ that satisfy $u(t_i) = u_*(t_i), i = 1, 2$ and are such that $\chi_{\mathbb{R}\setminus[t_1,t_2]}u_* + \chi_{[t_1,t_2]}u$ is homotopic to u^σ. For potentials corresponding to $\alpha \geq 2$, such a minimizer u_* can not have collisions and, depending on the particular sequence σ considered, it may exhibit several interesting and complex behaviors, including heteroclinic connections between periodic orbits and chaotic motion. For the Newtonian potential ($\alpha = 1$), at least under the assumption that the sequence σ satisfies condition (5.100), the discussion in Section 5 excludes the possibility of partial collisions. We conjecture that in many cases, actually infinitely many, the minimizers corresponding to such sequences σ do not have total collisions too and therefore are genuine solutions of the classical N–body problem.

Returning to the setting of periodic motions, we note that a natural generalization of the situation discussed in Section 2 where we have just a single generating particle that determines the motion of all the other particles, is obtained by considering $M \geq 1$ generating particles.

Given M positive constants $m_h, h = 1, .., M$ and M cones $\mathcal{K}_h \subset \Lambda_0^{(\mathsf{a})}, h = 1, ..., M$ we can consider maps $U = (u^1, ..., u^M) \in (\Lambda_0^{(\mathsf{a})})^M$ and look for minimizers of the action

$$\mathcal{A}(U) = \frac{1}{2}\int_0^T \left(N \sum_{h=1}^{M} |\dot{u}_1^h(t)|^2 + \sum_{\substack{R,R' \in \mathcal{R} \\ (R',h) \neq (R',k)}} \frac{m_h m_k}{|R_i u_1^h - u_1^k|} \right) dt \quad (6.8)$$

on the cone $\mathcal{K} = \mathcal{K}(u^1) \times \mathcal{K}(u^2) \times \ldots \mathcal{K}(u^M) \subset (\Lambda_0^{(\mathsf{a})})^M$. In (6.8) $u_1^h : \mathbb{R} \to \mathbb{R}^3$ is the motion of the h–th generating particle. The analogous of Proposition 4.1 holds: $\mathcal{A}(U)|_{\mathcal{K}}$ is coercive.

We conclude this Section by a few remarks on the method used for numerical simulation of the orbits discussed in the present paper. The method is based on a rather simple and natural idea already used in ([22]): we consider the L^2 gradient dynamics defined by the action functional

$$u_\theta = -grad_{L^2}\mathcal{A}(u) \quad (6.9)$$

with periodic boundary conditions. Here θ is a fictitious time and the true time t plays the role of a space variable. Suppose that $\mathcal{K} \in \Lambda_0^{(\mathsf{a})}/\!\sim$ is such that if $u_* \in \mathcal{K}$ is a minimizer then

$$\mathcal{A}(u_*) < a_c := \inf\{\mathcal{A}(u),\, u \in \overline{\mathcal{K}},\, u \in \mathfrak{S}\} \quad (6.10)$$

It follows that u_* is collision free and that, if $\overline{u} \in \mathcal{K}$ satisfies the condition $\mathcal{A}(\overline{u}) < a_c$ then the dynamics defined by (6.9), with initial datum $\overline{u}$, automatically preserves the cone $\mathcal{K}$:

$$u(\cdot, \theta, \overline{u}) \in \mathcal{K},\ \forall \theta \geq 0 \quad (6.11)$$

$\theta \to u(\cdot, \theta, \overline{u})$ being the solution of (6.9) through $\overline{u}$, and remains away from $\mathfrak{S}$. This follows from $\mathcal{A}(u(\cdot, \theta, \overline{u})) \leq \mathcal{A}(\overline{u}) < a_c$ and from (6.10) which imply $u(\cdot, \theta, \overline{u})$ can not cross the boundary of $\mathcal{K}$.

Once (6.11) is established, the general theory of infinite dimensional dissipative dynamical systems (see Lemma 3.8.2 in [19]) implies that the ω–limit set of $\overline{u}$ is contained in the set E of equilibria of (6.9), $E := \{u \in \mathcal{K} : \mathrm{grad}_{L^2}\mathcal{A}(u) = 0\}$, that is, in the set of periodic solutions of the classical Lagrange's equations which are the objects of our interest.

Based on (6.9) one can set up an automatic procedure for the numerical computation of a minimizer $u_* \in \mathcal{K}$ for each given $\mathcal{K} \in \Lambda_0^{(\mathsf{a})}/\!\sim$. To do this it suffices to develop a systematic way for constructing a suitable initial condition $\bar{u} \in \mathcal{K}$. Our choice is

$$\bar{u} = \mathrm{v}^{(\nu,n)}$$

where (ν, n) is the pair corresponding to $\mathcal{K}$ in the sense of Proposition 4.3 and $\mathrm{v}^{(\nu,n)}$ is defined in (4.14). We have developed a routine that, given a sequence ν of vertexes of an Archimedean polyhedron $\mathcal{Q}_{\mathcal{R}}$ and a number $n \in \mathbb{N}$ computes a minimizer u_* of the action on the cone $\mathcal{K}$ corresponding to (ν, n). A sample of motions computed in this way can be found at `http://adams.dm.unipi.it/~gronchi/nbody/`.

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