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SCHOOL ON DYNAMICAL SYSTEMS

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Notes on Dynamical Systems

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CHAPTER I. TRANSFORMATION THEORY

1. Differential Equations and Vector Fields.

(a) The flow of a system of differential equations.

The object of these lectures are systems of ordinary differential equations of the form

$$\frac{dx}{dt} = f(x) \tag{1.1}$$

or in components,

$$\frac{dx_j}{dt} = f_j(x) \qquad (j = 1,\ldots,n), \tag{1.1'}$$

defined in an open domain $D \subset R^n$. The right-hand side $f(x)$ is a vector valued function mapping D into R^n, belonging to $C^r(D,R^n)$ for $r \geq 1$. We recall the well known fact which will not be proven here that the system (1.1) has a unique solution $x(t)$ for a given initial value $x(0) \in D$, where the solution $x(t)$ is defined on an interval $|t| < \delta$, $\delta > 0$. More precisely, if K is a compact subset of D then there exists a $\delta > 0$ depending on K and f such that the solution $x(t)$ with initial values $x(0) \in K$ exists for the interval

$$I = \{t \mid |t| < \delta\}$$

To indicate the dependence on the intial value $x(0)$ we denote this solution by

$$x(t) = \phi^t(x(0)).$$

Then, according to the standard existence theorem

$$\phi^t(x(0)) \in C^r(I \times K, D) .$$

For fixed $t \in I$ we can view ϕ^t as a mapping of K into D, which satisfies

$$(1.2) \qquad \phi^t \circ \phi^s = \phi^{t+s}$$

for sufficiently small values of $|t|$, $|s|$, and

$$(1.3) \qquad \phi^t = \text{identity for } t = 0 .$$

This one parameter family of mappings ϕ^t is called the "flow" of the system (1.1). Clearly we have

$$\frac{d\phi^t}{dt} = f(\phi^t)$$

for sufficiently small $|t|$. Setting $t = 0$ we see that ϕ^t, in turn, determines f uniquely.

Setting $s = -t$ in (1.2) we see that ϕ^t has an inverse (defined on $\phi^t(K)$) which is also in C^r. We will call a C^r-mapping which has a C^r inverse a C^r-diffeomorphism. Thus ϕ^t is a C^r diffeomorphism, where defined.

We will also consider C^∞ systems, in which case ϕ^t is a C^∞-diffeomorphism as well as C^ω systems, for which f(x) is real analytic in which case ϕ^t is also real analytic, which we call a C^ω-diffeomorphism.

Geometrically one interprets the system (1.1) as a vector field, which assigns to each point $x \in D$ the vector f(x). The solution $x(t) = \phi^t(x(0))$ is then a curve which at every point is tangent to this vector field. We will use the term system of differential equations and vector field interchangeably.

(b) Transformation properties

We subject the system (1.1) to an invertible coordinate transformation

$$x = u(y)$$

where we assume that the Jacobian matrix

$$\left(\frac{\partial u_i}{\partial y_j}\right) = u_y$$

is invertible. Then (1.1) goes over into a new system, say,

$$\frac{dy}{dt} = g(y)$$

where

$$(1.4) \qquad g(y) = u_y^{-1} f(u(y)) .$$

This is the transformation law for vector fields. If we denote by ψ^t the flow belonging to g we have

$$(1.5) \qquad \psi^t = u^{-1} \circ \phi^t \circ u$$

where $\circ$ indicates composition of the various diffeomorphisms and u^{-1} denotes the inverse map of u. Of course, the above relations have to be restricted to domains in which the mappings are defined.

To verify (1.5) we simply define ψ^t by (1.5) and then show that it agrees with the flow for g. Clearly ψ^t = identity for $t = 0$ and differentiating the relation

$$u \circ \psi^t = \phi^t \circ u$$

we get

$$u_y \frac{d\psi^t}{dt} = f(\phi^t \circ u) = f(u \circ \psi^t) = u_y g(\psi^t)$$

hence

$$\frac{d\psi^t}{dt} = g(\psi^t) \ .$$

Since ψ^t is uniquely determined by this relation and its initial value (1.5) is proven.

The transformation law (1.4) is the same as that for the partial differential operators

$$X = \sum_{j=1}^{n} f_j(x) \frac{\partial}{\partial x_j} \ .$$

To describe the transformation laws under $x = u(y)$ we ~~and~~ observe that for any $h = h(x) \in C^1$ the expression

$$(Xh) \circ u$$

must be expressible in terms of a differential operator Y acting on $h \circ u$, i.e.

$$(Xh) \circ u = Y(h \circ u) \ .$$

We call Y the transformed differential operator and denote it by u^*X, so that

(1.6) $$(u^*X)(h \circ u) = (Xh) \circ u \ .$$

If we write

$$u^*X = \sum g_k(y) \frac{\partial}{\partial y_k}$$

we find for the vector $g = (g_k)$ readily

$$g = u_y^{-1} f \circ u \ ,$$

as we claimed.

There is a more direct relationship between the vector field (1.1) and X, namely

$$Xh = \frac{d}{dt} h(\phi^t)\big|_{t=0}$$

i.e. X is the directional derivative of a function h along the vector field.

The operator X is determined by the vector field f and conversely X determines f; indeed for $h = x_j$ we find

$$Xx_j = f_j(x) \ .$$

Therefore we will also use the operator X to describe the vector field. This is merely another notation which, however, has the advantage to reflect the transformation law under coordinate transformations. For this reason this notation is preferred in differential geometry and in the global study of vector fields on manifolds.

Incidentally, this notation shows that the vector fields in D form a Lie algebra since the commutator

$$XY - YX = [X,Y]$$

of two vector fields X,Y defines again a vector field. Indeed if

$$X = \sum_{j=1}^{n} f_j(x) \frac{\partial}{\partial x_j} \ , \qquad Y = \sum_{k=1}^{n} g_k(x) \frac{\partial}{\partial x_k}$$

then

$$[X,Y] = \sum_{j,k=1}^{n} \left(f_j g_{kx_j} - g_j f_{kx_j}\right) \frac{\partial}{\partial x_k}$$

since the second order derivatives cancel. It is an almost obvious consequence of the definition (1.6) that

$$u^*[X,Y] = [u^*X,u^*Y]$$

so that the definition of $[X,Y]$ is independent of the choice of the coordinates.

(c) Local equivalence of vector fields.

Two vector fields f,g which can be transformed into each other will be considered as equivalent, i.e., f,g are considered equivalent in some domains D_1,D_2 resp., if there exists a diffeomorphism $u: D_2 \to D_1$ for which

$$g = u_y^{-1} f \circ u \ .$$

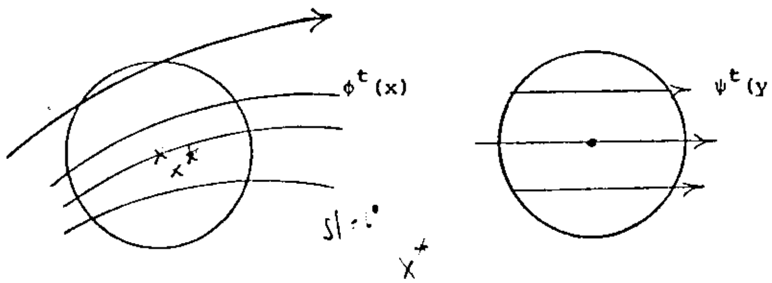
Only properties which are preserved under such transformation are of interest.

Therefore it is important to realize that locally, in the neighborhood of a point at which $f \neq 0$ it is equivalent to any other vector field with this property, e.g. to the vector field

$$\frac{dy}{dt} = e_1$$

where e_1 is the unit vector in the y_1 direction. Geometrically this statement simply means that in a small neighborhood of a point x^* where $f(x^*) \neq 0$ the flow can be mapped into the parallel flow

$$\psi^t(y) = y + te_1 \ .$$



<u>Lemma 1</u>. If f, g define two vector fields with $f(x^*) \neq 0$, $g(y^*) \neq 0$ then there exist two neighborhoods D_1,D_2 of x^*,y^* respectively and a map $u: D_2 \to D_1$ such that $u_y^{-1} f \circ u = g$.

<u>Proof</u>: We may take $x^* = y^* = 0$ by applying a translation. By appropriate choice of the coordinate axes we may assume $f_1(0) = \langle f(0),e_1\rangle \neq 0$. If $\phi^t(x)$ defines the flow of f we set

$$u(y) = \phi^{y_1}(0,y_2,\ldots,y_n).$$

Then one computes readily

$$\det(u_y(0)) = f_1(0) \neq 0$$

so that $x = u(y)$ defines a diffeomorphism near $x = 0$. Moreover, with $\psi^t = y + te_1$ we have

$$u \circ \psi^t(y) = \phi^{y_1+t}(0,y_2,\ldots,y_n) = \phi^t \circ u(y)$$

so that u maps ϕ^t into the parallel flow ψ^t. By differentiation we find

$$u_y^{-1} f \circ u = e_1 \ .$$

Similarly, we can construct a diffeomorphism v with

$$v_y^{-1} g \circ v = e_1 \ ,$$

thus $u \circ v^{-1}$ takes f into g. □

The assumption $f(x^*) \neq 0$ in the lemma is crucial. A point x^* at which $f(x^*) = 0$ is called a *singular point* (equilibrium point, stagnation point), of the vector field. If $x = u(y)$ maps a point y^* into $x^* = u(y^*)$ then

$$g = u_y^{-1} f \circ u$$

has a singular point at $y = y^*$ and the Jacobian is

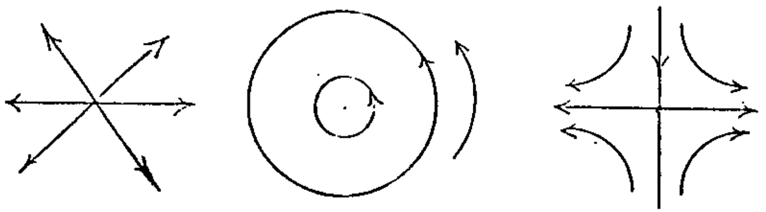
$$g_y(y^*) = u_y^{-1} f_x(x^*) u_y$$

where $u_y = u_y(y^*)$. Thus the Jacobians $f_x(x^*)$, $g_y(y^*)$ at a stationary point are similar. Hence the eigenvalues of $f_x(x^*)$ are invariant and must be essential for the vector field. In fact they are basic for the stability theory of vector fields at a singular point which was developed by Lyapunov.

It has to be mentioned that the above lemma is valid only "in the small" and fails in large domains. This is illustrated by the three simple examples in the plane:

(i) $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$ (ii) $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1$ (iii) $\dot{x}_1 = x_1$, $\dot{x}_2 = -2x_2$.

The corresponding flows are plotted below.



and it is obvious that there is no diffeomorphism taking any of these flows into any other — although this is possible locally near any point different from the origin.

The properties "in the large" are of principal interest. Examples of such properties are the existence of singular points, of periodic orbits, their stability or instability behavior, etc., which will be investigated in these lectures.

Systems of differential equations of the form (1.1) are usually called "autonomous" to distinguish them from systems

$$\frac{dx}{dt} = f(t,x)$$

which depend on t as well, and are called nonautonomous. These systems can easily be reduced to (1.1) by introducing $x_0 = t + \text{const.}$ as independent variable so that we obtain a system

$$\frac{dx_0}{dt} = 1\ , \qquad \frac{dx_j}{dt} = f_j(x_0,x) \qquad (j=1,2,...,n)$$

in n+1 dimensional space.

Also systems of second order

$$\frac{d^2x}{dt^2} = g(x, \frac{dx}{dt})$$

can easily be reduced to (1.1), simply introducing x and dx/dt as independent variables. As a rule we will therefore assume that this reduction has been carried out and study systems of the form (1.1). The domain D is called the "phase space", in which we visualize the motion.

(c) Examples.

We illustrate this concept with some simple examples.

(i) The differential equation

$$\frac{d^2x}{dt^2} + \sin x = 0 , \qquad x \in R^1$$

describes the motion of a pendulum, where x denotes the angle of deflection from the vertical.

The phase space in this case is the plane with coordinates x and $\dot{x} = \frac{dx}{dt}$. Multiplying the equation by $\dot{x}$ and integrating we obtain the energy relation

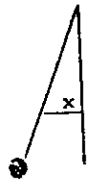


Fig. 1

$$\frac{1}{2}\dot{x}^2 - \cos x = E$$

where E is a constant along each orbit. This equation defines a set of curves (see Fig. 2) on which the solutions travel.

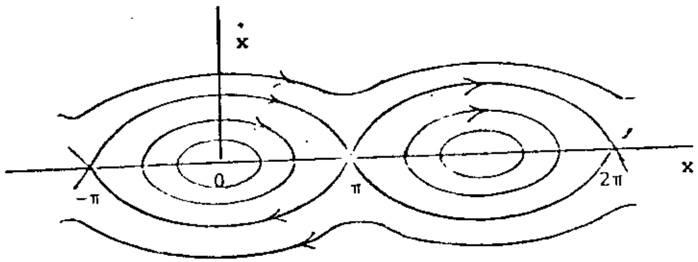


Fig. 2

Without determining the solutions explicitly (they are given in terms of elliptic functions) we can read off the figure the nature of the motion: The oscillations about the down position is given by the islands, the motion of the pendulum swinging over the top by the wavy lines on top and bottom and the "separatrices" which describe a motion where the pendulum just goes from the top position, falls down, and asymptotically returns to the top position.

Since x is an angle we should identify point $(x,\dot{x})$ and $(x_1,\dot{x}_1)$ if $x - x_1 = 2\pi j$, $\dot{x} - \dot{x}_1 = 0$ for any integer j. Thus the phase space becomes a cylinder and the many "islands" are identified to one.

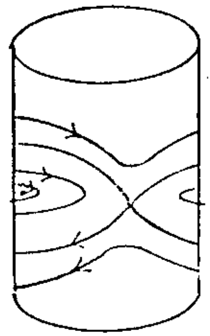


Fig. 3

(ii) Geodesics on S^2.

The two-dimensional sphere S^2 can be given by the equation

$$|x|^2 = x_1^2 + x_2^2 + x_3^2 = 1 ,$$

and the geodesics on it are the greatest circles. They are described by the differential equation

$$\frac{d^2}{dt^2} x = \lambda x$$

where the scalar λ is so determined that the equation $|x| = 1$ remains valid for all t. This requires $\lambda = -|\dot{x}|^2$ since

$$0 = (\tfrac{d}{dt})^2 |x|^2 = 2(\langle x,\ddot{x}\rangle + |\dot{x}|^2) = 2(\lambda + |\dot{x}|^2) ,$$

and the differential equation becomes

$$\frac{d^2}{dt^2} x + |\dot{x}|^2 x = 0 ,$$

where we have to restrict ourselves to

$$|x|^2 = 1 , \qquad \langle x,\dot{x}\rangle = 0 .$$

More precisely, if the last two conditions hold for $t = 0$ then they hold for all t. We see this by applying the uniqueness theorem for the initial value problem to the system

$$\frac{d}{dt} |x|^2 = 2\langle x,\dot{x}\rangle$$
$$\frac{d}{dt} \langle x,\dot{x}\rangle = |\dot{x}|^2(1-|x|^2).$$

What is the phase space in this case? If we set $y = \dot{x}$ we have the system of first order equations

(1.6) $\dot{x} = y , \qquad \dot{y} = -|y|^2 x$

where $|x| = 1$, $\langle x,y\rangle = 0$. Thus the element of the phase space are the vectors y attached at the point $x \in S^2$. The set of tangent vectors

$$\left\{(x,y) \in R^6 \;\middle|\; |x| = 1, \langle x,y\rangle = 0\right\}$$

form a manifold which is called the tangent bundle $T(S^2)$ of the sphere. Thus $T(S^2)$ is the phase space in this case.

The speed $|\dot{x}| = |y|$ is a constant along any solution since

$$\frac{d}{dt} |y|^2 = 2\langle y,\dot{y}\rangle = -2|y|^2\langle y,x\rangle = 0$$

and we may restrict ourselves to the case $|\dot{x}| = |y| = 1$ in which case t is the arc length. Then the phase space is given by

$$\left\{(x,y) \in R^6 \;\middle|\; |x| = 1, \langle x,y\rangle = 0, |y| = 1\right\} ,$$

the unit tangent bundle $T_1(S^2)$ of the sphere.

Clearly all solutions are periodic of period 2π.

To give a better picture of this flow and its phase space we show that $T_1(S^2)$ can be mapped one to one onto SO(3), the group of 3 by 3 orthogonal matrices U with determinant +1 and the differential equation becomes

(1.7) $\frac{d}{dt} U = U A \quad$ where $\quad A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The solutions of this system are clearly

$$(1.8)\quad U(t) = U(0)\, e^{tA} = U(0) \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The required mapping is obtained as follows: For $(x,y) \in T_1(S^2)$ we construct the orthonormal frame $x, y, z = x \wedge y$ (*) and define $\mathcal{U}$ by

$$Ue_1 = x , \quad Ue_2 = y , \quad Ue_3 = z ,$$

i.e., x,y,z can be taken as the column vectors of U. Then writing $U = (x,y,z)$ we have

$$\frac{d}{dt} U = (\dot{x},\dot{y},\dot{z}) = (y,-x,0) = U A .$$

Thus both the representation of the differential equation (1.6) on the unit tangent bundle $T_1(S^2)$ and (1.7) on SO(3) are equivalent, in the sense that one can be transformed into the other. This illustrates the concept of equivalence of vector fields, but shows the lack of our previous definition, since we have to extend our concepts from the local representation to the global one on manifolds. We return to the definition of vector fields on manifolds later.

(*) $x \wedge y$ denotes the vector product in R^3.

(iii) Kepler problem in the plane.

It is described by the system

$$(1.9)\qquad \frac{d^2x}{dt^2} = -\frac{x}{r^3} , \qquad \frac{d^2y}{dt^2} = -\frac{y}{r^3} , \qquad r^2 = x^2 + y^2 .$$

This system of second order differential equations has as its phase space the four-dimensional space R^4 (with coordinates $x,y,\dot{x},\dot{y}$) minus the plane $x = y = 0$, where the system is singular. This system possesses the energy integral

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{r}$$

which is constant, say E, along each orbit. It is well known that the solutions correspond to conic sections in the x,y plane; hyperbolae for $E > 0$, parabolas for $E = 0$ and ellipses for $E < 0$. Thus if we consider the energy surface

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{r} = E < 0$$

for a fixed negative E all solutions are periodic and have, as it turns out, a fixed period (namely $2\pi(-2E)^{-3/2}$).

In the course of this chapter we will show that after an appropriate change of t and an appropriate compactification this flow of the Kepler problem on a fixed negative energy surface is equivalent to the flow (1.8) of the geodesics on S^2. In particular, it will follow that the energy surface properly compactified is equivalent to SO(3).

For the following we will extend the concept of equivalence of two vector fields

$$\frac{dx}{dt} = f(x) \quad \text{and} \quad \frac{dy}{ds} = g(y)$$

in domains D_1 and D_2 respectively. We will say that f is *equivalent in the extended sense* if there exists a diffeomorphism $u: D_2 \to D_1$ and a positive function $\lambda = \lambda(y) \in C^r(D_2, R)$ such that

$$(1.7) \qquad g = \lambda u_y^{-1} f \circ u \ .$$

The factor $\lambda = \lambda(y)$ corresponds to a change of the independent variable. More precisely, if the independent variable for the g-vector field is called s, i.e. if

$$\frac{dy}{ds} = g(y)$$

and ψ^s the corresponding flow then we have

$$(1.8) \qquad \psi^s = u^{-1} \circ \phi^t \circ u$$

where s and t are related by

$$t = v(s,y) = \int_0^s \lambda(\psi^\sigma(y)) \, d\sigma \ .$$

This shows that the solutions of one system are mapped into those of the other with a change of parametrization.

We have clearly

$$u_y g = u_y \frac{d\psi^s}{ds} = \lambda \frac{d\phi^t}{dt} \circ u = \lambda f \circ \phi^t \circ u \ ,$$

showing (1.7). In other words, here we subject the vector field to the transformation

$$x = u(y) \ , \qquad t = v(s,y)$$

of the n+1 dimension space $R^n \times I$. We will simply write

$$\frac{dt}{ds} = \lambda .$$

Remark. We assume throughout that $u \in C^r$, $r \geq 1$, but frequently one studies also "topological equivalence" of vector fields when u is assumed to be a homeomorphism only. In that case (1.4) loses meaning and topological equivalence is defined through that of the flows (see (1.8)) and an appropriate t-transformation. We will hardly be concerned with this concept and discuss it when it comes up.

Exercises - 1

1. (a) Show with the example $\frac{dx}{dt} = x^2$ for $x \in R^1$ that the flow ϕ^t is not defined for all t.

   (b) Show if in the system (1.1) with $D = R^n$ and

   $$|f(x)| < M \text{ in } R^n$$

   then $\phi^t(x)$ is defined for all real t.

   (c) Show, if in the system (1.1), with $D = R^n$

   $$|f_x(x)| < M \text{ in } R^n$$

   then ϕ^t is again defined for all real t.

2. (a) Let f(x) be a C^1-vector field satisfying

   (i) $f(x) = c$ for $|x| > r$

   (ii) $\langle f(x), c \rangle > 0$ for all $x \in R^n$

   where c is a constant vector in R^n and r a positive constant. Let ϕ^t, ψ^t denote the flows corresponding to the vector fields $\dot{x} = f(x)$, $\dot{y} = g(y) = c$. Show that ϕ^t, ψ^t are defined for all t and that

   $$u = \lim_{t\to\infty} \phi^{-t} \circ \psi^t$$

   is a diffeomorphism satisfying

   $$g = u_y^{-1} f \circ u.$$

   (b) Use this result to give a proof of Lemma 1, by setting $f(x) = c$ and modifying f outside a small ball so that $f = c$ there.

3. (a) If ϕ^t is the flow of X and

   $$Z = \sum_{h=1}^{n} h_k \frac{\partial}{\partial x_k}$$

   any other vector field then

$$\frac{d}{dt} (\phi^t)^* Z \Big|_{t=0} = [X, Z] .$$

Hint: Differentiate

$$\phi_y^{-t} h(\phi^t(y)),$$

with respect to t at $t = 0$.

   (b) Show: If $[X, Z] = 0$ in some neighborhood then the flows ϕ^t, ψ^s belonging to X, Z respectively commute, i.e.

   $$\phi^t \circ \psi^s = \psi^s \circ \phi^t$$

   for small $|t|, |s|$. Show also the converse.

4. Verify (e.g. using any book (*) on this subject) that the solutions of the Kepler problem (1.9) for negative energy can be represented in terms of an auxiliary variable u (~~mean~~ eccentric anomaly) as

$$x = a(e + \cos u)\cos\alpha - a\sqrt{1-e^2} \sin u \sin\alpha$$

$$y = a(e + \cos u)\sin\alpha + a\sqrt{1-e^2} \sin u \cos\alpha$$

$$t = a^{3/2}(u + e \sin u)$$

where

$$a = -(2E)^{-1} > 0$$

is the semimajor axis of the ellipse and e $(0 \le e \le 1)$ is the eccentricity of the ellipse. The angle determines the position of the ellipse.

Moreover, the distance $r = (x^2+y^2)^{1/2}$ is given by

$$r = a(1 + e \cos u).$$

(*) E.g. A. Wintner, with some change of notation.

2. Variational Principles, Hamiltonian Systems

In many fields of application, the differential equations are derived from a variational principle and consequently have special features which we will investigate in this section.

(a) Variational Principles.

We begin with the standard variational principle and its Euler equation. We consider a function $F(t,q,r)$ of 2n+1 variables ~~t, q ∈ R^n, r ∈ R^n~~ defined in a domain $\Omega = I \times R^{2n}$, $I = \{t, t_1 \le t \le t_2\}$, and form the functional

$t \in R, q, r \in R^n$

$$(2.1) \quad \Phi[q] = \int_{t_1}^{t_2} F(t, q(t), \dot q(t))\, dt \; ; \qquad \dot q = \frac{dq}{dt} .$$

for curves $q \in C^2(I, R^n)$ and fixed end points $q(t_1) = a_1$, $q(t_2) = b$. The rather ugly choice of the letters $q = (q_1, q_2, \ldots, q_n)$ is dictated by tradition!

This functional assigns to any curve $q \in C^2(I, R^n)$ a real number. The stationary values of this functional are obtained by forming the first variation

$$\Phi'[q, \hat q] = \int_{t_1}^{t_2} (\langle F_q, \hat q\rangle + \langle F_{\dot q}, \dot{\hat q}\rangle)\, dt$$

of Φ and requiring that it vanishes for every choice of $\hat q \in C^1(I, R^n)$. Integration by parts leads to the Euler equations

$$(2.2) \qquad \frac{d}{dt}\left(F_{\dot q}(t,q,\dot q)\right) = F_q(t,q,\dot q)$$

for $q = q(t)$, $\dot q = \frac{d}{dt} q(t)$ which we will investigate. This is a system of second order

$$F_{\dot q \dot q} \frac{d^2 q}{dt^2} + F_{\dot q q} \frac{dq}{dt} - F_q = 0 \, , \qquad + F_{\dot q t}$$

where we omitted the arguments of the derivatives of F. One calls a point $(t, q, \dot q) \in I \times R^{2n}$ "regular" if

$$(2.3) \qquad \det(F_{\dot q \dot q}) \neq 0$$

and we will impose this assumption of regularity. It implies that we can write the Euler equation in the explicit form

$$\ddot q = G(t, q, \dot q)$$

and hence as a system of first order

$$\frac{d}{dt}\begin{pmatrix} q \\ \dot q \end{pmatrix} = \begin{pmatrix} \dot q \\ G(t,q,\dot q) \end{pmatrix}$$

(b) Legendre Transformation.

There is a more elegant way to achieve this reduction to a first order system of differential equations which leads to the Hamiltonian form of the equation. This is achieved by so-called Legendre transformation according to which one introduces instead of $\dot q$ the vector variable

(2.4) $p = F_{\dot q}$ or $p_j = F_{\dot q_j}(t,q,\dot q)$.

All the following considerations are "local" in nature since we appeal to the implicit function theorem. The Legendre transformation maps

$$(t,q,\dot q) \to (t,q,p)$$

where p is defined by (2.4). The Jacobian of this mapping is clearly $\det F_{\dot q \dot q}$ which by (2.3) is assumed not to vanish. Therefore the inverse mapping exists locally:

$$(t,q,p) \to (t,q,\dot q) \quad \text{when} \quad \dot q = V(t,q,p).$$

This mapping can be expressed effectively if one introduces the Hamilton function

$$H(t,q,p) = \langle p,V\rangle - F(t,q,V) ,$$

or equivalently,

$$F(t,q,\dot q) = \langle F_{\dot q},\dot q\rangle - H(t,q,F_{\dot q}) .$$

In other words we have

$$H(t,q,p) = \langle p,\dot q\rangle - F(t,q,\dot q)$$

if $p = F_{\dot q}$ or if $\dot q = V$. Therefore taking the differential, where we consider at first $\dot q$ and p independently and then set $p = F_{\dot q}$, we find

$$dH = \langle p,d\dot q\rangle + \langle \dot q,dp\rangle - \langle F_q,dq\rangle - \langle F_{\dot q},d\dot q\rangle - F_t\,dt$$

$$= \langle p - F_{\dot q},\, d\dot q\rangle + \langle \dot q,dp\rangle - \langle F_q,\, dq\rangle - F_t\,dt$$

$$= \langle \dot q,dp\rangle - \langle F_q,dq\rangle - F_t\,dt .$$

Thus

(2.5) $$H_p = \dot q , \quad H_q = - F_q , \quad H_t = - F_t .$$

The first relation shows that

$$V(t,q,p) = H_p(t,q,p).$$

These calculations show that the Euler equations are transformed into the system

(2.6) $$\begin{cases} \dot q = H_p \\ \dot p = \dfrac{d}{dt}(F_{\dot q}) = F_q = - H_q . \end{cases}$$

This form of the equations is called Hamilton's form of the differential equations, or short Hamiltonian systems. We have to keep in mind that this derivation was only local in nature — but we will see that in many applications the Legendre transformation is linear in $\dot q$ and is meaningful in the large. Moreover, one frequently considers the Hamiltonian system as the primary object, rather than the variational principle.

(c) Autonomous case.

If we assume F to be independent of t the same is true for H and we obtain an autonomous Hamiltonian system, which we will write in the form (1.1) of §1. For this purpose we introduce the 2n vector x and the 2n by 2n matrix J by

$$x = \begin{pmatrix} q \\ p \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where $I = I_n$ is the n by n identity matrix.
Then the Hamiltonian system (2.6) can be written in the form

$$\frac{dx}{dt} = J H_x \tag{2.7}$$

where H_x is the vector with components H_{x_j}, i.e. the gradient of H. In other words, H_x is defined by the differential relation

$$dH = \langle H_x, dx\rangle .$$

Thus Hamiltonian systems are vector fields (1.1) where

$$f = JH_x$$

is defined in terms of a single function, the Hamiltonian. This suggests that these systems are "special" in nature. This is indeed the case as we shall see.

(d) Canonical transformations.

If we subject the system (2.7) to an arbitrary transformation the special form of the equation will be destroyed and we will determine a class of transformations which preserve the class of Hamiltonian systems.

Since $J^2 = -I_{2n}$ we can write (2.7) also in the form

$$-J\dot{x} = H_x . \tag{2.8}$$

If we subject this system to an arbitrary transformation

$$x = u(y), \quad K(y) = H(u(y))$$

we find

$$\dot{x} = u_y \dot{y}, \qquad K_y = (u_y)^T H_x \circ u$$

where $(\ )^T$ denote the transposed matrix. Thus (2.8) goes into

$$-u_y^T J u_y \dot{y} = K_y .$$

Hence if we require the identity

$$u_y^T J u_y = J \tag{2.9}$$

then the transformed system is

$$\dot{y} = JK_y$$

with the Hamiltonian $K = H \circ u$ obtained from H simply by transformation. Thus transformations $x = u(y)$ satisfying the identity (2.9) do preserve the class of Hamiltonian system.

Definition. A *diffeomorphism* $x = u(y)$ is called *canonical* if it satisfies (2.9).

The canonical transformations form a group which we will investigate in the next section.

(e) Examples.

Ex. 1: If $n = 1$ and

$$H = \frac{1}{2} p^2 - \cos q$$

then the Hamiltonian system gives rise to the pendulum equation $\ddot{q} + \sin q = 0$.

Ex. 2: The equations of particle mechanics are written in Hamiltonian form

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2m_j} + U(q)$$

where $U(q)$ is called the potential and $m_j > 0$ are the masses. The differential equations become

$$m_j \ddot{q}_j = - U_{q_j} \qquad (j=1,2,\dots,n). \tag{2.10}$$

Notice that in this case the Legendre transformation is $p_j = m_j \dot{q}_j$ which is globally invertible.

The N-body problem in R^3 is contained in the above formulation: If $n = 3N$, $m_{3j-2} = m_{3j-1} = m_{3j} = \mu_j > 0$ is the mass and

$$Q_j = \begin{pmatrix} q_{3j-2} \\ q_{3j-1} \\ q_{3j} \end{pmatrix}$$

the position of the j^{th} mass point we have the potential

$$U(q) = - \sum_{j,k=1}^{N} \frac{\mu_j \mu_k}{|Q_j - Q_k|}$$

then (2.10) becomes

$$\ddot{Q}_j = - \sum_{k \neq j} \frac{\mu_k}{|Q_j - Q_k|^3} (Q_j - Q_k)$$

which are the equations of motion of the N-body problem.

Ex. 3: Geodesics.

If we introduce the metric

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x)\, dx_i\, dx_j ,$$

where $g(x) = \langle g_{ij}(x) \rangle$ is a positive definite symmetric matrix the geodesics can be defined as the extremals of the functional

$$\int_{t_1}^{t_2} \frac{1}{2} \langle g(x)\dot{x}, \dot{x} \rangle \, dt .$$

The Euler equations

$$\sum_j \frac{d}{dt} (g_{ij}(x)\, \dot{x}_j) = \frac{1}{2} \sum_{j,k} g_{jkx_i} \dot{x}_j \dot{x}_k$$

can be written as

$$\sum_j g_{ij}\ddot{x}_j + \sum_{j,k} (g_{ijx_k} - \frac{1}{2} g_{jkx_i})\dot{x}_j\dot{x}_k = 0$$

Using the obvious identity

$$\sum_{j,k} g_{ijx_k}\dot{x}_j\dot{x}_k = \sum_{j,k} g_{ikx_j}\dot{x}_j\dot{x}_k$$

we can write the equation in the form

$$\sum_j g_{ij}\ddot{x}_j + \sum_{j,k} \Gamma_{jik}\dot{x}_j\dot{x}_k = 0$$

where

$$\Gamma_{jik} = \frac{1}{2}\left(g_{ijx_k} + g_{ikx_j} - g_{jkx_i}\right).$$

We use the notation of differential geometry and denote the inverse matrix g^{-1} by (g^{ij}) and introduce the Christoffel symbols

$$\Gamma^i_{jk} = \sum_\ell g^{i\ell}\Gamma_{j\ell k} = \frac{1}{2}\sum_\ell g^{i\ell}\left(g_{\ell j x_k} + g_{\ell k x_j} - g_{jkx_\ell}\right)$$

so that the Euler equation become

$$\ddot{x}_i + \sum_{j,k} \Gamma^i_{jk}\dot{x}_j\dot{x}_k = 0 .$$

We were inconsistent in the notation describing points by the letter x instead of q. The Legendre transformation is given by

$$p = g(x)\dot{x}$$

which is clearly invertible since $\det g > 0$ and the Hamiltonian

is given by

$$H(x,p) = \frac{1}{2} \langle g^{-1}(x)p,p\rangle.$$

Ex. 4: Charged particle in an electromagnetic field.

We consider a particle of mass $m > 0$ and charge e in an electromagnetic field where the electric potential is $\phi(q)$ (scalar) and the magnetic vector potential is $A(q)$ (3 vector). The position of the particle is given by $q \in R^3$. Then the motion of the particle is governed by the equation

$$m\left(\frac{d}{dt}\right)^2 q = -e\phi_q + \frac{e}{c}\dot{q} \wedge B$$

where

$$B(q) = \nabla \wedge A(q) \quad (*)$$

is the magnetic field, and c is the speed of light.

Also these equations are the Euler equation of a variational problem with the Lagrange function

(2.12) $$F(q,\dot{q}) = \frac{m}{2}|\dot{q}|^2 - e\phi + \frac{e}{c}\langle\dot{q},A\rangle .$$

Therefore they can also be written in Hamiltonian form if

$$p = F_{\dot{q}} = m\dot{q} + \frac{e}{c}A$$

is introduced. Again the Legendre transformation is linear

(*) $\nabla \wedge A$ or curl A is the vector with components $A_{2q_3} - A_{3q_2}$, $A_{3q_1} - A_{1q_3}$, $A_{1q_2} - A_{2q_1}$.

in $\dot q$ and globally invertible. One computes

$$H(q,p) = \frac{1}{2m}\left|p - \frac{e}{c}A(q)\right|^2 + e\phi.$$

As a special case we describe the equation of a charged particle in a magnetic dipole field (like that of the earth). In this case the vector potential is

$$A = C \wedge \nabla(|q|^{-1})$$

where C has the direction of the dipole axis and $|C|$ measures its strength.

If $C = e_3$ one verifies that

$$B = \nabla \wedge A = -\frac{\partial}{\partial q_3}\nabla(|q|^{-1}) = \nabla(q_3|q|^{-3})$$

so that the differential equations are

$$m\ddot q = \frac{e}{c}\,\dot q \wedge \nabla\left(\frac{q_3}{|q|^3}\right).$$

The Hamiltonian is in this case

$$H = \frac{1}{2m}\left|p - \frac{e}{c}A\right|^2$$

$$= \frac{1}{2m}\left\{\left(p_1 - \frac{e}{c}\frac{q_2}{r^3}\right)^2 + \left(p_2 + \frac{e}{c}\frac{q_1}{r^3}\right)^2 + p_3^2\right\}$$

where $r = |q|$.

Ex. 5: The relativistic equations.

Actually the above equations are valid only for velocities $|\dot q|$ small compared to the speed of light c. The relativistic equations are described by the Lagrange function

$$F(q,\dot q) = mc^2\left(1 - \sqrt{1 - c^{-2}|\dot q|^2}\right) - e\phi + \frac{e}{c}\langle \dot q, A\rangle .$$

If one expands this function in powers of $c^{-1}|\dot q|$ and drops terms of at least fourth order one obtains (2.12). Therefore the Euler equations of these two variational problems are "close" if $c^{-1}|\dot q|$ is small.

The variable p is introduced by

$$p = F_{\dot q} = \frac{m\dot q}{\sqrt{1-c^{-2}|\dot q|^2}} + \frac{e}{c}A .$$

This relation is not linear in $\dot q$ any more; moreover, it has to be restricted to speed $|\dot q| < c$. Since

$$\left(m^2c^2 + \left|p - \frac{e}{c}A\right|^2\right)\left(1 - c^{-2}|\dot q|^2\right) = m^2c^2$$

it is easy to determine the inverse map as

$$\dot q = \left(m^2c^2 + \left|p - \frac{e}{c}A\right|^2\right)^{-1/2} cp - eA .$$

The Hamiltonian becomes

$$H(q,p) = c\left(m^2c^2 + \left|p - \frac{e}{c}A\right|^2\right)^{1/2} + e\phi - m c^2$$

where the constant is, of course, irrelevant.

(f) Generalized canonical transformations.

Under (d) we required that the transformation $x = u(y)$ takes the Hamiltonian system

$$\dot{x} = JH_x \qquad \text{into} \qquad \dot{y} = JK_y$$

where $K = H \circ u$. One can ask for the more general class of transformations $x = u(y)$ which take any system (2.8) in Hamiltonian form into another such system where we do not specify the relation between these Hamiltonians.

Theorem 2.1 If $x = u(y)$ is a diffeomorphism taking every system (2.8) in Hamiltonian form into another then there exists a constant $\mu \neq 0$ such that

(2.11) $$u_y^T J u_y = \mu J .$$

The new Hamiltonian is given by $\mu^{-1} H \circ u$.

We shall call a transformation $y \to x = u(y)$ *generalized canonical* if it satisfies (2.11) with some constant $\mu \neq 0$.

We indicate the proof of Theorem 2.1. If we set $K = H \circ u$ then the transformed system takes the form

$$\dot{y} = J M K_y$$

where $M = M(y)$ is given by

$$M^{-} = u_y^T J u_y J^{-1} .$$

It remains to characterize those matrices $M = M(y)$ for which

$$M K_y$$

is a gradient whichever function K is chosen.

Exercise 1 Prove that the most general matrix $M = M(y)$ for which $M K_y$ is a gradient for all functions $K = K(y)$ is of the form

$$M(y) = \mu I$$

where μ is a constant.

Hint: (i) Use that $(M K_y)_y$ must be a symmetric matrix.

(ii) Using $K = \langle Sy, y\rangle$, $S = S^T$ show that

$$M(0)S = SM(0)^T$$

for every symmetric matrix S. Taking $S = I$ and then arbitrarily show

$$M(0) = \text{const.}\ I$$

hence

$$M(y) = \mu(y) I .$$

(iii) Using $K = \langle b, y\rangle$ with any constant vector b show that $\mu_y = 0$, i.e. μ is a constant.

<u>Exercise 2</u> Let $F(x,\lambda) \in C^2$, $x \in R^n$, $\lambda \in R^m$ such that

$$\det\left(\frac{\partial^2 F}{\partial x_i\, \partial x_j}\right) \neq 0$$

and define the Legendre transformation $L(F) = G$ by

$$G(y,\lambda) = \langle y,\, u(y,\lambda)\rangle - F\big(u(y,\lambda),\lambda\big)$$

where $x = u(y,\lambda)$ is defined implicitly by

$$y = \frac{\partial F}{\partial x}(x,\lambda)\ .$$

Prove

(i) $\frac{\partial G}{\partial y} = u\ , \qquad \frac{\partial G}{\partial \lambda} = -\frac{\partial F}{\partial \lambda}$

(ii) $\det G_{yy} \det F_{xx} = 1$

(iii) $L^2(F) = L(G) = F$.

<u>Exercise 3</u> Show that the motion of a particle in a constant magnetic field given by the equations

$$\ddot{q} = \dot{q} \wedge B$$

takes place along helices.

<u>Exercise 4</u> Study the geodesic problem in R^n, $|x| < 1$, defined by the matrix

$$\langle g(x)\dot{x},\dot{x}\rangle = \left(1-|x|^2\right)^{-1}|\dot{x}|^2 + \left(1-|x|^2\right)^{-2}\langle x,\dot{x}\rangle^2 = 2F(x,\dot{x})\,.$$

Show that this metric is positive definite in $|x| < 1$ where the Legendre transformation

$$y = F_{\dot{x}}$$

is invertible. Determine the Hamiltonian $H = H(x,y)$.

Answer: $2H(x,y) = \left(1-|x|^2\right)\left(|y|^2 - \langle x,y\rangle^2\right)$

Can you show that the geodesics are straight line segments?

3. Canonical Transformation

The group of canonical transformations which were defined at the end of the previous section preserve the class of Hamiltonian vector fields. They are of basic importance and will be studied in this section. However, we will restrict ourselves here to local considerations. We will extend these concepts to manifolds in Section 8.

(a) Symplectic Geometry.

We begin with the study of canonical mappings which are linear and of the form

(3.1) $$x = U y$$

where U is a 2n by 2n matrix. By (2.9) this mapping is canonical if the identity

(3.2) $$U^T J U = J$$

holds. Matrices of this nature are also called symplectic — and for this reason the term canonical transformation has sometimes been replaced by symplectic transformation. We will generally use the older term, and reserve symplectic for linear maps.

To interpret (3.2) it is good to introduce the bilinear form

(3.3) $$[v,w] = \langle v,Jw\rangle,$$

which is (i) antisymmetric, i.e.,

$$[v,w] = - [v,w]$$

and (ii) is nondegenerate, i.e.,

$$[v,w] = 0 \text{ for all } w \in R^{2n} \text{ implies } v = 0 .$$

The latter condition follows from det $J \neq 0$.

The condition (3.2) is then tantamount to the condition that the mapping $y \to x = Uy$ preserves the bilinear form $[v,w]$. Indeed if $v' = Uv$, $w' = Uw$ then

$$[v',w'] = [Uv,Uw] = \langle v,U^TJ\,Uw\rangle$$

which agrees with $[v,w]$ if and only if (3.2) holds. Thus we can define symplectic maps as those linear maps which preserve the bilinear form $[v,w]$.

From this formulation it is clear that the symplectic maps form a group, which is denoted by $Sp(R,2n)$. Moreover, taking the determinant of (3.2) we find

$$(\det U)^2 = 1 .$$

Since J satisfies $J^2 = -I$ it is clearly symplectic and by (3.2)

$$U^T = J\,U^{-1}J^{-1}$$

is also symplectic.

Euclidean geometry is characterized by the group of transformations $x \to Ux$ preserving an inner product

$$\langle v,w\rangle$$

i.e. by the group of orthogonal transfmrations:

$$U^TU = I .$$

Similarly the domain of symplectic geometry are properties invariant under symplectic transformations.

From now on we will not insist that the bilinear form $[v,w]$ is given by the above representation with the matrix J but allow an arbitrary bilinear form which is (i) antisymmetric, and (ii) nondegenerate on a finite dimensional real vector space V. (We will see presently that this, in fact, is not more general.) We call V equipped with this form $[\ ,\ ]$ a symplectic space.

We will say that $v \perp w$, or "v is orthogonal to w with respect to the symplectic form $[\ ,\ ]$" if $[v,w] = 0$. If E is a linear subspace of V we define

$$E^{\perp} = \{w \in V \mid [v,w] = 0 \text{ for all } v \in E\}.$$

This is clearly a linear subspace and since $v \perp w$ is equivalent to $w \perp v$ we see that

$$(E^{\perp})^{\perp} = E .$$

Since $[\ ,\ ]$ is nondegenerate we have

$$\dim E + \dim E^{\perp} = \dim V$$

However, the concept of orthogonality in symplectic geometry differs sharply from that in Euclidean geometry: E and $E^{\perp}$ need not be complementary; for example, every vector is orthogonal to itself, since $[v,v] = -[v,v]$, hence if $\dim E = 1$ we have

$$E \subset E^{\perp} .$$

We can restrict the bilinear form $[\ ,\ ]$ to a linear subspace. This restricted form will obviously be antisymmetric

but in general fails to be nondegenerate. It is nondegenerate if and only if

$$(3.4) \qquad E \cap E^{\perp} = (0)$$

i.e. precisely if E and $E^{\perp}$ are complementary.

Indeed, in order that the restriction of $[\ ,\ ]$ to E is nondegenerate, we require that for $v \in E$

$[v,w] = 0$ for all $w \in E$ implies $v = 0$.

This is clearly equivalent to

$$v \in E \cap E^{\perp} \Rightarrow v = 0 .$$

The converse follows the same way.

Thus in case (3.4) we can call E a symplectic subspace. Because of the symmetry of (3.4) in E and $E^{\perp}$ we see: E is symplectic if and only if $E^{\perp}$ is symplectic.

~~LEMMA I.~~ Lemma 3.1 For a symplectic space V, dim V is even, say 2n, and there exists a basis $v_1, v_2, \cdots, v_{2n}$ satisfying

$$[v_j, v_{k+n}] = \delta_{jk} \quad \text{for } 1 \le j < k+n \le 2n$$

Such a basis will be called a "canonical basis".

Proof: We take any nonzero vector v_1. Since $[\ ,\ ]$ is nondegenerate there exists a vector, which we call v_{n+1} such that $[v_1, v_n] = 1$. If $\dim V = 2$ the proof is finished. If $\dim V \ge 4$ we set $E = \text{span}(v_1, v_n)$ which is a symplectic subspace of V, and so is $E^{\perp}$. Since $\dim E^{\perp} = 2n-2 < 2n$ we can use induction on the dimension to complete the proof.

If we represent x in terms of this basis, i.e.

$$x = \sum_{j=1}^{2n} x_j v_j \,, \qquad x' = \sum x_j' v_j$$

we find

$$[x,x'] = \sum x_j x_k' J_{jk}$$

where $J_{jk} = \pm 1$ for $k = j \pm n$ and 0 otherwise. Thus in this basis the bilinear form is represented by the matrix J. Thus it is no loss of generality to operate with the symplectic form (3.3).

<u>Corollary to Lemma 1.</u> If $[\ ,\ ]_1$ and $[\ ,\ ]_2$ are two nondegenerate antisymmetric bilinear forms on vector spaces V_1, V_2 of equal dimensions, then there exists an invertible linear map $T: V_1 \to V_2$ such that

(3.5) $$[Tv, Tw]_2 = [v,w]_1 \,.$$

Indeed, we just select canonical bases $v_j^{(1)}, v_j^{(2)}$ in V_1, V_2 respectively and define

$$Tv_1^{(j)} = v_2^{(j)} \,.$$

Thus all symplectic spaces of equal dimension are equivalent in this sense.

We introduce some vocabulary: A linear subspace E of a symplectic space V is called

(3.6)
$$\begin{aligned} &\text{isotropic} && \text{if } E \subset E^{\perp} \\ &\text{coisotropic} && \text{if } E \supset E^{\perp} \\ &\text{Lagrange space} && \text{if } E = E^{\perp} \\ &\text{symplectic} && \text{if } E \cap E^{\perp} = (0)\,. \end{aligned}$$

The last definition was introduced before. Any one-dimensional space E is isotropic. Since $\dim E + \dim E^{\perp} = \dim V$ it is clear that for an isotropic space

$$\dim E \le n$$

and for a coisotropic space $\dim E \ge n$. If $\dim E = n$ and E is isotropic then it is a Lagrange space.

We illustrate these concepts with the bilinear form

$$[x,x'] = \langle Jx, x' \rangle = \langle p,q' \rangle - \langle q,p' \rangle$$

in R^{2n}, where $x = \{q,p\}$ and $x' = \{q',p'\}$. The manifold

$$\{x \mid q_1 = q_2 = \cdots = q_r = 0\}$$

is coisotropic for $r \le n$, a Lagrange space for $r = n$.

The space

$$\{x \mid q_1 = p_1 = 0, \ldots, q_r = p_r = 0\}$$

is symplectic, and the space

$$\{x \mid q_1 = \cdots = q_n = 0,\ p_1 = \cdots = p_r = 0\}$$

is isotropic.

(b) Differential Forms.

To express the condition (2.9) for canonical maps, in a concise form suitable for coordinate transformations we use the exterior algebra of differential forms. A two form α in R^m is written

(3.2) $$\alpha(x) = \sum_{i,j=1}^{m} a_{ij}(x)\, dx_i \wedge dx_j \,.$$

We will simply use the formalism of differential forms (see) but remind the reader of the rule $dx_i \wedge dx_j = - dx_j \wedge dx_i$. If we view $dx_i \wedge dx_j$ as antisymmetric bilinear forms on R^m,

$$dx_i \wedge dx_j (v,w) = v_i w_j - v_j w_i ,$$

for every pair $v = (v_i)$ and $w = (w_j)$ of vectors in R^m, then the above two form α associates to every $x \in R^m$ an antisymmetric bilinear form on R^m. In order to remind the reader of the transformation law of two forms, we write for the moment the form α in the handy and suggestive form

$$\alpha(x) = \langle A(x)\, dx \wedge dx\rangle,$$

where $A = (a_{ij})$, $dx = (dx_i)$.

For a differentiable map $u: y \mapsto x = u(y)$ from R^k into R^m we call the resulting two form in R^k

$$\begin{aligned} u^*\alpha(y) &= \langle A(u(y))u_y\, dy \wedge u_y\, dy\rangle \\ &= \langle u_y^T (A \circ u) u_y \quad dy \wedge dy\rangle \\ &= \langle B(y)\, dy \wedge dy\rangle, \end{aligned}$$

so that $u^*\alpha$ is represented by the matrix

(3.8) $$B(y) = u_y^T (A \circ u)\, u_y .$$

More conceptually the two form $u^*\alpha$ at y is the antisymmetric bilinear form on R^k,

$$u^*\alpha(y)\ (v,w) = \alpha(x)\ (Uv, Uw) ,$$

$x = u(y)$, where $U = u_y$, and $v,w \in R^k$. In particular, if u is a coordinate transformation in R^m with u_y nonsingular, we have recalling the transformation law for vector fields $f(x)$, $g(x)$ on R^m,

$$u^*\alpha(y)\ \big(u^*f(y), u^*g(y)\big) = \alpha(x)\big(\ f(x), g(x)\big),$$

$x = u(y)$, expressing the coordinate independence.

We apply these remarks to the special two form on R^{2n},

$$\begin{aligned} \omega &= \tfrac{1}{2} \langle J\, dx \wedge dx\rangle \\ &= \tfrac{1}{2} \sum_{j=1}^{n} \big(dp_j \wedge dq_j - dq_j \wedge dp_j\big) \\ &= \sum_{j=1}^{n} dp_j \wedge dq_j , \end{aligned}$$

with $x_j = q_j$, $x_{j+n} = p_j$, $j = 1,2,\dots,n$, i.e., $\omega(v,w) = \langle Jv,w\rangle$ for $v,w \in R^{2n}$. This leads to the equivalent

<u>Definition.</u> A mapping $x = u(y)$ is canonical precisely if

(3.9) $$u^*\omega = \omega ,$$

where

(3.10) $$\omega = \sum_{j=1}^{n} dp_j \wedge dq_j .$$

This differential form, called the symplectic form, takes the place of the antisymmetric nondegenerate bilinear form in the linear case. It is a closed two-form, i.e. $d\omega = 0$, which is nondegenerate. We will see that every two-form

Lemma 3. The Poisson bracket satisfies the following identities:

$$\{F,G\} = -\{G,F\}$$

$$\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = 0$$

for all functions F,G,H.

The first relation is obvious, the second, called the Jacobi identity, requires a calculation which we leave to the reader.

With the notation of Section 1 we will associate with any function $F = F(x)$ a Hamiltonian system

$$\dot{x} = JF_x$$

or the partial differential operator

$$X_F = \sum_{j=1}^{n} \left(F_{p_j} \frac{\partial}{\partial q_j} - F_{q_j} \frac{\partial}{\partial p_j} \right)$$

which also represents our vector field. With this notation .

$$X_F G = \langle F_p, G_q \rangle - \langle F_q, G_p \rangle = -\{F,G\}$$

Hence we have

$$(3.11) \qquad X_G F = \{F,G\} = -X_F G .$$

It is easy to see now that the Hamiltonian vector fields form a Lie algebra, i.e. that the commutator

$$[X_F, X_G] = X_F X_G - X_G X_F$$

is again a Hamiltonian vector field. In fact

with these two properties can be transformed by appropriate local coordinates into the form (3.10). But in this section we shall continue to work with the differential form (3.10).

(c) Poisson Brackets.

There is a third and most important way to characterize canonical mapping. For any two scalar C^1-function F,G we define the function

$$\{F,G\} = \langle F_q, G_p \rangle - \langle F_p, G_q \rangle$$

where again $x_j = q_j$, $x_{j+n} = p_j$ for $j = 1,2,\cdots,n$, or

$$\{F,G\} = \langle JF_x, G_x \rangle .$$

Lemma 2. A mapping $x = u(y)$ is canonical if and only if the Poisson brackets transform like

$$\{F,G\}\circ u = \{F\circ u, G\circ u\}$$

for all functions F,G.

Proof: Since

$$(F \circ u)_y = u_y^T F_x \circ u$$

we find

$$\begin{aligned}\{F\circ u, G\circ u\} &= \langle J u_y^T F_x \circ u, u_y^T G_x \circ u \rangle \\ &= \langle u_y J u_y^T F_x \circ u, G_x \circ u \rangle\end{aligned}$$

and the condition of the lemma requires that u_y^T is a symplectic matrix. But we saw that U is symplectic if and only if U^T is, proving the lemma.

(3.12) $[X_F, X_G] = X_H$ with $H = -\{F,G\}$.

To verify this apply both sides to a function, say h, and apply (3.9) and the Jacobi identity.

(d) Flow of a Hamiltonian vector field.

THEOREM 3.1. A vector field

$$X = \sum_{j=1}^{m} f_j(x) \frac{\partial}{\partial x_j}$$

is Hamiltonian if and only if the corresponding flow ϕ^t is canonical.

Proof: Since ϕ^t satisfies the differential equation

$$\frac{d}{dt}\phi^t = f(\phi^t)$$

the Jacobian matrix

$$\Phi(t,x) = \phi_x{}^t(x)$$

satisfies

$$\frac{d}{dt}\Phi(t,x) = F\left(\phi^t(x)\right)\Phi(t,x)$$

where $F(x) = f_x(x)$.

If the vector field is Hamiltonian then

$$f = J\,H_x$$

and

$$F = J\,H_{xx}$$

where H_{xx} is the matrix of second derivatives and hence symmetric. Thus

$$\frac{d}{dt}\left(\Phi^T J\,\Phi\right) = \Phi^T F^T J\,\Phi + \Phi^T J\,F\Phi = \Phi^T\left(-(J\,F)^T + JF\right)\Phi = 0\ ,$$

since $-JF = H_{xx}$ is symmetric. Therefore

$$\Phi^T J\,\Phi$$

is independent of t and since it is equal to J for $t = 0$ the matrix Φ is symplectic, hence ϕ^t canonical.

Conversely, if ϕ^t is canonical, hence Φ symplectic we conclude from the above formula that

$$-J\,F = -J\,f_x = g_x$$

is symmetric where $g = -J\,f$. In other words we have

$$g_{jx_k} = g_{kx_j}\ ;$$

this is the integrability condition for the existence of a function H, in a sufficiently small neighborhood for which

$$g = H_x$$

holds. Hence

$$f = J\,g = J\,H_x\ ,$$

i.e. the vector field is Hamiltonian.

Later we will replace this somewhat clumsy argument ~~with~~ by a more elegant derivation using differential forms.

Corollary. The flow of a Hamiltonian vector field is volume preserving, i.e. preserves the 2n-form $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{2n}$.

Proof: We have to show that

$$\det\left(\phi_x{}^t\right) = 1\ .$$

Since ϕ_x^t is symplectic its determinant is ± 1 and for $t = 0$ it is $+1$ hence the statement.

Actually the determinant of *any* real symplectic matrix is $+1$. If ϕ^t is the flow of the Hamiltonian vector field X_H, then for every function G

$$\frac{d}{dt} G \circ \phi^t = \frac{d}{ds} G \circ \phi^{t+s}\Big|_{s=0}$$

$$= X_H G \circ \phi^t = \{G,H\} \circ \phi^t ,$$

and in particular for $G = H$

$$\frac{d}{dt} H \circ \phi^t = 0 ,$$

hence $H \circ \phi^t = H$ is independent of t i.e. H is an integral of the motion — which usually corresponds to the energy integral. This means that the flow leaves the surfaces

$$\{x \mid H(x) = c\}$$

invariant.

It is useful to note that on these so-called energy surfaces one also has an invariant volume form, provided $dH \neq 0$. Such an invariant form, say $\omega = \tilde{\omega}_{2n-1}$ is defined by

$$dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{2n} = \tilde{\omega} \wedge dH .$$

Of course, $\tilde{\omega}$ is defined only modulo dH by this formula, but this does not matter for the restriction of $\tilde{\omega}$ to the surf. $H = 0$ on whose tangent space one has $dH = 0$. Since both $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{2n}$ and dH are preserved under the flow we have

$$(\phi^t)^* \tilde{\omega} = \tilde{\omega} \pmod{dH}$$

and the restriction of $\tilde{\omega}$ to $\{H(x) = c\}$ is preserved.

If σ is the surface element on $\{H = c\}$ which belongs to the metric $ds^2 = \sum_{j=1}^{2n} dx_j^2$ then we can take

$$\tilde{\omega} = \frac{\sigma}{|H_x|} , \qquad |H_x|^2 = \sum_{j=1}^{2n} H_{x_j}^2 .$$

Section 8 — Exercises

Exercise 1 If a 2n by 2n matrix U is written in block form

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

it is symplectic if and only if

$$A^T C,\ B^T D \text{ are symmetric and}$$
$$A^T D - C^T B = I_n .$$

In particular, such a matrix with B = 0 is symplectic if and only if A is nonsingular and U can be written as

$$U = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}$$

with some symmetric matrix S.

Exercise 2. It is known that every real nonsingular matrix U can be uniquely represented as (polar decomposition)

$$U = P\,O$$

where $P = P^T$ is positive definite and O orthogonal.

Show: If U is symplectic then both P and O are symplectic.

Exercise 3. Show that a positive symmetric matrix P is symplectic if and only if it is of the form

$$P = e^M \quad \text{where} \quad M = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

and A, B are symmetric n by n matrices.

Exercise 4. Show that an orthogonal matrix O is symplectic if and only if it is of the form

$$O = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

where $A^T B$ is symmetric and

$$A^T A + B^T B = I .$$

These conditions are also necessary and sufficient for the complex matrix

$$A + i\,B$$

to be unitary. Hence we have established an isomorphism

$$SP(2n,R) \cap O(2n) \simeq U(n) .$$

Hint: Note that the relation

$$O^T K\, O = K$$

holds for $K = I_{2n}$ and J hence also for

$$K = I_{2n} + i\,J_{2n} .$$

Write out this condition in terms of $E = A + iC$, $F = B + iD$ as

$$E^* E = F^* F = I , \quad E^* F = iI .$$

From the first and last relation get

$$E^*(E + iF) = 0$$

hence $E + iF = 0$ which gives the result.

Exercise 5 Show that with

$$T = \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}$$

the matrix O of Exercise 4 is transformed into

$$T \circ T^{-1} = \begin{pmatrix} A-iB & 0 \\ 0 & A+iB \end{pmatrix}$$

showing that the mapping of that exercise is a homomorphism, i.e. respects the group operation.

Exercise 6 Show that for any symplectic matrix U one has

$$\det U = +1 .$$

Remark: It is only a matter of showing that det U is positive. One can use Exercies 2 and 5 to show this.

Exercise 7 Show that the most general symplectic mapping $x \to Ux$ which preserves the two Lagrange subspaces

$$E_1 = \{x \mid x_1 = x_2 = \cdots = x_n = 0\}$$
$$E_2 = \{x \mid x_{n+1} = \cdots = x_{2n} = 0\}$$

has the form

$$U = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$$

Exercise 8 A transformation $x_j = u_j(y)$ is canonical if and only if

$$\{u_j, u_k\} = J_{jk}$$

where $J_{j,j\pm n} = \pm 1$, $J_{jk} = 0$ otherwise.

Hint: Apply Lemma 2 to the function

$$F(x) = x_j , \qquad G(x) = x_k .$$

G (cmr)

Exercise 9 Show: given two vectors $v \neq 0$, $w \neq 0$ in a symplectic space (V, [,]) there exists a symplectic mapping U with

$$Uv = w$$

(i.e. the symplectic group acts transitively on V - (0)).

Hint: Use the argument of Lemma 1 to construct two canonical bases v_j and w_j with $v_1 = v$ and $w_1 = w$.

Exercise 10 (i) Let E be a Lagrange subspace of the symplectic space V then there exists a complementary Lagrange subspace F, i.e., $V = E + F$, $E \cap F = (0)$.

(ii) Given two complementary Lagrange subspaces E, F of V and a basis $e_1, e_2, \ldots, e_n$ of E show there exists a unique basis $f_1, f_2, \ldots, f_n$ of F such that

$$[e_i, f_j] = \delta_{ij}.$$

Exercise 11 Let E be an isotropic subspace of V with basis $e_1, e_2, \ldots, e_d$. Show that $e_{d+1}, \ldots, e_n$, $f_1, \ldots, f_n$ can be chosen such that e_i, f_j form a symplectic basis of V.

Exercise 12 Let

$$L = \frac{d^2}{dx^2} + q(x), \quad q \in C[a,b]$$

be an ordinary differential operator acting on $u \in C^2[a,b]$, and let $V = R^4$ be the space of boundary values, i.e., $x \in V$:

$$x = (x_1 = u(a),\ x_2 = u'(a),\ x_3 = u(b),\ x_4 = u'(b)).$$

Then

$$\int_a^b \left\{ (Lv)u - v(Lu) \right\} dx = [u,v]_a^b$$

depends only on the boundary values and defines a bilinear form on V. Prove that this bilinear form is a symplectic form, hence V a symplectic vector space.

Let $E \subset V$ be the two dimensional subspace defined by the pair of linearly independent and real boundary conditions for L:

$$\sum_{j=1}^{4} \alpha_j x_j = 0 , \qquad \sum_{j=1}^{4} \beta_j y_j = 0 .$$

Prove that the operator L belonging to these boundary conditions is symmetric precisely if $E \subset V$ is a Lagrangian subspace.

Generalize to ordinary differential operators of even order.

4. Hamilton-Jacobi Equations

In this section we give other representations of canonical transformations which will be of importance in the following chapters — and which will lead us to the first order partial differential equations, called the Hamilton-Jacobi equation.

(a) Generating Functions.

From now on we will abandon the notation (q,p) and replace it by (x,y), $x,y \in R^n$. This may lead to a confusion, since x does not stand for the 2n-vector any longer, but for an n-vector. Similarly we will denote by $(\xi,\eta) \in R^{2n}$ another point and consider the mapping

$$u: (\xi,\eta) \mapsto (x,y)$$

where

$$x = a(\xi,\eta) , \quad y = b(\xi,\eta) . \tag{4.1}$$

This mapping is canonical if and only if

$$u^*\omega = \omega \quad \text{where} \quad \omega = \sum_{j=1}^{n} d\eta_j \wedge d\xi_j$$

for which we also write

$$\sum_{j=1}^{n} dy_j \wedge dx_j = \sum_{j=1}^{n} d\eta_j \wedge d\xi_j . \tag{4.2}$$

It would be more precise to write

$$\sum_{j=1}^{n} db_j \wedge da_j = \sum_{j=1}^{n} d\eta_j \wedge d\xi_j .$$

If we assume

$$(4.3) \qquad \det(a_\xi) \neq 0$$

then locally the first equation in (4.1) can be solved for $\xi = \alpha(x,\eta)$. If we insert this into the second equation of (4.1) we get

$$(4.4) \qquad \begin{cases} \xi = \alpha(x,\eta) \\ y = \beta(x,\eta) \end{cases}$$

where

$$(4.5) \qquad \det(\alpha_x)\,\det(a_\xi) = 1 .$$

The advantage of scrambling the variables in this manner is that the condition for u to be canonical is much easier to express in terms of the α,β than in terms of the a,b. In fact, under condition (4.3), u is canonical if and only if there exists a scalar function $W = W(x,\eta)$ with

$$\xi = \alpha(x,\eta) = W_\eta(x,\eta)$$
$$y = \beta(x,\eta) = W_x(x,\eta)$$

i.e. (α,β) must be the gradient of a function. The necessary compatibility conditions for α,β are *linear* in contrast to the quadratic conditions for a,b. We emphasize again that this statement is of *local* nature.

To prove this statement one uses the differential form

$$\sigma = \sum_{j=1}^{n} y_j\, dx_j + \xi_j\, d\eta_j$$

whose exterior derivative is

$$d\sigma = \sum_{j=1}^{n} \left(dy_j \wedge dx_j + d\xi_j \wedge d\eta_j\right)$$
$$= \sum_{j=1}^{n} \left(dy_j \wedge dx_j - \sum_{j=1}^{n} d\eta_j \wedge d\xi_j\right).$$

This differential vanishes precisely if x,y,ξ,η are related by (4.1) or equivalenty by (4.4) provided u is canonical. This means that

$$\sum_{j=1}^{n} \beta_j(x,\eta)\, dx_j + \alpha_j(x,\eta)\, d\eta_j$$

is closed, and therefore locally exact i.e. it is of the form

$$dW(x,\eta) = \sum_{j=1}^{n} (W_{x_j}\, dx_j + W_{\eta_j}\, d\eta_j)$$

with some function $W = W(x,\eta)$. Comparison of the coefficients of dx_j, $d\eta_j$ gives the result which we summarize:

THEOREM 4.1. Every canonical transformation (4.1) satisfying (4.3) can locally be represented in the implicit form

$$\xi = W_\eta(x,\eta)$$
$$y = W_x(x,\eta)$$

where

$$\det W_{x\eta} \neq 0 .$$

Conversely, any smooth function $W = W(x,\eta)$ satisfying this inequality defines implicitly a canonical transformation near a point.

The proof has been given above except for the last statement which follows by reversing the steps. The last inequality, which follows from (4.5), allows us to return to the explicit representation (4.1).

The function $W = W(x,\eta)$ is called a generating function of the canonical transformation. This theorem shows that canonical transformations satisfying (4.3) can locally be described in terms of a single function showing how severely the 2n functions a_j, b_j are restricted.

The identity map corresponds to $W = \langle x,\eta \rangle$. Thus any function

$$W = \langle x,\eta \rangle \quad w(x,\eta)$$

with a function $w(x,y)$ which is small with its first derivatives defines a canonical transformation near the identity. We can use the function $w(x,\eta)$ to "parametrize" canonical transformations near the identity.

There are variations on this theme and other generating functions can be constructed: If we single out x_j, ξ_j as independent variables — where we assume

$$\det(a_{\eta}) \neq 0 \tag{4.6}$$

instead of (4.3) — then we can represent the canonical transformation u in the implicit form

$$\begin{aligned} \eta &= -V_{\xi} \\ y &= V_x \end{aligned}, \qquad \det V_{x\xi} \neq 0$$

with a generating function $V = V(\xi,x)$. The proof proceeds as above but starts with the differential form

$$\sum_{j=1}^{n} y_j \, dx_j - \sum_{j=1}^{n} \eta_j \, d\xi_j \ .$$

For example, the identity map violates condition (4.6) but satisfies (4.3), while the simple rotation

$$x_j = \eta_j \ , \qquad y_j = -\xi_j$$

which is canonical satisfies (4.6) but violates (4.3). However, for any canonical transformation there is some generating function respresenting it, as follows from the following lemma. To formulate it we introduce the group of "elementary canonical transformations" which are generated by the rotation

$$x_1 = \eta_1 \ , \quad y_1 = -\xi_1 \ , \quad x_k = \xi_k \ , \quad y_k = \eta_k \quad \text{for } k \geq 2$$

and the mappings

$$x_j = \xi_{\pi(j)} \ , \quad y_j = \eta_{\pi(j)} \qquad (j = 1,2,\ldots,n)$$

where π is a permutation of the numbers $1,2,\ldots,n$. It is clear that these transformations are canonical and can therefore be represented by symplectic matrices, say E.

Lemma 4.1. Given any symplectic matrix U there exists an elementary canonical transformation E such that

$$U\,E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $\det A \neq 0$.

We will leave the proof to the reader (Exercise 11).

Applying this lemma to a canonical transformation u with Jacobian U at some point we see that one can achieve that condition (4.3) holds for $u \circ \kappa$, for some elementary canonical transformation, and thus find a representation in terms of a generating function near this point.

(b) Extending a transformation to a canonical one.

We apply Lemma 1 to determine the most general canonical transformation (4.1) for which

$$x = a(\xi) \tag{4.7}$$

is prescribed, where $\det(a_\xi) \neq 0$, and $a(\xi)$ is independent of η. This represents a transformation in "configuration space" which is to be extended to a canonical transformation in phase space.

If we apply Lemma 1 to our problem the generating function $W(x,\eta)$ has to satisfy

$$\xi = W_\eta(x,\eta) = \alpha(x)$$

where $\alpha(x)$ is the inverse mapping to $x = a(\xi)$. Thus

$$W = \langle\alpha(x),\eta\rangle + w(x)$$

with an arbitrary scalar function $w(x)$.

It is more efficient to represent the canonical transformation in terms of another generating function $V = V(\xi,y)$ using ξ_j, y_j as independent variables. This canonical transformation is given by

$$x = V_y(\xi,y) \quad , \qquad \eta = V_\xi(\xi,y).$$

By (4.7) we have

$$V_y(\xi,y) = a(\xi)$$

or

$$V(\xi,y) = \langle a(\xi),y\rangle + v(\xi)$$

or

$$\eta = V_\xi = a_\xi^T(\xi)y + v_\xi(\xi)$$

and the desired explicit representation is

$$\begin{cases} x = a(\xi) \\ y = (a_\xi^T)^{-1}(\eta - v_\xi) \end{cases} \tag{4.8}$$

where $v = v(\xi)$ is an arbitrary scalar function. This is the most general canonical transformation compatible with (4.7). This generalizes Exercise 1 of Section 3 to the nonlinear case.

We will frequently apply (4.8). It is cumbersome to determine the inverse of the Jacobian a_ξ. But this difficulty disappears for conformal mappings $\xi \leftrightarrow a(\xi)$ for which

$$|dx|^2 = \sum dx_j^2 = \lambda(\xi) \sum d\xi_j^2$$

or

$$a_\xi^T\, a_\xi = \lambda(\xi)\, I\ .$$

Even if the ξ_j represent an "orthogonal" coordinate system, i.e. if

$$|dx|^2 = \sum_{j=1}^{n} g_j(\xi)\, d\xi_j^2$$

holds we have

$$a_\xi^T\, a_\xi = g(\xi) = \operatorname{diag}\,(g_1,g_2,\ldots,g_n)$$

and (4.8) can be written in the form

$$(4.9)\quad \begin{cases} x = a(\xi) \\ y = a_\xi g^{-1}(\eta - v_\xi) \end{cases} .$$

For example, we apply this statement to the conformal inversion on the sphere

$$x = \xi|\xi|^{-2}, \qquad \xi \neq 0$$

for which

$$a_\xi^T a_\xi = |\xi|^{-4} I .$$

Taking $v = 0$ in (4.9) one obtains the canonical transformation u:

$$(4.10)\quad \begin{aligned} x &= |\xi|^{-2}\xi \\ y &= |\xi|^2 \eta - 2\langle\xi,\eta\rangle\xi \end{aligned}$$

as one easily verifies. This mapping is, like the inversion, an involution, i.e. satisfies $u \circ u = id$. It played a central role in K. Sundman's investigation of the three-body problem.

In generalization of the extension (4.8) of (4.7) we ask when the relations

$$(4.11)\quad x_j = a_j(\xi,\eta) \qquad (j = 1,2,\dots,n)$$

can be extended to a canonical transformation. Clearly we have to require that

(i) $\{a_j, a_k\} = 0$, and

(ii) the matrix

$$(a_{j\xi_k}, a_{j\eta_k})_{j,k=1,2,\dots,n}$$

has rank n.

These conditions are also sufficient, as we now show. By applying an elementary canonical transformation we can assume that

$$\det(a_{j\xi_k}) \neq 0$$

and solve the equations (4.11) for

$$\xi_j = \alpha_j(x,\eta) .$$

We claim that the matrix α_η is symmetric as a consequence of condition (i). That condition can be written as

$$a_\xi a_\eta^T - a_\eta a_\xi^T = 0 ,$$

i.e. $a_\xi a_\eta^T$ is a symmetric matrix. Multiplying this equation by a_ξ^{-1} from the left and its transposed from the right we find

$$a_\eta^T (a_\xi^T)^{-1} - a_\xi^{-1} a_\eta = 0 ,$$

i.e. $a_\xi^{-1} a_\eta$ is a symmetric matrix.

On the other hand, differentiating the identity

$$\xi = \alpha(a(\xi,\eta),\eta)$$

yields

$$\begin{aligned} I &= \alpha_x a_\xi \\ 0 &= \alpha_x a_\eta + \alpha_\eta \end{aligned}$$

or

$$\alpha_\eta = -a_\xi^{-1} a_\eta .$$

This shows that α_η is a symmetric matrix and therefore locally there exists a function, say $W = W(x,\eta)$ with

$$W_{\eta_j}(x,\eta) = \alpha_j(x,\eta) .$$

Moreover,

$$\det(W_{x\eta}) = \det a_x = (\det a_\xi)^{-1} \neq 0$$

and therefore $W = W(x,\eta)$ is a generating function of a canonical transformation extending (4.11).

As a corollary to the extension of (4.11) we have

Lemma 4.2. If $a_0, a_1, \cdots, a_n$ are n+1 functions for which

$$(4.12) \qquad \{a_j, a_k\} = 0 \qquad j,k=0,\cdots,n$$

and

$$\operatorname{rank}\left(a_{j\xi}, a_{j\eta}\right)_{j=1,2\cdots n} = n$$

then a_0 can be expressed as a function of $a_1, a_2, \cdots, a_n$. In other words there are at most n independent functions satisfying (4.12).

Proof: Applying the canonical extension of (4.11) we can find a canonical transformation u such that

$$a_j \circ u(x,y) = x_j .$$

We set

$$a_0 \circ u = f(x,y)$$

Since the Poisson brackets are preserved under u we have

$$0 = \{a_j, a_0\} = \{x_j, f\} = f_{y_j}$$

i.e. f is independent of y. Hence

$$a_0 = f(a_1, a_2, \cdots, a_n)$$

proving Lemma 4.2.

(c) Local Equivalence.

In Section 1 we saw that any two vector fields are locally, near a nonsingular point, equivalent. The same applies, in particular, to Hamiltonian vector fields but it is not obvious that the mapping u establishing the equivalence can be chosen as a canonical mapping. That this is the case is the content of

THEOREM 4.2. If $dH \neq 0$ at a point $(x,y) = (x^*,y^*)$ and $H \in C^{r+1}$ $(r \geq 1)$ in a neighborhood of (x^*,y^*) then there exists a canonical C^r-mapping $u: (\xi,\eta) \mapsto (x,y)$ near a point (ξ^*,η^*) which is mapped into (x^*,y^*) such that $H \circ u = \eta_1$. Thus the transformed vector field

$$\dot{\xi} = e_1 , \qquad \dot{\eta} = 0$$

is parallel to the ξ_1-axis.

This theorem follows from the somewhat sharper

Lemma 4.3. If $H \in C^{r+1}$ near the origin $(x,y) = (0,0)$ and

$$dH = \langle a,dx\rangle + \langle b,dy\rangle \neq 0 \quad \text{at} \quad x = y = 0$$

then there exists a canonical map u near the origin with

$$u(0) = 0 , \quad \text{Jacobian } u(0) = \left(u_\xi(0), u_\eta(0)\right) = I$$

such that

$$H \circ u(\xi,\eta) = \langle a,\xi\rangle + \langle b,\eta\rangle ,$$

Indeed, to deduce Theorem 2 from this lemma one has to construct a symplectic map, say v, taking $\langle a,\xi\rangle + \langle b,\eta\rangle$ into η_1. That this is possible follows from Exercise 9 of Section 3. Before proving Lemma 4.3 we need an extension of the formula

$$(\phi^t)^*\omega = \omega\,, \qquad \omega = \sum_{j=1}^{n} dy_j \wedge dx_j$$

for the flow ϕ^t of a Hamiltonian vector field X_H. This formula simply expresses that ϕ^t is canonical, and therefore is a consequence of Theorem 3.1.

We wish to extend this formula to the mapping $\tilde\phi$ in R^{2n+1},

$$\tilde\phi: (t,\zeta) \mapsto (t, \phi^t(\zeta)), \qquad \zeta = (\xi,\eta)\ .$$

We define the extended differential form on R^{2n+1}

~~(4.12)~~ $$\tilde\omega = d\left(\sum_{j=1}^{n} y_j\, dx_j - H\, dt\right)$$

which is equal to

$$\sum_{j=1}^{n} dy_j \wedge dx_j - dH \wedge dt = \sum_{j=1}^{n} (dy_j + H_{x_j} dt) \wedge (dx_j - H_{y_j} dt)\ ,$$

and which agrees with ω on $dt = 0$.

Lemma 4.4. If ϕ^t is the flow of X_H, and $\tilde\phi$ the above extension we have

(4.13) $$\tilde\phi^*\tilde\omega = \omega\ .$$

Proof: One can simply verify this statement by a matrix calculation. The form is represented by the (2n+1) by (2n+1) matrix

$$\tilde A = \begin{pmatrix} 0 & H_z^T \\ -H_z & J \end{pmatrix}, \qquad \text{where} \qquad H_z = \begin{pmatrix} H_x \\ H_y \end{pmatrix},$$

and the Jacobian of $\tilde\phi$ by the matrix

$$\tilde\Phi = \begin{pmatrix} 1 & 0 \\ JH_z & \Phi \end{pmatrix},$$

where Φ is the Jacobian of ϕ^t. Then one computes readily that

$$\tilde\Phi^T \tilde A\, \tilde\Phi = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix},$$

since Φ is symplectic and $\langle H_z, JH_z\rangle = 0$. This formula is in agreement with (4.13).

Since $z = (x,y) = \phi^t(\xi,\eta)$ we write the identity (4.13) in the suggestive form

(4.14) $$\sum_{j=1}^{n} dy_j \wedge dx_j - dH \wedge dt = \sum_{j=1}^{n} d\eta_j \wedge d\xi_j\ .$$

Proof of Lemma 4.3: We can assume $H(0) = 0$ and that $(a,b) = e_{n+1}$, using Exercise 9 of Section 3, so that

$$H = y_1 + \hat H(x,y)$$

where $\hat H$ and its first derivatives vanish at 0. We wish to construct a canonical mapping

$$z = u(\xi,\eta) \qquad \text{or} \qquad \begin{aligned} x &= a(\xi,\eta) \\ y &= b(\xi,\eta) \end{aligned}$$

so that

(4.15) $$H \circ u = \eta_1 ,$$

The resulting vector field $\dot{\xi} = e_1$, $\dot{\eta} = 0$ has the flow

$$\psi^t(\xi,\eta) = (\xi+te_1,\eta)$$

and u has to satisfy

(4.16) $$\phi^t \circ u = u \circ \psi^t .$$

There is a large freedom in the choice of u and we could attempt to take u when restricted to the hypersurface $\xi_1 = 0$ equal to the identity. However, this would not be compatible with (4.15). Therefore we require for $\xi_1 = 0$ only

$$a_j(\xi,\eta) = \xi_j \quad \text{for} \quad j = 1,2,\cdots,n$$
$$b_j(\xi,\eta) = \eta_j \quad \text{for} \quad j \geq 2$$

while $b_1 = b_1(\xi_1,\eta)$ will be determined by

(4.17) $$H(0,\xi',b_1,\eta') = \eta_1 .$$

Here we write $\xi' = (\xi_2,\cdots,\xi_n)$, $\eta' = (\eta_2,\cdots,\eta_n)$. This equation determines b_1 uniquely with $b_1(0) = 0$ since $H_{y_1}(0) \neq 0$. Note that b_1 is independent of ξ_1. Next we set $\xi_1 = 0$ in (4.16) and find

(4.18) $$\phi^t(0,\xi',b_1,\eta') = u(t,\xi',\eta_1,\eta') .$$

Therefore we will define $u = u(\xi,\eta)$ by

$$u(\xi,\eta) = \phi^{\xi_1}(0,\xi',b_1,\eta')$$

where b_1 is defined by (4.17). We have to verify that u has the required properties.

First u satisfies (4.16) since

$$\phi^t \circ u = \phi^t \circ \phi^{\xi_1}(0,\xi',b_1,\eta') = \phi^{t+\xi_1}(0,\xi',b_1,\eta') = u \circ \psi^t .$$

Second, one computes easily from (4.17) that

$$(b_{1\xi},b_{1\eta})(0) = e_{n+1}$$

and from this that the Jacobian of u at 0 is the identity.

Third, we have to show that u is canonical. For this we use Lemma 4.4 which we apply in the notation (4.14). Note that u is obtained from ϕ^t by setting $\xi_1 = 0$ and replacing t by ξ_1 (see (4.18)) so that (4.14) restricted to $\xi_1 = 0$ becomes

$$\sum_{j=1}^{n} dy_j \wedge dx_j - dH \wedge d\xi_1 = \sum_{j=2}^{n} d\eta_j \wedge d\xi_j .$$

But since H is an integral under the flow ϕ^t we have

$$H \circ u(\xi,\eta) = H \circ \phi^{\xi_1}(0,\xi',b_1,\eta') = H(0,\xi',b_1,\eta') = \eta_1$$

so that

$$\sum_{j=1}^{n} dy_j \wedge dx_j = \sum_{j=1}^{n} d\eta_j \wedge d\xi_j .$$

This shows that u is canonical and the proof of Lemma 4.3 is complete.

To summarize the seemingly tricky construction: We introduce the time t elapsed from passing the hyperplane

$\xi_1 = 0$ and the Hamiltonian H as independent variables (which we called ξ_1, η_1 above) together with ξ_j, η_j $(j \geq 2)$ to obtain the desired canonical transformation. An alternate proof of Theorem 4.2 is indicated in Exercise 6 below.

<u>Corollary to Lemma 4.3</u>. If $H_y(0) \neq 0$ then there exists a canonical transformation $u: (\xi, \eta) \to (x,y) = (a(\xi,\eta), b(\xi,\eta))$ with

$$\det a_\xi(0) \neq 0$$

and

$$H \circ u = \eta_1 .$$

<u>Proof</u>: By Lemma 4.3 it suffices to construct a symplectic map

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \longrightarrow U \begin{pmatrix} \xi \\ \eta \end{pmatrix} , \quad U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $\det A \neq 0$ taking

$$K(\xi,\eta) = \langle a,\xi\rangle + \langle b,\eta\rangle$$

into η_1, where we required $b \neq 0$. With a map given by

$$U = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} , \quad A D^T = I$$

we can replace b by $D^T b = e_1$. A second symplectic map of the form

$$U = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} , \quad S^T = S$$

keeps $b = e_1$ fixed and replaces a by $a+Se_1$ which can be made equal to zero by choice of S. Thus K is replaced by η_1.

(d) Hamilton - Jacobi Equation

The canonical transformation u of the corollary to Lemma 4.3 can by Theorem 4.1 be represented in the form

$$\xi = W_\eta , \quad y = W_x , \quad \det W_{x\eta} \neq 0$$

Hence the relation $H \circ u = \eta_1$ turns into

$$H(x, W_x(x,\eta)) = \eta_1 , \tag{4.19}$$

which is called the Hamilton-Jacobi equation. Thus we have established the following existence theorem:
If $H_y(0,0) \neq 0$ then there exists a function $W = W(x,\eta)$ satisfying the equation (4.19) and $\det W_{x\eta} \neq 0$ in a neighborhood of the origin. Notice that $\eta_2, \eta_3 \cdots \eta_n$ are just parameters not given by the equation. Any solution of this first order partial differential equation depending on these parameters such that $\det W_{x\eta} \neq 0$ is called a complete integral.

Jacobi's approach, however, was the opposite: He tried to find a complete integral W of (4.19) and used it to solve the Hamiltonian system of ordinary differential equations. In fact, if $W(x,\eta)$ is known, it defines a canonical transformation, say u, which transforms the system into

$$\dot\xi = e_1 , \quad \dot\eta = 0$$

with the flow

$$\psi^t(\xi,\eta) = (\xi+te_1, \eta).$$

Thus the flow of the original system is given by

$$\phi^t(x,y) = u \circ \psi^t(\xi,\eta) \quad \text{where} \quad (x,y) = u(\xi,\eta) \ ,$$

which defines the solution of the given system.

Finding a complete solution of (4.19) is therefore equivalent to solving the system of ordinary differential equations — but it has to be kept in mind that the solution we gave is only local. Many of the difficulties which we will encounter later on are tied to the fact that a global solution of (4.19) generally does not exist.

Actually it suffices to find a solution of the equation

$$H(x,W_x) = \phi(\eta) \ , \quad \det W_{x\eta} \neq 0$$

since in this case the differential equations become

$$\dot{\xi} = \phi_\eta \ , \qquad \dot{\eta} = 0$$

which is easily integrated by

$$\xi(t) = \xi(0) + t\,\phi_\eta(\eta(0)) \ , \quad \eta(t) = \eta(0) \ .$$

Exercise 1 Show that

$$x = \sqrt{2\eta}\cos\xi$$
$$y = \sqrt{2\eta}\sin\xi$$

defines a canonical transformation of $(\xi,\ \eta > 0) \rightarrow (x,y)$.

Exercise 2 Show that

$$x_1 = \xi_1 \cos\xi_2 \ , \qquad y_1 = \eta_1 \cos\xi_2 - \xi_1^{-1}\eta_2 \sin\xi_2$$
$$x_2 = \xi_1 \sin\xi_2 \ , \qquad y_2 = \eta_1 \sin\xi_2 - \xi_1^{-1}\eta_2 \cos\xi_2$$

is a canonical transformation.

Exercise 3 Transform the Hamiltonian

$$H = \frac{1}{2}\left|p - \frac{e}{c}A\right|^2$$

for the vector potential of the dipole

$$A = e_3 \wedge \nabla\left(\frac{1}{q}\right) = |q|^{-3}\begin{pmatrix} -q_2 \\ +q_1 \\ 0 \end{pmatrix}$$ 191

into polar coordinates

$$q_1 = \rho\cos\theta \ , \qquad p_1 = p_\rho \cos\theta - \frac{p_\theta}{\rho}\sin\theta$$
$$q_2 = \rho\sin\theta \ , \qquad p_2 = p_\rho \sin\theta + \frac{p_\theta}{\rho}\cos\theta$$
$$q_3 = z \ , \qquad p_3 = p_z$$

where ρ, θ, z and p_ρ, p_θ, p_z are the new variables.

(Answer:

$$2H = \left(\frac{p_\theta}{\rho} - \frac{e}{c}\frac{\rho}{r^3}\right)^2 + p_\rho^2 + p_z^2$$

with $r^2 = \rho^2 + z^2$.)

Exercise 4 Show that in Exercise 3 $\dot{p}_\theta = 0$, i.e., p_θ is a constant along solutions and the differential equations can be written

$$\ddot{\rho} = -V_\rho , \quad \ddot{z} = -V_z , \qquad V = \frac{1}{2}\left(\frac{p_\theta}{\rho} - \frac{e}{c}\frac{\rho}{r^3}\right)^2 .$$

Exercise 5 Discuss the solutions of Exercise 4 which lie on the plane $z = 0$, for $e/c = 1$. Show, in particular, that for $p_\theta \le 0$ all solutions escape and for $p_\theta > 0$ there is a periodic circular solution of radius $\rho = 2p_\theta^{-1}$.

Exercise 6 Prove Theorem 4.2 in case $H_{y_1}(0) \neq 0$ by constructing u in the form

$$u = \lim_{t\to\infty} \phi^{-t} \circ \psi^t$$

where ψ^t is the flow corresponding to $K = y_1$ and ϕ^t the flow corresponding to a modified H (see Section 1, Exercise 2).

Exercise 7 Consider a function $H = H(q,p)$ which is invariant under the translations

$$q_j \to q_j + a , \quad p_j \to p_j \qquad (j=1,2,\dots,n),$$

and let $m_1, m_2, \dots, m_n$ be positive numbers. Introduce the "relative coordinates" $x_1, x_2, \dots, x_n$ by

$$x_j = q_{j+1} - q_j \quad \text{for} \quad j \le n-1$$

$$x_n = M^{-1} \sum_{k=1}^{n} m_k q_k , \qquad M = \sum_{k=1}^{n} m_k .$$

Extend this to a canonical transformation u: $(q,p) \to (x,y)$ homogeneous in the p_j. Show that

$$y_n = \sum_{j=1}^{n} p_j$$

and that $F = H \circ u^{-1}$ is independent of x_n , hence

$$\dot{y}_n = -F_{x_n} = 0 .$$

In case the Hamiltonian function is of the form

$$H(q,p) = T(p) + U(q)$$

where

$$T(p) = \frac{1}{2} \sum_{j=1}^{n} m_j^{-1} p_j^2$$

is a positive definite quadratic form there is a more effective way to introduce relative coordinates via a canonical transformation which keeps T in diagonal form. This transformation is due to Jacobi and is based on the Gram-Schmidt orthogonalization process with respect to $T(p)$, which is presented in the following two exercises.

Exercise 8 Introduce x by the linear transformation $q \to x = Aq$ given by

$$x_j = \sum_{k=1}^{j} a_{jk}q_k + q_{j+1} \qquad (j \le n-1)$$

$$x_n = M^{-1}\sum_{j=1}^{n} m_j q_j$$

and extend it to a canonical map u: $(q,p) \mapsto (x,y)$ by

$$y = (A^T)^{-1}p$$

We write A in terms of its row vectors $a_j = (a_{jk})_{k=1,2\ldots n}$ e.g.

$$a_n = M^{-1}(m_1, m_2, \ldots, m_n) ,$$

and consider the bilinear form

$$T(v,w) = \sum_{j=1}^{n} m_j^{-1} v_j w_j .$$

The vectors $a_1, a_2, \cdots, a_{n-1}$ will be chosen in such a way that

$$T(a_i, a_j) = 0 \quad \text{for} \quad i \ne j, \quad i,j=1,2,\cdots,n.$$

Show that the a_j, hence A, are uniquely determined by this requirement, that $x_1, x_2, \cdots, x_{n-1}$ are invariant under translation $q_j \to q_j + a$ and that

$$y_n = \sum_{j=1}^{n} p_j .$$

Moreover, $F = H \circ u^{-1}$ has the form

$$F(x,y) = \frac{1}{2}\sum_{j=1}^{n} \mu_j^{-1} y_j^2 + V(x),$$

where V is independent of x_n and $\mu_j > 0$.

Exercise 9 Verify that the matrix A constructed above is given by

$$x_j = -\left(\sum_{k=1}^{j} m_k\right)^{-1}\left(\sum_{k=1}^{j} m_k q_k\right) + q_{j+1} \quad \text{for} \quad j \le n-1$$

and the coefficients μ_j of F in Exercise 8 are

$$\mu_j = m_{j+1}\left(\sum_{k=1}^{j} m_k\right)\left(\sum_{1}^{j+1} m_k\right)^{-1} , \qquad j \le n-1$$

$$\mu_n = M .$$

Note that the x_j can be interpreted as the vector from the center of mass of $q_1, q_2, \cdots, q_j$ to q_{j+1}. The μ_j are called the reduced masses. This construction applies equally well for vectors q_j.

Exercise 10 Let u: $(\xi,\eta) \to (x,y)$ given by

$$x = a(\xi,\eta)$$
$$y = b(\xi,\eta)$$

be a canonical transformation with generating function $W(x,\eta)$ and

$$M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} , \qquad A^T D = I$$

a symplectic matrix. Show that u commutes with M if and only if

$$W(Ax, D\eta) = W(x,\eta) + \text{const}$$

<u>Exercise 11</u> Let (A,B) be an n by 2n matrix of rank n, with two n by n matrices A and B satisfying

$$AB^T = BA^T.$$

Prove there is a symplectic matrix S such that (A,B)S = (A',B') satisfies

$$\det(A') \neq 0 .$$

Prove that S can be chosen to be an elementary symplectic matrix E. (The definition of elementary symplectic was given before Lemma 4.1.)

5. <u>Integrals and Group Actions</u>

(a) Integrals.

For any vector field X with flow ϕ^t we call a function $G = G(x)$ an integral if $G(\phi^t(x))$ is independent of t, i.e.,

$$G(\phi^t(x)) = G(x)$$

but the gradient G_x of G is not zero. Since

$$\frac{d}{dt} G(\phi^t(x))\Big|_{t=0} = X\,G$$

this is equivalent with the following

Definition. A C^1-function in a domain D is called an integral of the vector field X if

$$X\,G = 0 , \qquad dG \neq 0 \text{ in } D .$$

If $X = X_H$ is a Hamiltonian vector field then

$$X_H G = -\{H,G\} = \{G,H\} \tag{5.1}$$

and G is an integral if and only if

$$\{H,G\} = 0 , \qquad dG \neq 0 \text{ in } D .$$

On account of (5.1) the conditions

$$X_H G = 0 \quad \text{and} \quad X_G H = 0$$

are equivalent and this equivalence is the basis of the principle relating group invariance of H and the existence of integrals to be discussed in this section. We begin with a simple example.

(b) Example.

We consider a function H in R^6 and assume that it is rotation invariant, in the sense that

$$H(Rx,Ry) = H(x,y)$$

for any orthogonal 3 by 3 matrix R with $\det R = 1$. It is well known that X_H has the integrals

$$(5.2) \qquad G_1 = x_2y_3 - x_3y_2\,, \qquad G_2 = x_3y_1 - x_1y_3\,, \qquad G_3 = x_1y_2 - x_2y_1$$

which are the components of the angular momentum vector $x \wedge y$.

To show this we recall that the group of all rotations is generated by rotations about the three axes, i.e. by

$$R_3(\tau) = \begin{pmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and two similar matrices $R_1(\tau_1)$, $R_2(\tau_2)$, defining rotations about the x_1-axis and the x_2-axis respectively. The matrix $R_3(\tau)$ can also be written as

$$R_3(\tau) = e^{\tau A_3}\,, \qquad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and is the solution of the differential equation

$$\frac{dR_3}{d\tau} = A_3R_3\,, \qquad R_3(0) = I\,.$$

Similarly, we have

$$R_j = e^{\tau_j A_j}\,, \qquad j = 1,2\,,$$

with

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and an arbitrary orthogonal matrix with determinant +1 can be written as

$$R = e^A \quad \text{with} \quad A = \sum \tau_j A_j = \begin{pmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{pmatrix}$$

Thus the matrices A so obtained constitute the antisymmetric 3 by 3 matrices which represent the Lie algebra of the group of rotations.

Now we observe that the mapping

$$\psi_3^\tau : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} R_3(\tau)x \\ R_3(\tau)y \end{pmatrix} = \begin{pmatrix} e^{\tau A_3}x \\ e^{\tau A_3}y \end{pmatrix}$$

is a canonical mapping which is the flow corresponding to the Hamiltonian vector field

$$\frac{dx}{d\tau} = A_3x$$
$$\frac{dy}{d\tau} = A_3y$$

with the Hamiltonian $G_3 = \langle A_3x, y\rangle$. Since by assumption

$$H \circ \psi_3^\tau = H$$

is invariant under this flow we conclude by differentiation with respect to τ at $\tau = 0$,

$$X_{G_3}H = 0\,.$$

By (5.1) this implies that

$$X_H G_3 = 0$$

i.e. G_3 is an integral of X_H in $R^6 \setminus 0$.

By the same reasoning

$$G_j(x,y) = \langle a_j x, y \rangle$$

are integrals which are the same as (5.2).

(c) Group Actions.

The above example illustrates the connection between the invariance of H under a finite dimensional Lie group and the existence of integrals, which was formulated by E. Noether (1918) as a general principle in much greater generality than we will admit. We shall restrict ourselves to translation groups and to groups G represented by matrices $g \in G$, where the group multiplication agrees with matrix multiplication. If $g(\tau)$, $\tau \in R$, describes a one parameter smooth subgroup of a matrix group, i.e.

$$g(\tau_1 + \tau_2) = g(\tau_1) \cdot g(\tau_2) ,$$

then

$$(5.3) \qquad \frac{dg}{d\tau}(\tau) = A\, g(\tau) \quad \text{or} \quad g(\tau) = e^{\tau A} ,$$

where the matrix A is an element of the corresponding Lie algebra $\mathfrak{g}$ of G, with

$$[A_1, A_2] = A_1 A_2 - A_2 A_1 \in \mathfrak{g} \quad \text{if} \quad A_1, A_2 \in \mathfrak{g} .$$

We speak of a canonical group action of G, if

$$\psi : G \times R^{2n} \to R^{2n}$$

is a smooth map satisfying, with the notation $\psi^g = \psi(g,\cdot)$,

$$(5.4) \qquad \psi^{gh} = \psi^g \circ \psi^h , \qquad g,h \in G ,$$

and in addition ψ^g is canonical for every $g \in G$.

For any one parameter subgroup given by $g(\tau)$ with (5.3), $\tau \mapsto \psi^{g(\tau)}$ represents a flow on R^{2n} which by Theorem 3.1 is generated by a Hamiltonian vector field X. This vector field $X = X^A(z)$ is defined by

$$(5.5) \qquad X = \frac{d}{d\tau} \psi(g(\tau), z) \Big|_{\tau=0} , \quad z = (x,y) \in R^{2n} .$$

If we write

$$\psi^{g(\tau)} = \exp(\tau X)$$

for the flow generated by $X = X^A$, we have the suggestive formula

$$\psi^{\exp \tau A} = \exp(\tau X) , \quad X = X^A .$$

Thus we have associated to every A in the Lie algebra $\mathfrak{g}$ of G a vector field X^A. From the definition (5.5) one sees easily that this mapping is linear, i.e.

$$X^{\lambda_1 A_1 + \lambda_2 A_2} = \lambda_1 X^{A_1} + \lambda_2 X^{A_2} .$$

Moreover, as we shall show,

(5.6) $$[X^A, X^B] = X^{[B,A]} ,$$

such that the mapping $A \mapsto X^A$ is a Lie algebra homomorphism of $\mathcal{G}$ into the vector fields on R^{2n} with negative brackets. To verify this formula we first note that

$$h\ (\exp A)\ h^{-1} = \exp\ (h\ A\ h^{-1}) , \quad h \in G .$$

This action of the group G in its Lie algebra is called the adjoint mapping, and is denoted by

$$Ad_h\ A = h\ A\ h^{-1} .$$

To prove (5.6) we consider together with the one-parameter group $g(\tau) = \exp\ (\tau A)$ also

$$\hat{g}(\tau) = h\ g(\tau)\ h^{-1} = \exp\ (\tau\hat{A}) \quad \text{with} \quad \hat{A} = h\ A\ h^{-1}.$$

Since we have a group action, we have

$$\psi^{\hat{g}(\tau)} = \psi^h \circ \psi^{g(\tau)} \circ \psi^{h^{-1}} ,$$

and differentiating with respect to τ at $\tau = 0$

(5.7) $$X^{\hat{A}} = (\psi^{h^{-1}})^* X^A \quad \text{where} \quad \hat{A} = h\ A\ h^{-1} .$$

To complete the proof we set $h = \exp(sB)$ in (5.7) and differentiate with respect to s at $s = 0$. By Exercise 1.3 the right-hand side gives

$$[X^A, X^B]$$

while the left-hand side being linear in $\hat{A}$ gives

$$X^{[B,A]} \quad \text{since} \quad \frac{d}{ds} \hat{A}\Big|_{s=0} = [B,A] .$$

As already pointed out, the vector fields X^A are by Theorem 3.1 Hamiltonian, and therefore there exist functions $G = G_A$ such that

$$X^A = X_G , \quad G = G_A .$$

Actually G is defined only up to a constant. We thus have associated with every A a function (modulo constants) and it follows from (5.6)

$$\{G_A, G_B\} = G_{[A,B]} + c(A,B) ,$$

with constants $c(A,B) \in R$. All these concepts can be generalized to symplectic manifolds. The indetermined constants give rise to interesting phenomena (see for example V. Arnold, Mathematical Methods of Classical Mechanics, Appendix 5). However we will be content to study systems for which these constants can be chosen to be zero. Such a canonical action is sometimes called Poisson action, i.e., if

(5.8) $$\{G_A, G_B\} = G_{[A,B]} .$$

Examples of Poisson actions are given by group actions which are not only canonical but preserve also the one-form on R^{2n}:

$$\theta = \sum_{j=1}^{n} y_j\ dx_j ,$$

i.e.

$$(\psi^g)^* \theta = \theta .$$

Clearly, since $\omega = d\theta$ this implies

$$(\psi^g)^* \omega = \omega ,$$

so that the action is canonical. In this case we can associate with a vector field X a *unique* Hamiltonian by

$$G = \theta(X) , \tag{5.9}$$

where $\theta(X)(z) = \theta(z)(X(z))$. Here we use the concept of an "inner product" of a one form, say $\alpha = \sum_{j=1}^{m} \alpha_j(x)\, dx_j$ on R^m and a vector field on R^m, say

$$X = \sum_{j=1}^{m} a_j(x) \frac{\partial}{\partial y_j} ,$$

as

$$\alpha(X) = \sum_{j=1}^{m} \alpha_j(x)\, a_j(x) .$$

In our case, if $m = 2n$ and

$$X = \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j} \right)$$

we have

$$\theta(X) = \sum_{k=1}^{n} y_k a_k .$$

To verify that this is a Hamiltonian for our vector field X we have to use that the corresponding flow ϕ^s preserves θ, i.e.

$$\frac{d}{ds} (\phi^s)^* \theta \Big|_{s=0} = 0 ,$$

hence

$$\sum_{j=1}^{n} (\dot{y}_j\, dx_j + y_j\, d\dot{x}_j) = \sum_{j=1}^{n} (b_j\, dx_j + y_j\, da_j) = 0 ,$$

or expressing this in terms of dx_j and dy_j :

$$b_j + \sum_{k=1}^{n} y_k a_{kx_j} = 0 , \quad \sum_{k=1}^{n} y_k a_{ky_j} = 0 \tag{5.10}$$

Thus the Hamiltonian vector field X_G with $G = \theta(X)$ is given by

$$\dot{x}_j = G_{y_j} = a_j + \sum_{k=1}^{n} y_k a_{ky_j}$$

$$\dot{y}_j = -G_{x_j} = - \sum_{k=1}^{n} y_k a_{kx_j} .$$

Because of (5.10), this agrees with

$$\dot{x}_j = a_j , \quad \dot{y}_j = b_j$$

i.e. $X_G = X$ as we wanted to show. Now a simple computation shows

$$\theta([X_1, X_2]) = \{\theta(X_2), \theta(X_1)\} ,$$

and therefore (5.8) follows from (5.6), i.e. the action is indeed a Poisson action. An example of such an action leaving the one form θ invariant is provided by the example (b), i.e.

$$\psi(R,z) = (Rx, Ry) , \quad R \in SO(3) .$$

More generally, every mapping $u: R^{2n} \to R^{2n}$ of the form u:

$$x = v(\xi) \qquad y = (v_\xi^T)^{-1} \eta \tag{5.11}$$

leaves θ invariant, and actually this property characterizes mappings of the form (5.11) (see Exericse 3).

If we consider a function H which is preserved under the group action ψ^g i.e. $H \circ \psi^g = H$ for all $g \in G$ then it follows that

$$X_G H = 0 \quad \text{for all} \quad G = G_A \, , \quad A \in \mathcal{G} \, .$$

Since

$$X_H G = -\{H,G\} = \{G,H\} = -X_G H = 0$$

we conclude that all functions $G = G_A$ are integrals of the motion. If $A_1, A_2, \cdots, A_r$ are a basis of the Lie algebra and

$$A = \sum_{j=1}^{r} c_j A_j$$

then

$$G_A = \sum_{j=1}^{r} c_j G_{A_j}$$

and we obtain r integrals G_{A_j} which generate the other G_A. In order that these integrals are independent one has to impose a nondegeneracy condition on the group action.

We illustrate this principle which allows us to associate r integrals with an r-dimensional canonical group action leaving H invariant with several examples.

(d) Rotation and Translation Group.

We consider the example

$$(5.12) \quad H = \sum_{j=1}^{N} \frac{1}{2m_j} |P_j|^2 + \sum_{1 \le i < j \le N} U_{ij}(|Q_i - Q_j|) \, ,$$

where $Q_j, P_j \in R^3$, m_j positive scalars and U_{ij} smooth scalar functions. Then X_H describes the motion of N mass points in R^3

under forces which depend only on the distance of the mass points.

Therefore this Hamiltonian is invariant under the translations in R^3:

$$\psi^g : Q_j \to Q_j + g \, , \quad P_j \to P_j \quad \text{for} \quad g \in R^3$$

which is a canonical group action for the additive group $G = R^3$. The corresponding Lie algebra is given by R^3 again and for $a \in R^3$ the corresponding Hamiltonian is

$$G_a = \sum_{j=1}^{N} \langle a, P_j \rangle = a_1 G_1 + a_2 G_2 + a_3 G_3$$

where

$$G_\alpha = \langle e_\alpha , \sum_{j=1}^{N} P_j \rangle \, , \qquad \alpha = 1,2,3.$$

These represent the integrals of the system corresponding to the translation invariance. They are the components of the total momentum

$$\sum_{j=1}^{N} P_j = \sum_{j=1}^{N} m_j \dot{Q}_j$$

Its constancy implies that the center of mass moves with constant velocity.

Clearly the example (5.5) is also invariant under the group SO(3,R) of rotations in R^3 represented by the orthogonal 3 by 3 matrices with determinant 1. The group action ψ^g is given by

$$(Q_j, P_j) \to (R\, Q_j, \, R\, P_j)$$

where $R \in SO(3,R) = G$. This example is a slight generalization of the one discussed above and leads to the integrals

$$G_\alpha = \langle e_\alpha, \sum_{j=1}^{N} P_j \wedge Q_j \rangle , \qquad \alpha = 1,2,3.$$

Thus the total angular momentum vector

$$\sum_{j=1}^{N} P_j \wedge Q_j = \sum_{j=1}^{N} m_j \dot{Q}_j \wedge Q_j$$

is conserved under the flow of X_H.

(e) Example with SL(2,R)-action.

We consider the example

(5.13) $$H(x,y) = \frac{1}{2}(|x|^2|y|^2 - \langle x,y\rangle^2) = \frac{1}{2}\det S(x,y)$$

where

$$S = \begin{pmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{pmatrix} .$$

We consider the group $G = SL(2,R)$ given by the matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1 ,$$

and the group action

$$\psi^g(x,y) = (ax + by,\ cx + dy)$$

which is clearly canonical. Since

(5.14) $$S(\psi^g(x,y)) = g\, S\, g^T$$

it follows that

$$H \circ \psi^g = (\det g)^2 H = H ,$$

i.e. H is invariant under this action. We compute the corresponding integrals.

The Lie algebra corresponding to SL(2,R) consists of all real 2 by 2 matrices of trace 0 i.e., matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The corresponding vector field is given by

$$\dot{x} = \alpha x + \beta y$$
$$\dot{y} = \gamma x - \alpha y$$

which belongs to the Hamiltonian

$$G = \alpha\langle x,y\rangle + \frac{\beta}{2}|y|^2 - \frac{\gamma}{2}|x|^2 .$$

Thus these integrals are generated by

$$G_1 = |x|^2 , \quad G_2 = \langle x,y\rangle , \quad G_3 = |y|^2$$

With this information it is very easy to integrate the differential equations and show that all solution orbits with $H > 0$ are ellipses with period $2\pi\ (2H)^{-1/2}$, while for $H = 0$ all solutions are singular points (equilibrium points).

Indeed the differential equations for (5.13) are

(5.15) $$\dot{x} = -\langle x,y\rangle x + |x|^2 y$$
$$\dot{y} = -|y|^2 x + \langle x,y\rangle y .$$

Although this system is nonlinear it is easily integrated since all coefficients are integrals and hence constant along solutions. Therefore the system can be integrated like a linear one and the solutions are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\,(c\, e^{\lambda t}) , \qquad c \in C^{2n}$$

where λ is an eigenvalue of the coefficient matrix, i.e.

$$\lambda^2 + 2H = 0$$

or

$$\lambda = \pm\, i \sqrt{2H} .$$

These solutions represent ellipses for $H > 0$. By the Schwarz inequality one has always $H \geq 0$ and one verifies easily that for $H = 0$ one has

$$\dot{x} = 0 \ , \ \dot{y} = 0$$

which proves our claim.

We discuss the case $H > 0$ further and show that it leads to the geodesic flow on the sphere S^{n-1}. For this purpose we consider the solutions on the surface

$$G_1 = |x|^2 = c_1 , \quad G_2 = \langle x,y \rangle = c_2 , \quad G_3 = |y|^2 = c_3 ,$$

where $c_1 c_3 - c_2^{\,2} = 2H > 0$.

If we make use of the group action ψ^g we can by (5.14) replace the symmetric matrix

$$S = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}$$

by

$$S \circ \psi^g = g^T S\, g .$$

This is the transformation law of quadratic forms and we can choose g so that

$$g\, S\, g^T = \begin{pmatrix} 1 & 0 \\ 0 & c_3 \end{pmatrix} \quad \text{where} \quad c_3 = 2H .$$

Thus we can restrict attention to the case

$$|x|^2 = c_1 = 1 , \quad \langle x,y \rangle = c_2 = 0$$

which we recognize as the tangent bundle TS^{n-1} of the sphere $|x| = 1$. The differential equation (5.15) become

$$\dot{x} = y$$
$$\dot{y} = -|y|^2 x$$

i.e. they define the geodesic flow on the sphere $|x| = 1$. In particular, for $H = \frac{1}{2}$ we obtain $|y| = 1$ and we get the geodesic flow on the unit tangent bundle:

$$\dot{x} = y \qquad (5.16)$$
$$\dot{y} = -x$$

with the flow

$$\phi^t(x,y) = \begin{pmatrix} x \cos t + y \sin t \\ -x \sin t + y \cos t \end{pmatrix} \qquad (5.17)$$

Finally we observe that the Hamiltonian (5.13) is also invariant under the group $G = SO(n,R)$ of n by n orthogonal matrices of determinant 1. If $R \in G$ the group action

$$(x,y) \to (Rx, Ry)$$

is clearly canonical and leaves (5.13) invariant. The corresponding Lie algebra consists of all antisymmetric matrices leading to the $n(n-1)/2$ integrals

$$G_{jk} = x_j y_k - x_k y_j \,, \qquad 1 \leq k < j \leq n \,,$$

which generalize the angular momentum vector to higher dimensions.

In the next section we will see that the three dimensional Kepler problem with the Hamiltonian

$$H = \frac{1}{2} |p|^2 - |q|^{-1} \,, \qquad p,q \in R^3$$

on a fixed negative energy surface does not only admit a canonical SO(3,R)-action which leads to the angular momentum conservation but an SO(4,R)-action which leads to 3 additional integrals. These group symmetries are rather unexpected at first, and it is ultimately responsible for the special feature of the inverse square force that all solutions of negative energy are closed.

(f) The Moment Map of a Canonical Action.

To a canonical action ψ: $G \times R^{2n} \to R^{2n}$ we have associated in section (c) a linear map from $\mathfrak{g}$, the Lie algebra of G, into the functions on R^{2n}, $A \mapsto G_A$. We therefore can define a mapping P:

$$P: R^{2n} \to \mathfrak{g}^*$$

from R^{2n} into $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$. ($\mathfrak{g}^*$ is precisely defined as the space of linear functionals on $\mathfrak{g}$, which is a Lie algebra again). This mapping P is defined as follows: for fixed $z \in R^{2n}$, the map $A \mapsto G_A(z) \in R$ is linear in A,

hence it is an element of $\mathfrak{g}^*$, which we call P(z), such that

(5.17) $$P(z)(A) = G_A(z) \,.$$

This map P is called the moment map of Souriau. If the Hamiltonian vector field X_H with flow ϕ^t is invariant under the action ψ, i.e. $H \circ \psi^g = H$, for $g \in G$, then the flow ϕ^t leaves P invariant, i.e.

(5.18) $$P \circ \phi^t = P \,.$$

We illustrate and motivate this general concept by example (b), where the action is given by

$$\psi(R,z) = (Rx, Ry) \,, \qquad R \in SO(3) \,.$$

We know

$$G_A(x,y) = \langle Ax, y \rangle \,, \qquad A \in \mathfrak{g}(SO(3)) \,.$$

Representing $A = \sum_{j=1}^{3} a_j A_j$ in the basis A_j, j = 1,2,3 of $\mathfrak{g}$(SO(3)) given there, we can write

$$G_A(x,y) = \langle x \wedge y, a \rangle \,,$$

$a = (a_1, a_2, a_3)$, hence the moment $P(x,y) = x \wedge y$ is simply the angular momentum, and (5.18) generalizes the fact, that the angular momentum vector is an integral. In example (e) of the SL(2,R) action, the moment is given by

$$G_A(x,y) = \alpha \langle x,y \rangle + \frac{\beta}{2} |y|^2 - \frac{\gamma}{2} |x|^2 \,,$$

where $A = \alpha A_1 + \beta A_2 + \gamma A_3$ is the representation of A in the given basis of the Lie Algebra of SL(2,R).

Ch. I, - § 4.

Exercise 1.

(a) Show that

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(\alpha_1^2 x_1^2 + \alpha_2^2 y_2^2)$$

with $\alpha_1 > 0$ and $\alpha_2 > 0$ has $G_j = y_j^2 + \alpha_j^2 x_j^2$, $j = 1,2$ and functions of them as integrals.

(b) Show that if α_2/α_1 is irrational, then there are no further integrals.

Hint: Introduce canonical polar coordinates:

$$\left(\sqrt{\alpha_j}\, x_j\, ,\ \frac{1}{\sqrt{\alpha_j}}\, y_j\right) = \left(\sqrt{2R_j}\,\sin\theta_j\, ,\ \sqrt{2R_j}\,\cos\theta_j\right)$$

and use that the orbits of $\dot\theta_j = \alpha_j$, $j = 1,2$, are dense on the torus T^2, if α_2/α_1 is irrational.

(c) If α_1/α_2 is rational, say p/q with p and q natural numbers, show there exist further integrals, e.g. the real and imaginary part of $z_1^q \bar z_2^p$ are polynomial integrals, where $z_j = y_j + i\alpha_j x_j$.

Exercise 2.

Show that if the map $A \mapsto G_A$ associated to the canonical action ψ satisfies

$$G_A \circ \psi^g = G_{\hat A}\, , \quad \hat A = g^{-1} A g\, ,$$

for every $g \in G$ and $A \in \mathfrak{g}$, then the action is a Poisson action, i.e. $\{G_A, G_B\} = G_{[A,B]}$. Does the converse also hold true?

Exercise 3.

Let θ be the one form on R^{2n}

$$\theta = \sum_{j=1}^{n} y_j\, dx_j\, ,$$

prove that every diffeomorphism ϕ satisfying $\phi^*\theta = \theta$ is of the form ϕ:

$$x_1 = f(x)$$
$$y_1 = (f_x(x)^T)^{-1} y$$

Hint: The 1-parameter group e^t: $(x,y) \mapsto (x, e^t y)$ satisfies

$$e^t \circ \phi = \phi \circ e^t\, ,$$

use this to prove that ϕ is of the form $\phi(x,y) = (f(x), g(x,y))$ and show then that $g(x,y) = \frac{\partial}{\partial y} g(x,0)y$, hence linear in y.

Exercise 4.

The Cayley-Klein metric on $\varepsilon|x| < 1$, $\varepsilon = \pm 1$, is defined by

$$\frac{d^2 s}{dt^2} = 2F(x,\dot x) = (1 - \varepsilon|x|^2)^{-2}\left\{(1 - \varepsilon|x|^2)|\dot x|^2 + \varepsilon\langle x,\dot x\rangle^2\right\}$$

where $\varepsilon = +1$ defines the hyperbolic, $\varepsilon = -1$ the elliptic geometry.

(a) Compute the Legendre transformation H of F. Result:

$$2H(x,y) = (1 - \varepsilon|x|^2)(|y|^2 - \varepsilon\langle x,y\rangle^2).$$

(b) Show that

$$G_{ij} = x_i y_j - x_j y_i\, , \qquad i < j$$
$$G_k = y_k - \varepsilon x_k\langle x,y\rangle, \qquad 1 \le k \le n$$

are integrals of H, and verify

$$\{G_j, G_k\} = -\varepsilon\, G_{jk}$$

We remark that for $\varepsilon = -1$ the Lie algebra generated by those integrals is that of SO(n+1), and for $\varepsilon = +1$ that of SO(n,1), the Lorentz group of index 1.

(c) Prove

$$2H = \sum_{k=1}^{n} G_k^2 - \varepsilon \sum_{i<j} G_{ij}^2 .$$

(d) Show that the flow of the Hamiltonian vector field X_{G_1} is given by

$$x_1(t) = \frac{\sinh t + x_1 \cosh t}{\cosh t + x_1 \sinh t}$$

$$x_j(t) = \frac{x_j}{\cosh t + x_1 \sinh t}, \quad 2 \le j$$

for $\varepsilon = +1$ and by

$$x_1(t) = \frac{\sin t + x_1 \cos t}{\cos t - x_1 \sin t}$$

$$x_j(t) = \frac{x_j}{\cos t - x_1 \sin t}, \quad 2 \le j$$

for $\varepsilon = -1$. Show that the flow of the vector field $\sum_{k=1}^{n} v_k X_{G_k}$ is given by

$$\frac{\{\sinh t + \langle x,v\rangle(\cosh t - 1)\}v + x}{\cosh t + \langle x,v\rangle \sinh t} \quad \text{if } |v| = 1, \quad \varepsilon = 1.$$

(e) Prove that the projections onto the x-space of the solutions of X_H are straight lines.

Hint: $\quad H_{y_k} = (1 - \varepsilon|x|^2)G_k.$

Exercise 5

(a) Prove that if $\varepsilon = +1$, the canonical transformation u:

$$x = \frac{2\xi}{1 + |\xi|^2},$$

$$y = \frac{1+|\xi|^2}{2}\eta + \frac{1+|\xi|^2}{1-|\xi|^2}\langle\xi,\eta\rangle\,\xi$$

takes $|\xi| < 1$ onto $|x| < 1$ and H into

$$K = H \circ u = \frac{1}{8}(1 - |\xi|^2)^2\, |\eta|^2 ;$$

and

$$G_{ij} \circ u = \xi_i\eta_j - \xi_j\eta_i$$

$$G_k \circ u = \frac{1+|\xi|^2}{2}\eta_k - \langle\xi,\eta\rangle\,\xi_k ,$$

where the latter is the analogue of the Runge-Lenz vector to be introduced in §6.

(b) Prove that the flow of X_K is the geodesic flow for

$$ds^2 = \frac{\langle d\xi, d\xi\rangle}{(1-|\xi|^2)^2}, \qquad |\xi| < 1 .$$

Remark: Exercises 4 and 5 are concerned with two different models of non-Euclidean Geometry, namely Exercise 4 with the projective model which is based on the projective metric introduced by Cayley in 1859. Later on F. Klein used this metric to construct a model for non-Euclidean geometry, $\varepsilon = 1$ corresponding to the hyperbolic, $\varepsilon = -1$ to the elliptic case. This metric is invariant under the group of projective transformations preserving the quadrics $\varepsilon \sum_{k=1}^{n} x_k^2 = 1$, (which for $\varepsilon = -1$ is complex!). The more familiar model based on the Poincaré metric of Exercise 5 is invariant under a group of conformal mappings. Both models are equivalent as Exercise 5 shows.

6. The SO(4) Symmetry of the Kepler Problem.

(a) The Kepler problem in R^3. This problem deals with the Hamiltonian

$$H = \frac{1}{2}|p|^2 - |q|^{-1} \tag{6.1}$$

where $p,q \in R^3$, $q \neq 0$ and it is well known that the solutions $q(t), p(t)$ when projected into the q-space are given by planar conic sections and circles when projected into the p-space. That the solution curves in the q-space are planar is true for any Hamiltonian of the form

$$H = \frac{1}{2}|p|^2 + U(|q|) \tag{6.2}$$

but that all orbits are closed if $H < 0$ is very special and remarkable property of the Kepler problem. It has other special features: From the discussion of Section 5 it is clear that the Kepler problem as well as any system (6.2) possesses the 3 components of the angular momentum

$$A = p \wedge q$$

as integrals, which follows from the SO(3) symmetry. However, it is surprising that the Kepler problem leaves another vector, the so-called Runge-Lenz vector

$$L = p \wedge A - \frac{q}{|q|}$$

invariant, so that we have altogether six integrals. They are not all independent functions and we have the relation

$$\langle L,A\rangle = \sum_{j=1}^{3} L_j A_j = 0$$

since L lies in the plane spanned by p and q while A is orthogonal to it. But L and A do define 5 independent integrals suggesting that a larger symmetry group leaves the Kepler problem invariant. To show this is the main aim of this section. We will restrict ourselves to orbits on a fixed energy surface

$$\{q,p \mid q \neq 0,\ H(q,p) = E\} \tag{6.3}$$

with negative energy, on which all orbits are ellipses. We will show that the flow on this energy surface — when appropriately compactified at the singularities $q = 0$ — admits the 6-dimensional rotation group SO(4). This unintuitive result will be derived by showing that the Kepler problem on the energy surface (6.3) for $E < 0$ is equivalent in the extended sense (Section 1) to the geodesic flow on the sphere

$$S^3 = \{\xi \in R^4,\ |\xi| = 1\},$$

which clearly admits the rotation group SO(4). This way we will be led to the Runge-Lenz vector following the ideas of Section 5.

The derivation presented here has other fringe benefits: It shows that the energy surface (6.3) when compactified is topologically equivalent to

$$T_1(S^3) = \{\xi,\eta \in R^4 \mid |\xi| = 1,\ \langle\xi,\eta\rangle = 0,\ |\eta| = 1\},$$

the unit tangent bundle of the sphere S^3.

Moreover, the equivalence will be established by an explicit mapping. The need for a compactification of the energy surface comes from the singularity at $q = 0$. Solutions approaching $q = 0$ are called collision orbits. They will be mapped into geodesics going through some point of S^3 and will be "regularized" that way, as we will explain below. Finally, since the geodesic flow on the sphere is given by linear equations, see (5.9), (5.10), we effectively transform the nonlinear Kepler problem on the negative energy surface into a linear system.

We will derive the proof for the above statements by beginning with the geodesic flow on the sphere S^n. We study the n-dimensional case since it is in no way more difficult than the 3-dimensional one. Using a canonical extension of the stereographic projection we will map the vector field to R^{2n} and after an appropriate change of the independent variable identify it with the Kepler problem.

This approach can be viewed as an illustration of the concept of equivalent vector fields, when the two vector fields, namely the Kepler problem and the geodesic flow, are considered in the large, in contrast to our previous considerations which were local.

(b) Stereographic Projection.

We begin with the Hamiltonian

(6.4) $\Phi(\xi,\eta) = \frac{1}{2}\{|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2\}$,

where $\xi = (\xi_0,\xi_1,\dots,\xi_n)$, $\eta = (\eta_0,\eta_1,\dots,\eta_n) \in R^{n+1}$.

We studied this system in Section 5 and saw that for $|\xi| = 1$, $\langle\xi,\eta\rangle = 0$ it defines the geodesic flow on the unit sphere S^n in R^{n+1}, all of whose orbits are periodic.

We use the stereographic projection to map the sphere punctured at the northpole into R^n and ask for the Hamiltonian system obtained that way. The usual stereographic map takes an (n+1)-vector ξ on $|\xi| = 1$ into a vector $x \in R^n$. In order to avoid this compression of dimension we use a "homogeneous" version of the stereographic mapping defined by

$$(6.5)\qquad \begin{cases} x_0 = |\xi| = a_0(\xi) \\ x_j = \dfrac{\xi_j}{|\xi|-\xi_0} = a_j(\xi) \end{cases}$$

which is defined for $|\xi| > \xi_0$. Denoting by L^+ the closed positive ξ_0 axis the equations (6.5) define an invertible mapping

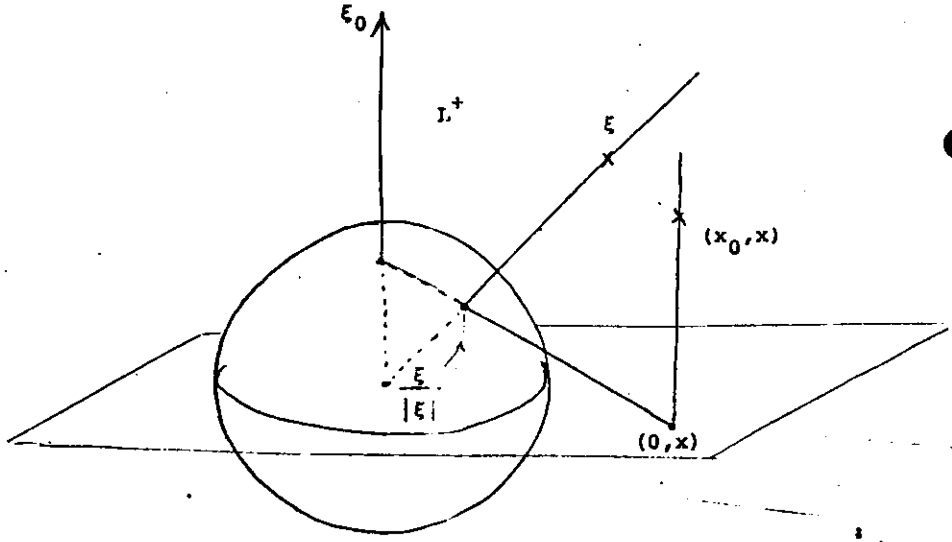
$$a:\ R^{n+1} \setminus L^+ \to R^{n+1}_+ = \{(x_0,x_1,\dots,x_n) \mid x_0 > 0\} ,$$

with the inverse

$$(6.6)\qquad \begin{cases} \xi_0 = x_0 \dfrac{|x|^2-1}{|x|^2+1} \\ \xi_j = x_0 \dfrac{2x_j}{|x|^2+1} \end{cases}$$

Here $x = (x_1,x_2,\dots,x_n) \in R^n$ is an n-vector, while $\xi \in R^{n+1}$. In order to avoid confusion we will write Greek letters, e.g., α,β, for subscripts running over $0,1,2,\dots,n$ and Latin letters, e.g., j,k for subscripts running over $(1,2,\dots,n)$. For example $\xi = (\xi_\alpha)$, $x = (x_j)$.

For $x_0 = |\xi| = 1$ the mapping (6.5) is the standard stereographic projection. For arbitrary $x_0 = |\xi|$ it takes rays through the origin $\xi = 0$ into vertical half-lines in R_+^{n+1} such that the heights x_0 agree with the distance $|\xi|$ on the ray. See the figure below.



We extend this mapping to a canonical mapping, using the results of Section 4. For example, we use a generating function

$$W(x,\eta) = x_0 \frac{|x|^2-1}{|x|^2+1} \eta_0 + \frac{2x_0}{|x|^2+1} \sum_{j=1}^{n} x_j \eta_j$$

and write the canonical transformation in the form

$$y_\alpha = W_{x_\alpha}, \qquad \xi_\alpha = W_{\eta_\alpha}.$$

The choice of W was made so that the second relation is compatible with (6.6) and the first gives

$$(6.7) \qquad \begin{cases} y_0 = \langle \xi, \eta \rangle |\xi|^{-1} \\ y_j = (|\xi| - \xi_0)\eta_j - (y_0 - \eta_0)\xi_j \end{cases}$$

where y_0 in the last equation is to be replaced by the first. The equations (6.5), (6.7) define the desired canonical mapping $u: (\xi,\eta) \mapsto (x_0, x, y_0, y)$ mapping the domain

$$(R^{n+1} \setminus L^+) \times R^{n+1} \to R_+^{n+1} \times R^{n+1}.$$

We note that

$$(6.8) \qquad |\xi|^2 = x_0^2, \qquad \langle \xi, \eta \rangle = x_0 y_0.$$

We restrict ξ, η to

$$\left\{ \xi \in R^{n+1} \setminus L^+,\ \eta \in R^{n+1} \mid |\xi| = 1,\ \langle \xi,\eta \rangle = 0 \right\} = T(\dot{S}^n)$$

which is the tangent bundle of the unit sphere punctured at the north pole $\xi^* = (1,0,\ldots,0)$ which we denote $\dot{S}^n$. By (6.8) it is clear that

$$(6.9) \qquad u: T(\dot{S}^n) \to R^n \times R^n$$

where the right-hand side is given by $x_0 = 1$, $y_0 = 0$. We ask for the image of the flow defined by Φ (see (6.4)) under this mapping. Since u is canonical we merely have to compute $F = \Phi \circ u^{-1}$.

For this purpose we calculate the rotation invariant quantities $|x|^2$, $\langle x,y\rangle$ and $|y|^2$. We find from (6.5) and (6.7)

$$|x|^2 = \frac{|\xi| + \xi_0}{|\xi| - \xi_0}$$

$$\langle x,y\rangle = x_0\eta_0 - y_0\xi_0 = |\xi|\eta_0 - \frac{\langle\xi,\eta\rangle}{|\xi|}\xi_0$$

$$|y|^2 = \left(\frac{|\xi| - \xi_0}{|\xi|}\right)^2 (|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2).$$

Since

$$|x|^2 + 1 = \frac{2|\xi|}{|\xi| - \xi_0}$$

we obtain from the equation for $|y|^2$

(6.10)
$$\begin{cases} \Phi(\xi,\eta) = \frac{1}{2}\ (|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2) \\ \qquad = \frac{1}{8}\ (|x|^2 + 1)^2|y|^2 = F(x,y)\ . \end{cases}$$

Thus the vector field X_Φ is mapped by u into X_F given by

(6.11)
$$\begin{cases} \frac{d}{d\tau} x_0 = \frac{d}{d\tau} y_0 = 0 \\ \frac{dx}{d\tau} = F_y\ , \quad \frac{dy}{d\tau} = -F_x\ , \end{cases}$$

where we use τ as independent variable. We see that x_0, y_0 are integrals which is also clear from the fact that $|\xi|^2$, $\langle\xi,\eta\rangle$ are integrals of X_Φ and from (6.8). In fact any Hamiltonian $\Gamma = \Gamma(\xi,\eta)$ with the integrals $|\xi|^2$, $\langle\xi,\eta\rangle$ is mapped into a function $\Gamma \circ u^{-1}$ which is independent of x_0, y_0; they are called ignorable variables.

Since x_0, y_0 do not enter into the above equation we can

restrict the system (6.11) to $x_0 = 1$, $y_0 = 0$ and have shown:

Proposition 6.1. The mapping (6.9) takes the geodesic flow given by

$$\frac{d}{d\tau}\xi = \eta\ , \qquad \frac{d}{d\tau}\eta = -|\eta|^2\xi$$

on $T(\dot{S}^n)$ into the system

$$\frac{dx}{d\tau} = F_y\ , \qquad \frac{dy}{d\tau} = -F_x$$

where

(6.12) $$F(x,y) = \frac{1}{8}\ (|x|^2 + 1)^2|y|^2\ ,$$

On $T_1(\dot{S}^n)$ we have

$$\Phi = \frac{1}{2}|\eta|^2$$

and it follows that u maps the unit tangent bundle $T_1(\dot{S}^n)$ into

(6.13) $$\left\{x,y \in R^{2n}\ \middle|\ F = \frac{1}{2}\right\},$$

which is an energy surface of $F(x,y)$.

(c) Change of Hamiltonian and of t-Variable.

The Hamiltonian function H is determined by the vector field X_H up to a constant, since X_H determines the gradient of H. If we restrict the vector field to the surface

$$H = 0 \quad \text{where} \quad dH \neq 0\ ,$$

then this restricted field allows much more freedom for the choice of H. As a matter of fact any function K which has the same zero set $K = 0$ and the same gradient on this set will determine the same vector field $X_K = X_H$ on $K = 0$. For example, if $\psi(\lambda)$ is any scalar C^2-function with

$$\psi(0) = 0, \quad \psi'(0) = 1,$$

then

$$K = \psi(H)$$

will determine the same vector field on $H = 0$. Indeed, for $H = 0$ one has $K = \psi(0) = 0$ and $dK = \psi'(0)\, dH = dH$ on $H = 0$.

We generalize this remark to a Hamiltonian vector field X_H with r integrals $G_1 = H, G_2, \ldots, G_r$. We restrict this vector field to the manifold defined by

$$G_1 = c_1, \quad G_2 = c_2, \ldots, \quad G_r = c_r \tag{6.14}$$

on which $dG_1, \ldots, dG_r$ are assumed to be linearly independent. If $\psi(\lambda_1, \ldots, \lambda_r)$ is a C^2-function with

$$\frac{\partial\psi(c)}{\partial\lambda_j} = \delta_{j1}, \quad c = (c_1, \ldots, c_r)$$

and

$$K = \psi(G_1, G_2, \ldots, G_r)$$

then

$$X_K = \sum_{j=1}^{r} \psi_{\lambda_j}(c)\, X_{G_j} = X_{G_1} = X_H$$

on (6.14).

A similar remark refers to the change of the independent variable. If we change the independent variable t to a new variable, say

$$s = \int_0^t \lambda\{\phi^\tau(x,y)\}\, d\tau \quad \text{or} \quad \frac{ds}{dt} = \lambda$$

then the new system

$$\frac{dx}{ds} = \lambda^{-1}(x,y)\, H_y(x,y); \qquad \frac{dy}{ds} = -\lambda^{-1}(x,y)\, H_x(x,y)$$

will not be in Hamiltonian form. But if we restrict ourselves to the energy surface

$$H = c \quad \text{where} \quad dH \neq 0$$

then the function

$$K = \lambda^{-1}(H - c)$$

serves as a Hamiltonian on $H = c$, since on $H = c$ we have

$$dK = \lambda^{-1}\, dH \quad \text{and} \quad X_K = \lambda^{-1} X_H .$$

Again, what is required is the knowledge of the gradient of the Hamiltonian on the energy surface which allows much freedom in modifying K. In other words, the vector fields X_H, X_K may be quite different from each other except on the energy surface.

(d) Transformation into the Kepler Problem.

We make use of these remarks to change the Hamiltonian F on the energy surface (6.13). First we choose the function

$$\psi(\lambda) = \sqrt{2\lambda}$$

which satisfies $\psi'(\tfrac{1}{2}) = 1$, $\psi(\tfrac{1}{2}) = 1$. Therefore we can replace the Hamiltonian F of (6.12), by

$$\psi(F) = \tfrac{1}{2}\left(|x|^2+1\right)|y|$$

when we restrict it to (6.13) or equivalently to $\psi(F) = 1$.

Next we introduce a new time variable t by

$$\frac{dt}{d\tau} = |y|$$

leading to the Hamiltonian

$$|y|^{-1}(\psi(F) - 1) = \frac{1}{2}(|x|^2+1) - |y|^{-1}$$

for $y \neq 0$. Finally if we introduce p,q by the symplectic transformation

$$q = -y, \quad p = x$$

we obtain the Hamiltonian

$$\frac{1}{2}(|p|^2+1) - |q|^{-1} = H(q,p) + \frac{1}{2}$$

of the Kepler problem (6.1) in R^n.

These formulae establish that the Kepler problem restricted to the energy surface (6.3) with $E = -\frac{1}{2}$ is equivalent in the extended sense (Section 1) to the linear system

(6.15) $\quad \xi' = \eta, \quad \eta' = -\xi \quad$ on $T_1(\dot{S}^n)$.

We can easily remove the restriction on E by rescaling the variables: With a positive constant ρ we set

$$q^* = \rho^2 q, \quad p^* = \rho^{-1} p, \quad t^* = \rho^3 t$$

and verify that this transformation takes the Kepler problem into itself and replaces E by

$$E^* = \rho^{-2} E .$$

Thus if $E < 0$ we can choose ρ so as to achieve $E^* = -\frac{1}{2}$.

Thus we have proven:

Theorem 6.1. The Kepler problem restricted to the energy surface (6.3) is equivalent in the extended sense to the geodesic flow on the punctured sphere $\dot{S}^n$ with unit velocity — which is given by (6.15).

We collect the formulae for the mapping

$$(\xi,\eta,\tau) \mapsto (q,p,t)$$

defining this equivalence mapping for $(\xi,\eta) \in T_1(\dot{S}^n)$:

$$(6.16) \qquad \begin{cases} p_j = \dfrac{\xi_j}{1-\xi_0} \\ -q_j = (1-\xi_0)\eta_j + \eta_0\xi_j \\ \dfrac{dt}{d\tau} = |q| = 1 - \xi_0 \end{cases}$$

(e) Geometrical Interpretation

The above theorem has the following immediate corollaries.

(i) The projection of the solutions $(q(t),p(t))$ of the Kepler problem on (6.3) into the p-space are circles.

Indeed by the first formula of (6.16) the point $\xi \in \dot{S}^n$ is related to $p = \dot{q}$ by the stereographic projection which takes circles into circles. Of course, the motion on this circle does not take place at uniform speed.

(ii) The energy surface (6.3) can be compactified so that it is topologically equivalent to $T_1 S^n$.

For this purpose we just have to observe that the mapping $(\xi,\eta) \mapsto (q,p)$ of $T_1(\dot{S}^n)$ onto the energy surface (6.3) is invertible; then the energy surface is compactified by compactifying $T_1(\dot{S}^n)$. This is achieved by restoring the north pole $\xi^* = (1,0,\ldots,0)'$ of S^n, i.e. by restoring the sphere of unit tangent vectors at ξ^*.

One can try to give a physical meaning to the ideal points of this compactification. The north pole ξ^* corresponds to the "point at infinity" in the p-space, or by the energy relation to $q = 0$. The unit tangent vectors at ξ^* can be interpreted as defining directions at $q = 0$. The corresponding solutions are degenerate ellipses lying on a line in the given direction. They are called collision orbits. In the picture of the motion on S^n they correspond simply to geodesics through ξ^*. One has to recall that the t-variable has been changed; for a collision orbit the velocity $\dot{q} = p$ tends to infinity as q approaches 0 while in the τ-variable it becomes finite.

(e) SO(n+1) Symmetry.

After the energy surface is compactified and identified with the unit tangent bundle

$$|\xi|^2 = 1 , \quad \langle\xi,\eta\rangle = 0 , \quad |\eta|^2 = 1 ,$$

and the flow is identified with the Φ-flow of (6.4) it is evident that it admits SO(n+1) symmetry. Since Φ is preserved under the symplectic group action

$$(\xi,\eta) \mapsto (R\xi,R\eta) ; \qquad R \in SO(n+1)$$

our system admits the integrals

$$\Gamma_{\alpha\beta} = \xi_\alpha\eta_\beta - \xi_\beta\eta_\alpha , \qquad 0 \le \alpha < \beta \le n ,$$

according to Section 4. From (6.5) and (6.7) we find

$$x_i y_j - x_j y_i = \xi_i\eta_j - \xi_j\eta_i = \Gamma_{ij} \quad \text{for} \quad 1 \le i < j \le n$$

which corresponds to the subgroup of rotations leaving the ξ_0-axis fixed, i.e. to the group SO(n). Therefore the interesting nonobvious integrals are

$$\Gamma_{0k} = \xi_0\eta_k - \xi_k\eta_0 \quad \text{for} \quad k = 1,2,\ldots,n .$$

One easily verifies that $\Gamma_{0k} \circ u^{-1} = G_{0k}$ is given by

$$G_{0k} = -\langle x,y\rangle x_k + \tfrac{1}{2}\,(|x|^2 - 1)y_k = -\langle p,q\rangle\, p_k + \tfrac{1}{2}\,(|p|^2-1)q_k$$

where we used the previous identities for $|x|^2$ and $\langle x,y\rangle$. These are therefore also integrals of the Kepler problem restricted to the fixed energy surface $H = -\frac{1}{2}$ but, of course, need not be preserved for other values of H. But one computes readily

$$X_H G_{0k} = -p_k(H + \tfrac{1}{2})$$

which confirms our statement. But it shows also that

$$\text{(6.17)} \qquad L_k = G_{0k} + q_k(H + \tfrac{1}{2}) = \langle p,q\rangle p_k + |p|^2 q_k - \frac{q_k}{|q|}$$

satisfy

$$X_H L_k = X_H G_{0k} + p_k(H + \tfrac{1}{2}) = 0 ,$$

hence are unrestricted integrals of the motion.

Thus we have found an n-vector $L = (L_k)$ which is conserved under the Kepler flow. This vector L is called the Runge-Lenz vector.

It is easy to give a geometrical interpretation of the Lenz-vector: L being a linear combination of p,q it lies in the orbit plane. At the perihelion, i.e. the point of minimal $|q|$ on the ellipse, one has $\langle p,q\rangle = \langle\dot{q},q\rangle = 0$ hence L lies on the line through the perihelion and the origin. Finally one computes

$$|L|^2 = 2H(|p|^2|q|^2 - \langle p,q\rangle^2) + 1 = \varepsilon^2$$

where ε turns out to be the eccentricity of the ellipse (see Exercise 3). Therefore, if $\varepsilon \neq 0$, we cay say, *the Runge-Lenz vector* (6.17) *points in the* (positive or negative) *direction of the perihelion and its length agrees with the eccentricity of the ellipse.*

The quantum-mechanical analogue of the Kepler problem, is the eigenvalue problem for the hydrogen atom,

$$(-\Delta - \frac{1}{|x|})\phi = \lambda\phi .$$

It was the discovery of V. Fock (1935) 1/ and V. Bargmann (1936) 2/

1/ "Zur Theorie des Wasserstoffatoms," Zeitschrift für Physik 98, 1935, pp. 145-154.

2/ " " 99, 1936, pp. 576-582.

that this problem can also be treated in a similar way by stereographic projection relating it to an eigenvalue problem on the sphere S^3. Therefore this problem also admits the SO(4) symmetry which accounts for the multiplicities of the negative eigenvalues — which are not explained by the obvious SO(3)-symmetry.

Strangely enough the classical Kepler problem was originally not put into relation to the stereographic projection. Runge 3/ found "his vector" in a simple vector analytical derivation of the solution of the Kepler problem. Later on W. Lenz 4/ used this vector in a provisional version of quantum theory. Therefore one often refers to it as the Runge-Lenz vector.

3/ C. Runge, Vektoranalysis, Hirzel, Leipzig, 1919, p. 70.

4/ Über den Bewegungsverlauf und die Quantenzustände der gestörten Keplerbewegung," Zeitschrift für Physik, 24 (1924), p. 197-207.

Exercise 1 Show for n=3 the Runge-Lenz vector can be written in the form

$$L = p \wedge A - \frac{q}{|q|}$$

where

$$A = p \wedge q$$

is the angular momentum vector.

Exercise 2 Show that the components L_k of the Runge-Lenz vector (6.17) and $G_{jk} = q_j p_k - q_k p_j$ satisfy

$$\{L_k, L_j\} = (p_k q_j - p_j q_k) 2H$$

$$\{G_{ij}, G_{k\ell}\} = \delta_{ik} G_{j\ell} - \delta_{i\ell} G_{jk} - \delta_{jk} G_{i\ell} + \delta_{j\ell} G_{ik}$$

$$\{L_i, G_{jk}\} = \delta_{ki} L_j - \delta_{ij} L_k .$$

Exercise 3 After a rotation in R^n every solution of (6.15) can be brought into the form

$$\xi_0 = -\sin\alpha\cos\tau , \qquad \eta_\alpha = \xi_\alpha' \qquad (' = \tfrac{d}{d\tau}) .$$

$$\xi_1 = -\sin\tau$$

$$\xi_2 = \cos\alpha\cos\tau$$

$$\xi_j = 0 \quad (j \geq 3)$$

where α is the inclination angle of the orbit plane against $\xi_0 = 0$.

Show that these solutions are mapped by (6.16) into

$$q_1 = \varepsilon + \cos\tau , \qquad p_1 = \frac{-\sin\tau}{1 + \varepsilon\cos\tau}$$

$$q_2 = \sqrt{1-\varepsilon^2}\sin\tau , \qquad p_2 = \frac{\sqrt{1-\varepsilon^2}\cos\tau}{1+\varepsilon\cos\tau}$$

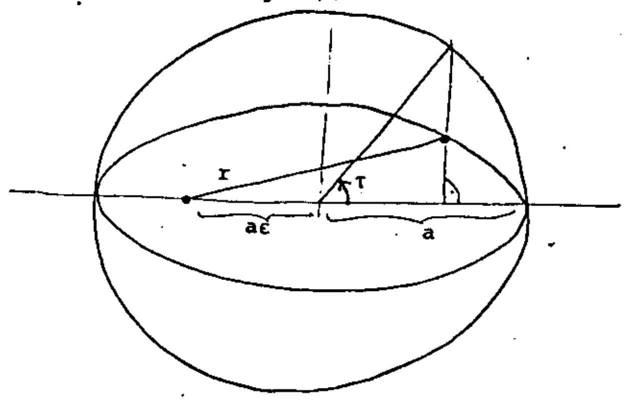
The q-equations describe an ellipse of eccentricity $\varepsilon = \sin\alpha$ and the p describe a circle about the center $p_1 = 0$, $p_2 = -\varepsilon(1-\varepsilon^2)^{-1/2}$. Finally,

$$t = \int_0^\tau (1 - \xi_0)\, d\tau = \tau + \varepsilon\sin\tau ,$$

corresponds to Kepler's equation.

Obtain the general solution of the Kepler problem of negative energy by rescaling and rotation.

In astronomy one uses three angles to describe the position of the mass point on an ellipse) which are called "anomalies". The *true anomaly* is the angle in the orbit plane measured from a fixed point (usually the perihelion). The *eccentric anomaly* is given by the angle τ in the above formula, or geometrically is given by the angle of the point on the circle (see figure).



Finally the *mean anomaly* is proportional to the time t elapsed from passage of the perihelion, normalized so that one period corresponds to the angle 2π.

Exercise 4 Show that (up to a constant) the angle on the geodesic on S^n is equal to the eccentric anomaly and the angle on the circle in the p-space is equal to the true anomaly. (By "angle" on the circle we mean the arc length divided by the radius of the circle.)

7. Symplectic Manifolds

(a) Manifolds

We have encountered various vector fields defined on a manifold, e.g. the tangent bundle $T(S^n) = M$ of the sphere as phase space. Also, a Hamiltonian vector field X_H is tangent to the level sets $M = \{x \in R^{2n} \mid H(x) = \text{const.}\}$ and hence defines a vector field on M. These subsets M of R^{2n} are examples of manifolds which are embedded in an Euclidean space. The aim of this section is to extend the concepts of the Hamiltonian formalism developed so far in R^{2n} to abstract manifolds which are not considered as subsets of Euclidean spaces.

It is not our aim to give a systematic derivation of differential calculus on manifold but rather introduce the relevant concepts and notations in a cursory manner, which should make it clear how to pass from the local considerations already studied to the global ones. We do not believe that it is opportune to use just one notation, e.g. the classical one using coordinates or the differential geometric one using differential calculus, since different mathematical problems require different tools and we would like to be free to use them if they help to a better understanding.

A systematic description of the calculus on manifolds can be found in many books on differential geometry (see e.g.

M. Berger and B. Gostiaux, "Geometrie differentielle" Armand Colin 1972, M. Spivak, "Differential Geometry.")

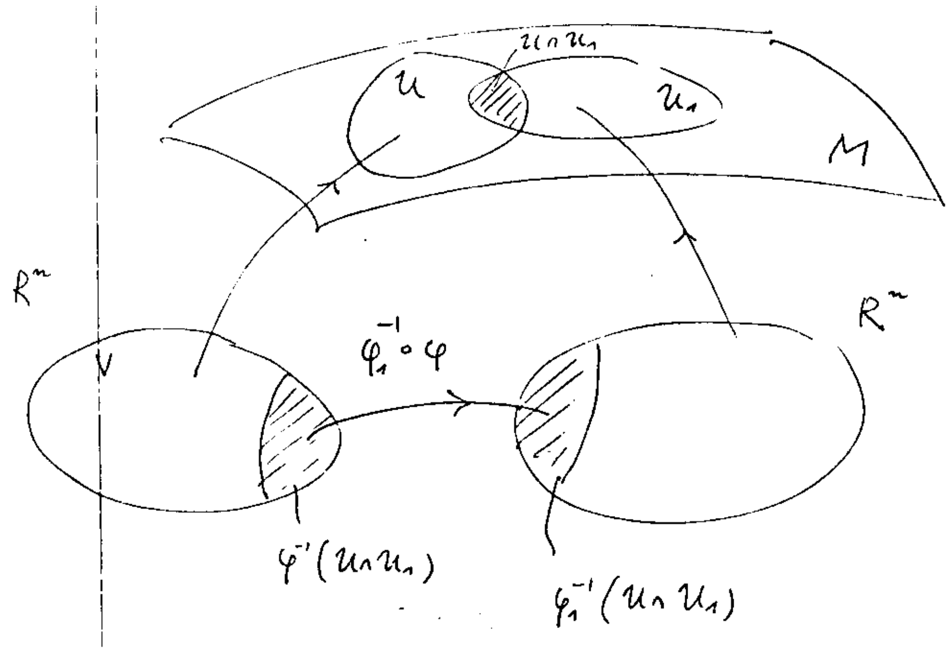
To set the notation we briefly recall the definition of a manifold. It is a topological Hausdorff space M which locally looks like an Euclidean space: every point p on M has an open neighborhood U which is described by local coordinates in R^n, i.e. there is a homeomorphism ϕ of an open set V in R^n onto U

$$\phi: V \subset R^n \to U \subset M .$$

Hence the points $p = \phi(x_1, \ldots, x_n)$ in U are in one-to-one correspondence to points $x = (x_1, \ldots, x_n)$ in the open set V of R^n. The pair (U,ϕ) is often called a chart and the points $x \in V$ the corresponding coordinates. Overlapping charts are patched together as follows: if (U_1,ϕ_1) is another chart at the point p such that $U_1 \cap U \neq \emptyset$, we require that the following map representing the coordinate transformation

$$(7.1) \quad \phi_1^{-1} \circ \phi: \phi^{-1}(U \cap U_1) \to \phi_1^{-1}(U \cap U_1)$$

is a diffeomorphism. In other words this map $y = u(x) = \phi_1^{-1} \circ \phi(x)$, which is a one-to-one map between open sets of R^n is required to be differentiable and also the inverse is required to be differentiable.



The space M together with a collection of charts covering M having this compatibility property is called a differentiable manifold of dimension, n, denoted by M^n. The compatibility property allows the extension of concepts of differentiable functions, vector fields and forms thus far considered in R^n to M.

A function $F: M \to R$ defined on M is called differentiable if for every point $p \in M$ there is a chart (U,ϕ) such that the local representation $F \circ \phi: x \mapsto F \circ \phi(x)$, which is a function in R^n, is differentiable. This then holds true for any other chart (U_1,ϕ_1) at p due to the compatibility condition (7.1) and the chain rule, since

$$F \circ \phi_1 = (F \circ \phi) \circ (\phi^{-1} \circ \phi_1) .$$

More generally, a map $F: M^n \to M^m$ between two manifolds is differentiable if for every point p there is a chart (U,ϕ) in M and a chart (U_1,ϕ_1) at f(p) with $F(U) \subset U_1$ such that in local representation

$$\phi_1^{-1} \circ F \circ \phi : \phi^{-1}(U) \subset R^n \to \phi_1^{-1}(U_1) \subset R^m ,$$

which is a map between Euclidean spaces, is differentiable.

In order to define the tangent space at a point p in M we consider a differentiable curve $c: R \to M$ through this point p with $c(0) = p$. In a chart (U,ϕ) at p, this curve is represented by the curve $\phi^{-1} \circ c: t \to \phi^{-1} \circ c(t) = x(t)$ in R^n which has a tangent vector at $x(0) = \phi^{-1}(p)$ given by

$$\frac{d}{dt} (\phi^{-1} \circ c)(0) = v \in R^n .$$

In other coordinates (U_1,ϕ_1) in which the curve is represented by $\phi_1^{-1} \circ c$ we find a different vector

$$\frac{d}{dt} (\phi_1^{-1} \circ c)(0) = v_1 \in R^n .$$

Since $\phi_1^{-1} \circ c = (\phi_1^{-1} \circ \phi) \circ (\phi^{-1} \circ c)$ these two vectors $v, v_1 \in R^n$ are related by the linear isomorphism

(7.2) $$v_1 = (\phi_1^{-1} \circ \phi)'(x)v , \quad x = \phi^{-1}(p) ,$$

where $(\phi_1^{-1} \circ \phi)'(x)$ is the derivative of the map $\phi_1^{-1} \circ \phi$ at x, which is a linear isomorphism of R^n. We view v and $v_1 \in R^n$ as coordinate representations of a tangent vector at p and

therefore define a tangent vector at $p \in M$ to be the equivalence class of all vectors $v \in R^n$ with the equivalence relation (7.2) under coordinate transformations. The collection of all tangent vectors is denoted by T_pM.

The derivative of a map between Euclidean spaces can now be extended to manifolds. If $f: M^n \to N^m$ is a differentiable map between two manifolds we define the tangent map at a point $p \in M$, denoted by

(7.3) $$df(p): T_pM \to T_{f(p)}N$$

as the linear map which in coordinates (U,ϕ) at p and (U_1,ϕ_1) at f(p) is represented by the linear map

$$v \to (\phi_1^{-1} \circ f \circ \phi)'(x)v , \quad x = \phi^{-1}(p)$$

from R^n to R^m, where $\phi_1^{-1} \circ f \circ \phi$ is the local representation of f.

A vector field X on a manifold associates to every point $p \in M$ a tangent vector $X(p) \in T_pM$. In local coordinates (U,ϕ) at p it is represented by a vector function $f(x) \in R^n$, $x = \phi^{-1}(p)$;

$$f: \phi^{-1}(U) \subset R^n \to R^n .$$

Similarly if (U_1,ϕ_1) are other coordinates at p, the vector X(p) is represented by $g(y) \in R^n$, $y = \phi_1^{-1}(p)$, and by (7.2) the vector functions f(x) and g(y) are related by

$$g(y) = (\phi_1^{-1} \circ \phi)'(x) \; f(x) , \quad y = \phi_1^{-1} \circ \phi(x) .$$

Hence with the notation $y = \iota(x) = \phi_1^{-1} \circ \phi(x)$ we have

$$g(y) = u' \cdot f \circ u^{-1}(y) \quad \text{and}$$
(7.4)
$$f(x) = u'^{-1} \cdot g \circ u(x) \quad ,$$

which is precisely the familiar transformation law of vector fields in R^n encountered in Section 1. Conversely, if there are vector fields given in all local coordinates, they define a vector field on M if they satisfy the compatibility conditions (7.4) for the corresponding transition maps.

Alternatively we can define a vector field X as a directional derivative, in local coordinates represented by

$$\sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j} \quad ,$$

where $f = (f_1, \ldots, f_n)$ is the corresponding vector function from above. The relation to the previous definition is as follows. We consider in local coordinates any curve $t \to \phi^{-1} \circ c(t) = x(t)$ through $x(0) = x$ and having there the tangent vector

$$\frac{d}{dt} (\phi^{-1} \circ c)(0) = f(x) \in R^n \ .$$

If then F is any function in R^n we have

$$\frac{d}{dt} F(\phi^{-1} \circ c(t))\Big|_{t=0} = \sum_{j=1}^{n} f_j(x) \frac{\partial}{\partial x_j} F(x) \quad ,$$

i.e. the derivation of F in the direction of the vector field f at x.

If one has a diffeomorphism $F: M \to M_1$ between two manifolds of the same dimension, it gives rise to a transformation of vector fields which however goes backwards from TM_1 to TM and hence is called a "pull back". It maps a vector field Y on M_1 into the vector field $X = F^*(Y)$ on M defined by

(7.5)
$$F^*(Y)(p) = d(F^{-1})Y \circ F(p) \ .$$

In coordinates (U,ϕ) at p and (U_1, ϕ_1) at F(p) this map between vector fields is represented by

$$f(x) = u'^{-1} \cdot g \circ u(x) \ ,$$

$u(x) = \phi_1^{-1} \circ F \circ \phi(x)$ being the representation of F and g being the representation of the vector field Y. That is again the transformation law of vector fields.

Finally differential forms can be defined on a manifold M. A k-form α associates to every point $p \in M$ a skew symmetric k-multilinear form α_p on the tangent space T_pM,

$$\alpha_p(X_1, \ldots, X_k) \ , \quad X_j \in T_pM \ .$$

In local coordinates $x_1, \ldots, x_n$ of M it is represented as

$$\alpha_x = \sum a_{i_1 \ldots i_k}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \ ,$$

where the form $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ on R^n is defined by

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_1,\dots,v_k) = \det \begin{pmatrix} dx_{i_1}(v_1) & \cdots & dx_{i_k}(v_1) \\ \\ dx_{i_1}(v_k) & \cdots & dx_{i_k}(v_k) \end{pmatrix}$$

here $dx_i(v_\alpha)$ stands for the i-th component of the vector $v_\alpha \in R^n$.

The exterior derivative d of forms is locally defined by

$$(d\alpha)_x = \sum \left(\sum_{s=1}^{n} \frac{\partial a_{i_1 \dots i_k}}{\partial x_s}(x)\, dx_s \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and it is invariant under coordinate transformations, that is $d(u^*\alpha) = u^*(d\alpha)$ holds for any diffeomorphism of R^n. Therefore d can be viewed as an operation on forms on M, which maps k-forms into (k+1)-forms.

A map $f: M \to M_1$ between manifolds gives rise to a "pull back" map f^*, mapping forms on M_1 backwards into forms on M. It is defined as follows: if α is a k-form on M_1, then $f^*\alpha$ is defined as the following k-form on M:

(7.6) $\quad (f^*\alpha)_p(X_1,\dots,X_k) = \alpha_{f(p)}\big(df(X_1),\dots,df(X_n)\big),$

for every $p \in M$ and $X_j \in T_pM$. This is well defined since by definition of the tangent map, $df(X_j) \in T_{f(p)}M_1$. We emphasize that in contrast to vector fields, the pull back of forms is defined for any map between manifolds, not only for those that are invertible.

(b) Lie Derivatives and Contractions.

If X_t is a time dependent vector field on a manifold M it has a flow ϕ^t defined by

$$\frac{d}{dt}\phi^t = X_t \circ \phi^t, \quad \phi^t = id;$$

conversely every one-parameter family of diffeomorphisms ϕ^t such that $\phi^0 = id$ defines a time dependent vector field X_t having ϕ^t as its flow by

$$\frac{d}{dt}\phi^t \circ (\phi^t)^{-1} = X_t \ .$$

A vector field X_t having the flow ϕ^t gives rise to an operation on forms which is called the Lie derivative with respect to X_t, denoted by L_{X_t}. If α and hence $(\phi^t)^*\alpha$ are k-forms on M, then the k-form $L_{X_t}\alpha$ is defined by

$$L_{X_0}\alpha = \lim_{t \to 0} \frac{1}{t}\big((\phi^t)^*\alpha - \alpha\big) = \frac{d}{dt}(\phi^t)^*\alpha\Big|_{t=0}$$

for t = 0, and we conclude

$$\frac{d}{dt}(\phi^t)^*\alpha = \frac{d}{dh}(\phi^{t+h})^*\alpha\Big|_{h=0} = (\phi^t)^* L_{X_t}\alpha \ .$$

For example, if f is a function viewed as a 0-form, then

(7.7) $\quad L_{X_0}f = \frac{d}{dt} f(\phi^t)\Big|_{t=0} = X_0(f)$

is the derivative of f in the direction of the vector X_0.

Similarly one can define the Lie derivative on vector-fields Y by

$$L_{X_0}(Y) = \frac{d}{dt}(\phi^t)^* Y\Big|_{t=0} ,$$

and one shows (Exercise 3 in Section 1) that

$$L_{X_0}(Y) = [X_0, Y] , \tag{7.8}$$

i.e. the commutator of X_0 and Y.

A vector field X also gives rise to another action on forms, the so called contraction (or inner product) denoted by i_X which maps a (k+1)-form α into a k-form $i_X\alpha$ and which is defined by

$$i_X\alpha(X_1,\dots,X_k) = \alpha(X,X_1,\dots,X_k) . \tag{7.9}$$

For example, if ω is a 2-form, then $i_X\omega$ is the 1-form $i_X\omega(\cdot) = \omega(X,\cdot)$, where the dot is to be replaced by an arbitrary vector field.

In order to evaluate Lie derivatives, the following important identity due to E. Cartan is very useful:

$$L_X\alpha = i_X(d\alpha) + d(i_X\alpha)$$

for any form α, i.e.

$$L_X = i_X \circ d + d \circ i_X . \tag{7.10}$$

There are other rules relating the operations contraction and Lie derivative, e.g.:

$$i_{[X,Y]} = [L_X, i_Y] . \tag{7.11}$$

Also the exterior derivative d can be expressed by the Lie derivative, e.g. if α is a 1-form, then

$$d\alpha(X,Y) = L_X\alpha(Y) - L_Y\alpha(X) + \alpha[L_X(Y)]$$

which in view of (7.7) and (7.8) is equal to

$$X\,\alpha(Y) - Y\,\alpha(X) + \alpha([X,Y]) .$$

Generally, if α is a k-form, then

$$d\alpha(X_0,X_1,\dots,X_k)$$
$$= \sum_{i=0}^{k} (-1)^i L_{X_i}\left(\alpha(X_0,\dots,\hat{X}_i,\dots,X_k)\right)$$
$$+ \sum_{0\le i<j\le k} (-1)^{i+j} \alpha\left(L_{X_i}(X_j),X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k\right) ,$$

where $\hat{X}_i$ denotes that X_i is deleted.

We shall use this formula for 2-forms ω later on, for which we find

$$d\omega(X_1,X_2,X_3) = X_1\omega(X_2,X_3) + \text{cyclic permutation} - \{\omega([X_1,X_2],X_3) + \text{cyclic permutation}\}. \tag{7.12}$$

For proofs we refer to the above cited books.

(c) Symplectic structures on manifolds.

In order to introduce the Hamiltonian formalism on a manifold, we first have to extend the symplectic structure

$$\omega = \sum_{j=1}^{n} dy_j \wedge dx_j \quad \text{on} \quad R^{2n} \tag{7.13}$$

introduced in Section 3 to even dimensional manifolds M^{2n}.

Definition. A symplectic structure on an even dimensional manifold $M = M^{2n}$ is a two form ω on M with the following two properties:

(i) ω is closed, i.e. $d\omega = 0$;

(ii) ω is nowhere degenerate, i.e. for every $X \in T_xM$, $x \in M$ there is a vector $Y \in T_xM$ with $\omega_x(X,Y) \neq 0$.

The pair (M,ω) is then called a symplectic manifold.

Thus every tangent space T_xM , which is a 2n-dimensional vector space becomes a symplectic vector space with respect to the antisymmetric nondegenerate bilinear form ω_x at x, and it is clear that M must have even dimension.

The basic examples of a symplectic manifold is the cotangent-bundle of any manifold N which we discuss now.

If N is a manifold of dimension n, and if T_pN is the tangent space of $p \in N$, we denote its dual space, the so called cotangent space, by T_p^*N ; it is the space of linear forms defined on the vector space T_pN. The union of all cotangent

spaces is called the cotangent bundle of N denoted by T^*N,

$$T^*N = \bigcup_{p \in N} T_p^*N .$$

In other words, a point α in the set $M = T^*N$ is a linear form α_p in the tangent space T_pN at some point $p \in N$, and we shall sometimes use the notation

$$\alpha = (p,\alpha_p)$$

for a point α in M. One can view T^*N as a differentiable manifold of dimension 2n by introducing local coordinates e.g. as follows. If $x_1,\dots,x_n$ are local coordinates in N, a one form $\alpha \in T_p^*N$ is represented by $\alpha = \sum_{i=1}^{n} y_j \, dx_j$ having the coordinates $y_1,\dots,y_n$, and together $(x_1,\dots,x_n,y_1,\dots,y_n)$ form local coordinates for T^*N.

In these local coordinates one can define a 1-form θ on T^*N by

$$\theta = \sum_{j=1}^{n} y_j \, dx_j \quad , \tag{7.13}$$

which has so far only a local meaning. It is remarkable and crucial for the following, that the above form θ has a general meaning which we explain briefly. Since a point of T^*N is represented by a 1-form α at a point $p \in N$, we can form $\alpha(V)$ for any vector $V \in T_pN$. To define a one form on T^*N, say β, one has to give its value $\beta(X)$ for any tangent vector X of T^*N. If X is such a tangent vector at a point $\alpha \in T^*N$ one can use the form α to define

$\alpha(d\pi X)$

as a linear functional. Here the map

$$\pi: T^*N \to N$$

is the projection map which maps the points $\alpha = (p,\alpha_p)$ into their base points $p \in N$, and therefore $d\pi$ maps the tangent space $T_\alpha(T^*N)$ at α into the tangent space T_pN at p. This form which is sometimes called the tautological form since it is defined in terms of itself, defines a 1-form θ on T^*N by

(7.14) $$\theta_\alpha(X) = \alpha(d\pi\, X), \quad X \in T_\alpha(T^*N) .$$

It is readily verified, that in the above local coordinates this form θ agrees with the form (7.13). In fact, if $X = \sum_j \left(a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j}\right)$, then

$$\pi(X) = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} ,$$

and with $\alpha = \sum_{j=1}^{n} y_j\, dx_j$ we have

$$\alpha(d\pi\, X) = \sum_{j=1}^{n} a_j y_j$$

which agrees with $\theta(X)$ if $\theta = \sum_{j=1}^{n} y_j\, dx_j$. From the local expression (7.13) we find $d\theta = \sum_{j=1}^{n} dy_j \wedge dx_j$, and we conclude that the form $\omega = d\theta$ is closed and nondegenerate and hence defines a symplectic two-form on T^*N.

One may ask whether the tangent bundle also has a symplectic structure. The answer is, of course, yes since

the tangent bundle can be mapped diffeomorphically onto the cotangent bundle, and the symplectic form ω on T^*N is mapped by the pull back into a symplectic form on TN. This 2-form, however, depends on the choice of the mapping of TN onto T^*N. Such a mapping can, for example, be constructed with the help of a metric on N which can always be found. A metric defines an inner product $\langle\ ,\ \rangle_p$ in T_pN and therefore an isomorphism $X \to \langle X,\cdot\rangle_p$ of the tangent space T_pN onto the cotangent space T_p^*N. This map gives a diffeomorphism of TN onto T^*N uniquely associated to the metric.

Cotangent bundles are examples of so called <u>exact</u> symplectic manifolds (M,ω) where the 2-form is not only closed, i.e. $d\omega = 0$ but exact, i.e. $\omega = d\theta$ for a 1-form θ on M.

We have already remarked that every manifold M can carry a Riemannian structure. In contrast not every even dimensional manifold allows a symplectic structure. For instance spheres S^{2n} do not admit a symplectic structure if $n \geq 2$. In fact assume ω is a symplectic structure, then $\Omega = \omega \wedge \ldots \wedge \omega$ (n times) is a volume form, since ω is nondegenerate. But $\omega = d\alpha$ for a 1-form α, since the second de Rham group vanishes, $H^2(S^{2n}) = 0$ for $n \geq 2$. Therefore $\Omega = d\beta$ with $\beta = \omega \wedge \ldots \wedge \omega \wedge \alpha$ and hence by Stokes' theorem,

$$\int_{S^{2n}} \Omega = \int_{\partial S^{2n}} \beta = 0 ,$$

which is, of course, not possible for a volume form. This argument evidently applies to any compact manifold M without boundary having $H^2(M) = 0$.

(d) A theorem by Darboux.

We shall prove that there are always local coordinates, in which the symplectic form ω is represented by the constant 2-form (7.13).

Theorem 7.1. Suppose ω is a nondegenerate 2-form on a manifold M of dimension 2n. Then $d\omega = 0$ if and only if at each point $p \in M$ there are coordinates (U,ϕ), $\phi: (x_1,\dots,x_n,y_1,\dots,y_n) \to q \in U$ such that $\phi(0) = p$ and

$$\phi^*\omega = \sum_{j=1}^{n} dy_j \wedge dx_j \; .$$

Remark. Such coordinates are sometimes called canonical (or symplectic) coordinates. Clearly they are not uniquely determined; the most general coordinates of this sort are related to x,y by a canonical transformation as it was discussed in Section 4.

Proof: We may assume that ω is a 2-form on $M = R^{2n}$ and that p corresponds to $x = 0$. By a linear change of coordinates we can achieve that this 2-form is in normal form at the origin, i.e.

$$\omega = \sum_{j=1}^{n} dy_j \wedge dx_j \quad \text{at} \quad x = 0 \; .$$

This is precisely the same as the statement that any nondegenerate skew symmetric bilinear form can be brought into normal form (see Section 3). With ω_0 we shall denote the constant 2-form $\sum dy_j \wedge dx_j$ on R^{2n}. The aim is to find a local diffeomorphism ϕ in a neighborhood of 0 leaving the origin fixed such that

$$\phi^*\omega = \omega_0 \; .$$

We shall solve this equation using a deformation argument. We interpolate ω and ω_0 by a family ω_t of forms defined by

$$(7.14) \qquad \omega_t = \omega_0 + t(\omega - \omega_0) \; , \qquad 0 \le t \le 1 \; ,$$

such that $\omega_t = \omega_0$ for $t = 0$ and $\omega_1 = \omega$, and look for a differentiable family ϕ^t of diffeomorphisms satisfying $\phi^0 = id$ and

$$(7.15) \qquad (\phi^t)^*\omega_t = \omega_0 \; , \qquad 0 \le t \le 1 \; .$$

The diffeomorphism ϕ^t for $t = 1$ will then be the solution to our problem. In order to find ϕ^t we shall construct a t-dependent vector field X_t generating ϕ^t. Differentiating (7.15) such a vector field X_t has to satisfy the following identity,

$$0 = \frac{d}{dt}(\phi^t)^*\omega_t = (\phi^t)^*\left\{L_{X_t}\omega_t + \frac{d}{dt}\omega_t\right\} \; .$$

Using Cartan's identity (7.10) now and $d\omega_t = 0$ we find by (7.14)

$$0 = (\phi^t)^* \left\{ d(i_{X_t}\omega_t) + \omega - \omega_0 \right\},$$

hence X_t has to be a solution of the linear equation

$$d(i_{X_t}\omega_t) + \omega - \omega_0 = 0 . \tag{7.16}$$

In order to solve this equation we observe that $\omega - \omega_0$ is closed hence locally exact by the Poincaré Lemma, and there is a 1-form λ such that

$$\omega - \omega_0 = d\lambda \quad \text{and} \quad \lambda(0) = 0 .$$

Since $\omega_t(0) = \omega_0$, the 2-forms ω_t, $0 \leq t \leq 1$, are nondegenerate in an open neighborhood of the origin and hence there is a unique vector field X_t determined by

$$i_{X_t}\omega_t = \omega_t(X_t,\cdot) = -\lambda , \tag{7.17}$$

$0 \leq t \leq 1$, which then solves the equation (7.16). One has to keep in mind that X_t is not uniquely determined by (7.16), but the choice of λ makes X_t uniquely determined by (7.17). Since we normalized $\lambda(0) = 0$ we have $X_t(0) = 0$ and there is an open neighborhood of the origin on which the flow ϕ^t of X_t exists for all t in $0 \leq t \leq 1$. It satisfies $\phi^0 = id$ and $\phi^t(0) = 0$. We can follow our arguments backward: by construction this family ϕ^t of diffeomorphisms satisfies

$$\frac{d}{dt}\left((\phi^t)^*\omega_t\right) = 0 , \qquad 0 \leq t \leq 1 ,$$

hence $(\phi^t)^*\omega_t = (\phi^0)^*\omega_0 = \omega_0$ for $0 \leq t \leq 1$ as we wanted to prove.

From Darboux's theorem we conclude that any two symplectic manifolds having the same dimension are locally indistinguishable; symplectic manifolds do not possess any <u>local</u> symplectic invariants other than the dimension. This is in sharp contrast to Riemannian manifolds: two different metrics generally are not locally isometric, e.g. the Gaussian curvature is an invariant.

(e) Symplectic maps

Next we introduce the general analogue of canonical maps considered in Section 3. A differentiable map

$$f: M_1 \to M_2$$

between two symplectic manifolds (M_1,ω_1) and (M_2,ω_2) is called symplectic (or canonical) if the pull back maps ω_2 into ω_1, i.e. if

$$f^*\omega_2 = \omega_1 . \tag{7.18}$$

In other words $\omega_1(X,Y) = \omega_2(df(X), df(Y))$, $X,Y \in T_pM_1$. As ω_1 is nondegenerate, the tangent map df must be injective at every point and hence $\dim M_2 \geq \dim M_1$, hence a canonical map is an immersion. If $\dim M_1 = \dim M_2$ then f is a local diffeomorphism.

In the special case that f maps a symplectic manifold (M,ω) into itself the condition (7.18) for f to be canonical becomes

(7.19) $$f^*\omega = \omega$$

i.e. f preserves the differential form ω.

We can express the condition $f^*\omega = \omega$ in <u>symplectic</u> coordinates at f and at f(p). It is simply

$$f^* \sum_{j=1}^{n} d\eta_j \wedge d\xi_j = \sum_{j=1}^{n} dy_j \wedge dx_j ,$$

which is in agreement with the definition of canonical maps in R^{2n} given in Section 3.

To illustrate this concept we consider a map

$$f: \left(R^n \times R^{n^*}, d\theta\right) \to \left(R^{n+1} \times R^{n+1^*}, d\theta_1\right)$$

with $\theta = \sum_{k=1}^{n} y_k \, dx_k$ and $\theta_1 = \sum_{\nu=0}^{n} y_\nu \, dx_\nu$. It is defined by $f(x,y) = (\xi,\eta)$:

(7.20) $$f: \quad \xi_0 = \frac{|x|^2 - 1}{|x|^2 + 1}, \qquad \eta_0 = \langle x,y \rangle$$
$$\xi_k = \frac{2x_k}{|x|^2 + 1}, \qquad \eta_k = \frac{|x|^2 + 1}{2} y_k - \langle x,y \rangle x_k ,$$

where $k = 1,2,\dots,n$. One readily verifies that $\langle \eta, d\xi \rangle = \langle y, dx \rangle$, i.e.

$$f^*\theta_1 = \theta$$

and therefore $f^* d\theta_1 = d\theta$ i.e. f is a symplectic map.

The image points of f, i.e. $(\xi,\eta) = f(x,y)$ satisfy

$$|\xi|^2 = 1 \quad \text{and} \quad \langle \xi,\eta \rangle = 0 ,$$

and one verifies easily that f maps the cotangent bundle $R^n \times R^{n^*}$ one to one onto $T^*\dot{S}^n$, the cotangent bundle of the n-dimensional unit sphere $|\xi|^2 = 1$ in R^{n+1} punctured at the north pole $\xi^* = (1,0,\dots,0)$. Moreover f maps fibers linearly into fibers.

We describe another example. As in the case $R^n \times R^n$ of Section 4, every diffeomorphism $\phi: N \to N$ of an n-dimensional manifold N can be extended to a symplectic diffeomorphism $f: T^*N \to T^*N$ of the cotangent bundle of N, which is a symplectic manifold with respect to the symplectic structure $\omega = d\theta$. This diffeomorphism f is defined by the formula

(7.21) $$f(\alpha) = \left(\phi(p), (d\phi^{-1})^* \alpha_p\right) \in T^*N$$

if $\alpha = (p,\alpha_p)$ is an element of T^*N. This mapping maps fibers onto fibers and one verifies readily that

$$f^*\theta = \theta ,$$

hence $f^* d\theta = d\theta$ and f is indeed symplectic. Conversely it can be shown that every diffeomorphism on T^*N leaving the one form θ invariant is given by the formula (7.21) for some diffeomorphism ϕ of N.

If (M,ω), $\omega = d\alpha$ is an exact symplectic manifold we call a map f exact symplectic if

$$f^*\alpha - \alpha \text{ is exact, i.e. } = dF,$$

where F is a function on M. Of course, every exact symplectic mapping is also symplectic since $f^*\alpha = \alpha + dF$ implies

$$f^*\omega = f^* d\alpha = d(f^*\alpha) = d\alpha = \omega.$$

The converse, however is not true in general. Indeed from $f^*\omega = \omega$, we conclude that $d(f^*\alpha - \alpha) = 0$, i.e. $f^*\alpha - \alpha$ is a closed one form, which need not be exact. But for simply connected, exact symplectic manifolds the two concepts coincide. We illustrate the difference of the two concepts in the plane, which is an exact symplectic manifold with $\alpha = y\,dx$ and the corresponding two form $\omega = d\alpha = dy \wedge dx$ is the area element. Any mapping whose Jacobian is identically =1 is symplectic. But in the non-simply-connected domain $R^2 \setminus \{0\}$ the mapping $f: (x,y) \to (x_1,y_1)$ given by

$$x_1 = x\left(1 + \frac{\varepsilon^2}{x^2+y^2}\right)^{1/2}, \quad y_1 = y\left(1 + \frac{\varepsilon^2}{x^2+y^2}\right)^{1/2}$$

is symplectic, but not exact symplectic for any $\varepsilon \neq 0$, as one easily verifies. In geometrical terms an exact symplectic mapping in the plane does not only preserve the area element ω but also the line integral $\int_C \alpha$ over any closed curve.

Similarly in higher dimensions: A symplectic diffeomorphism f on (M,ω) preserves the area

$$\int_{f(\sigma)} \omega = \int_\sigma f^*\omega = \int_\sigma \omega$$

for every compact two dimensional surface σ. If $\omega = d\alpha$ and if f is exact symplectic then we have, in addition, for every closed curve C:

$$\int_{f(C)} \alpha = \int_C f^*\alpha = \int_C \alpha.$$

8. Hamiltonian vector fields on symplectic manifolds.

(a) Hamiltonian vector fields.

We have seen that a Riemannian structure defines an isomorphism between tangent vectors and cotangent vectors and therefore gives rise to an isomorphism between vector fields and 1-forms. Similarly if (M,ω) is a symplectic manifold, the symplectic structure ω being nondegenerate defines an isomorphism

$$\alpha \to X_\alpha$$

between the 1-forms α and the vector fields X by the following formula:

$$\alpha = \omega(X_\alpha, \cdot) = i_{X_\alpha}\omega . \tag{8.1}$$

We therefore have two distinguished subspaces in the space of vector fields corresponding to the closed and to the exact 1-forms under this isomorphism, which leads to the following definition:

<u>Definition</u>. A vector field $X = X_\alpha$ is called Hamiltonian if the form

$$\alpha = \omega(X, \cdot)$$

is closed, i.e., $d\alpha = 0$. Since $d\omega = 0$, we obtain by formula (7.10) $d\alpha = L_X\omega$ and hence this is equivalent to $L_X\omega = 0$. The vector field is called exact Hamiltonian if the 1-form α is not only a closed but an exact 1-form, i.e. $\alpha = dH$ for a function H on M.

Clearly every exact Hamiltonian vector field is a Hamiltonian vector field and in view of Poincaré's lemma the converse also holds true locally, but in general not globally, unless the manifold is simply connected. For example, on the symplectic manifold $T^n \times R^n$, $T^n = R^n/Z^n$ with the symplectic form $\omega = \sum dy_j \wedge dx_j$ (x mod 1), the form

$$\alpha = \sum_{k=1}^{n} a_k\, dx_k ,$$

$a = (a_1,\ldots,a_n) \in R^n$, is closed but not exact if $a \neq 0$. Only in the covering space $R^n \times R^n$ we have $\alpha = dH$ with $H = \langle a,x\rangle$. The corresponding Hamiltonian vectorfield is given by

$$\dot{x} = 0 , \quad \dot{y} = a ,$$

and it is not an exact Hamiltonian vectorfield.

In order to relate the Hamiltonian vectorfields defined here to the Hamiltonian equations on R^{2n} introduced in section 2, we describe the definition in symplectic coordinates, where $\omega = \sum dy_j \wedge dx_j$. If X_α is the vectorfield

$$X_\alpha = \sum_{j=1} \left(a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j}\right)$$

and α the corresponding 1-form, locally described by

$$\alpha = -dH = -\sum_{j=1}^{n} (H_{x_j}\, dx_j + H_{y_j}\, dy_j) ,$$

then

$$i_X\omega = \sum_{j=1}^{n} ((i_X dy_j)\, dx_j - (i_X dx_j)\, dy_j) = \sum_{j=1}^{n} (b_j dx_j - a_j dy_j) .$$

Hence, since $i_X\omega = \alpha$, we conclude $a_j = H_{y_j}$ and $b_j = H_{x_j}$ and find

$$X_\alpha = \sum_{j=1}^{n} \left(H_{y_j} \frac{\partial}{\partial x_j} - H_{x_j} \frac{\partial}{\partial y_j}\right)$$

which corresponds to our old definition of Hamiltonian vectorfields in R^{2n}. We emphasize, that this representation holds in symplectic coordinates only where the symplectic form ω is given by (7.13), in other coordinates the vectorfield can look completely differently, as we will see in examples of Chapter III.

We remark that the space of Hamiltonian vectorfields is a subalgebra of the Lie Algebra of all vectorfields on the manifold. In fact the commutator of two Hamiltonian vectorfields X_α and X_β is not only a Hamiltonian but an exact Hamiltonian vectorfield. More precisely we shall prove that

(8.2) $\qquad i_{[X_\alpha, X_\beta]}\,\omega = df \; , \; f = \omega(X_\beta, X_\alpha)$

if the 1-forms α and β are closed, i.e. $d\alpha = 0$ and $d\beta = 0$.

In fact using the formula (7.11) we have

$$i_{[X_\alpha , X_\beta]}\,\omega = L_{X_\alpha} (i_{X_\beta}\,\omega) - i_{X_\beta}(L_{X_\alpha}\omega) = L_{X_\alpha}(i_{X_\beta}\omega)$$

since by assumption $d\alpha = L_{X_\alpha}\omega = 0$. By means of the formula (7.10) for the derivative this is equal to

$$d(i_{X_\alpha}\, i_{X_\beta}\,\omega) + i_{X_\alpha}\, d(i_{X_\beta}\,\omega) .$$

The second term vanishes, since by assumption $d\beta = d(i_{X_\beta}\omega) = 0$. Therefore $i_{[X_\alpha, X_\beta]}\omega = d(i_{X_\alpha}\, i_{X_\beta}\,\omega) = d(\omega(X_\beta , X_\alpha))$ proving the claim (8.2).

The following statement is the global version of Theorem 3.1.

<u>Theorem 8.1</u>

A vectorfield X with flow ϕ^t is a Hamiltonian vectorfield if and only if ϕ^t is symplectic for every t.

<u>Proof</u>: Assume X to be Hamiltonian, i.e. $L_X\omega = 0$, thus

$$\frac{d}{dt}\, \phi^{t*}\omega = \phi^{t*}(L_X\omega) = 0 \; ,$$

hence $\phi^{t*}\omega = \phi^{0*}\omega = \omega$, and reversing the arguments the converse follows.

In the following we shall denote the exact Hamiltonian vectorfield X defined by $\omega(X,\cdot) = -dH$, by

$$X = X_H .$$

The function H, determined only up to a constant by X if the manifold is connected, is an integral of the vectorfield X_H. In fact, if ϕ^t is its flow, then

$$\frac{d}{dt} H(\phi^t(x)) = dH(X_H) \circ \phi^t(x)$$

$$= -\omega(X_H , X_H) \circ \phi^t(x) = 0$$

for every $X \in M$ as ω is antisymmetric. The next theorem is the global version of the result in section 2d.

Theorem 8.2

Assume (M_1, ω_1) and (M_2, ω_2) are two symplectic manifolds and $\psi: M_1 \to M_2$ is a diffeomorphism. Then ψ is symplectic if and only if

$$\psi^* X_H = X_K \quad \text{where } K = H \circ \psi$$

for all functions H on M.

Proof: Assume ψ to be symplectic, i.e. $\phi^* \omega_2 = \omega_1$ and set $X = X_H$, then

$$d(H \circ \psi) = \psi^*(dH) = -\psi^*(i_X \omega_2)$$

$$= i_{\psi^* X}(\psi^* \omega_2) = -i_{\psi^* X}\, \omega_1$$

and therefore $\psi^* X = X_{H \circ \psi}$. The converse is proved similarly.

Finally we observe that in case (M, ω) is an exact symplectic manifold i.e. $\omega = d\alpha$, then the flow ϕ^t of an exact Hamiltonian vectorfield $X = X_H$ consists of exact symplectic maps, i.e.

$$(\phi^t)^* \alpha - \alpha = d(f_t)$$

for some function f_t. In fact, as $\phi^\circ = id$,

$$(\phi^t)^* \alpha - \alpha = \int_0^t \frac{d}{ds}(\phi^s)^* \alpha \, ds .$$

In view of (7.10), and since by definition $i_X \omega = -dH$, the

integrand is equal to

$$(\phi^s)^* L_X \alpha = (\phi^s)^*(i_X d\alpha + d\, i_X \alpha)$$

$$= (\phi^s)^*(-dH + d\alpha(X))$$

$$= d(\phi^s)^*(-H + \alpha(X))$$

$$= d([-H + \alpha(X)] \circ \phi^s)$$

proving our claim with

$$f_t = \int_0^t (-H + \alpha(X)) \circ \phi^s \, ds .$$

(b) The Poisson bracket with respect of a Symplectic Structure.

We consider first a nondegenerate two form ω on a manifold M which is not assumed to be a closed form. The manifold M must, of course, have even dimensions. If F is a function, we shall denote by X_F the unique vectorfield $X = X_F$ satisfying $\omega(X, \cdot) = -dF$.

Definition: The Poisson bracket for two functions F, G is defined as

$$\{F, G\} = -\omega(X_F, X_G) .$$

By the antisymmetry of ω,

$$\{F, G\} = -\{G, F\} ,$$

and the nondegeneracey of ω implies that

$$\text{if } \{F, G\} = 0 \ \forall G , \text{ then } dF = 0 .$$

Since $\omega(X_F, \cdot) = -dF$, we get $\{F,G\} = dF(X_G) = X_G(F)$, and by antisymmetry this agrees with $-dG(X_F) = -X_F(G)$. We record:

$$(8.3) \qquad \{F,G\} = -X_F(G) = X_G(F) \ .$$

In order to study the closedness of ω in terms of $\{\cdot\}$ we introduce for three functions F_1, F_2, F_3 the expression

$$J(F_1,F_2,F_3) = \{F_1,\{F_2,F_3\}\} + \{F_2\{F_3,F_1\}\} + \{F_3,\{F_1,F_2\}\},$$

and shall prove

<u>Lemma 8.1</u>

(i) $[X_{F_1}, X_{F_2}](F_3) = X_{\{F_2,F_1\}}(F_3) + J(F_1,F_2,F_3)$,

(ii) $(d\omega)(X_{F_1}, X_{F_2}, X_{F_3}) = -J(F_1,F_2,F_3)$.

<u>Proof</u>: To prove the first formula we simply insert the definitions and use (8.3).

$$[X_{F_1}, X_{F_2}](F_3) = X_{F_1}(X_{F_2}(F_3)) - X_{F_2}(X_{F_1}(F_3))$$

$$= \{F_1, \{F_2, F_3\}\} - \{F_2, \{F_1, F_3\}\}$$

$$= \{F_1, \{F_2, F_3\}\} + \{F_2\{F_3, F_1\}\} .$$

Adding the identity $\{F_3, G\} - X_G F_3 = 0$ with $G = \{F_1 F_2\}$ we obtain the result.

In order to verify the second indentity we make use of the formula (7.12) for the d-operator:

$$d\omega(X_{F_1}, X_{F_2}, X_{F_3}) = X_{F_1}(\omega(X_{F_2}, X_{F_3})) + \text{cycl. perm.}$$
$$- \omega([X_{F_1}, X_{F_2}], X_{F_3}) - \text{cycl. perm.}$$

Using the definition $\omega(X_F, X_G) = -\{F,G\}$ and (8.3) we find

$$d\omega(X_{F_1}, X_{F_2}, X_{F_3}) = J(F_1,F_2,F_3) - \omega([X_{F_1}, X_{F_2}], X_{F_3}) - \text{cycl. perm.}$$

Next, using the definition $\omega(X_F,\cdot) = -dF$ we get

$$\omega([X_{F_1}, X_{F_2}], X_{F_3}) = dF_3([X_{F_1}, X_{F_2}]) = [X_{F_1}, X_{F_2}](F_3)$$
$$= \{F_1, \{F_2, F_3\}\} + \{F_2,\{F_3, F_1\}\},$$

and by cyclic permutatives this contributes -2J to the above J, as we wanted to show.

From Lemma 8.1 we deduce the following characterization of the closedness of a nondegenerate two form.

<u>Theorem 8.3</u>:

If ω is a nondegenerate two form on M then the condition $d\omega = 0$ is equivalent to each of the following conditions :

(i) $[X_F, X_G] = X_{\{G,F\}}$

(ii) $\{F, \{G,H\}\} + \{H, \{F,G\}\} + \{G, \{H,F\}\} = 0$

for every pair F,G respectively for every triple of functions F,G,H on M. The relation (ii) is called Jacobi-identity.

Incidentally, Jacobi used this identity to derive from two integrals F, G of a Hamiltonian system a new one, namely {F,G} and so on. However, in many cases the new integrals could be constants or functions of the old ones.

If (M,ω) is a symplectic manifold, then we can associate to the symplectic structure a Poisson bracket by means of the above definition. Since $d\omega = 0$ in this case we conclude from the theorem that the map $F \to X_F$ from the space of functions into the algebra of the exact Hamiltonian vectorfields is a Lie-Algebra homeomorphism with the negative Poisson bracket as Lie Algebra structure.

We show how the Poisson-bracket is expressed in local coordinates $x_1, \ldots, x_{2n}$. The 2-form ω is given by

$$\omega_x = \sum_{i<j} a_{ij}(x)\, dx_i \wedge dx_j$$

where the matrix $A = (a_{ij}(x))$ is skew symmetric. Then for the vectorfield

$$X = \sum_{j=1}^{2n} f_j \frac{\partial}{\partial x_j}, \quad f = (f_1, \ldots, f_{2n})$$

we have

$$\omega(X,\cdot) = \langle Af, dx \rangle ,$$

hence the relation $\omega(X_F, \cdot) = -dF$ yields

$$f = -A^{-1} \nabla F ,$$

where ∇F is the vector with components F_{x_j}. The Poisson-bracket then becomes

$$\{F,G\} = \langle A^{-1}\nabla F , \nabla G \rangle .$$

If ω is a symplectic form, we have in symplectic coordinates

$$A = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

which yields the old definition of the Poissonbracket.

(c) Special submanifolds of a symplectic manifold.

We can generalize in a natural way the different concepts of subspaces of a symplectic vectorspace introduced in Section 3. If $N \subset M$ is a submanifold of a symplectic manifold (M,ω) of $\dim M = 2n$, then a tangent-space

$$T_xN \subset T_xM , \quad x \in N$$

is a subspace of the symplectic vectorspace T_xM which possesses the symplectic structure ω_x. If $L_x \subset T_xM$ is any linear subspace then we denote by $L_x^{\perp}$ its symplectic orthogonal complement, i.e.

$$L_x^{\perp} = \{X \in T_xM \mid \omega_x(X, Y) = 0 \text{ for all } Y \in L_x\}$$

The submanifold N is called

isotropic	if	$T_xN \subset T_xN^{\perp}$
coisotropic	if	$T_xN \supset T_xN^{\perp}$
Lagrange	if	$T_xN = T_xN^{\perp}$
symplectic	if	$T_xN \cap T_xN^{\perp} = \{0\}$

for every $x \in N$.

To put it differently we introduce by $j: N \to M$ the inculsive map. Thus $j^*\omega$, the restrictive of ω onto TN, is a two form on N, which is closed since $d(j^*\omega) = 0$. The condition $TN \subset TN^{\perp}$ requires $\omega(X,Y) = 0$ for all $X, Y \in TN$, therefore a submanifold N is isotropic precisely if $j^*\omega = 0$. If N is maximally isotropic, i.e. has half the dimension of M, then N is a Lagrangian submanifold. On the other hand, the condition $T_xN \cap T_xN^{\perp} = \{0\}$ is equivalent to the condition that $T_xN \subset T_xM$ is a symplectic subspace, i.e. $j^*\omega(x)$ is nondegenerate, as we have seen in section 3. Therefore N is a symplectic submanifold precisely if $(N, j^*\omega)$ is a symplectic manifold.

If N is locally given by

(8.4) $$N = \{ x \mid F_k(x) = 0, \quad k = 1,2,\ldots, m\}$$

for m functions $F_1, \ldots, F_m$ with linearly independent $dF_1, dF_2, \ldots, dF_m$ the tangentspace is

$$T_xN = \{Y \in T_xM \mid dF_k(Y) = \omega(X_{F_k}, Y) = 0, \quad k = 1,2,\ldots,m\}$$

and therefore

$$T_xN = \text{span}\{X_{F_1}, X_{F_2}, \ldots, X_{F_m}\}^{\perp}$$

so that

(8.5) $$T_xN^{\perp} = \text{span}\{X_{F_1}, \ldots X_{F_m}\} .$$

Therefore the condition $T_xN^{\perp} \subset T_xN$ requires the vectors

$X_{F_1}, \ldots, X_{F_m}$ to be tangent to N and we find that N given by (8.4) is coisotropic precisely if the functions $F_1, \ldots, F_m$ satisfy

$$\omega(X_{F_k}, X_{F_k}) = \{F_i, F_k\} = 0 \quad \text{on } N.$$

This can occur, of course, only if $m \le n$, i.e. $\dim N \ge n$. Indeed,

$$\dim T_xN + \dim T_xN^{\perp} = \dim T_xM = 2n$$

as we saw in section 3, and therefore, as $T_xN^{\perp} \subset T_xN$, $2 \dim T_xN^{\perp} \le 2n$, i.e. $\dim T_xN \ge n$.

Incidentally, every submanifold of codimension 1 is coisotropic.

We now assume that the submanifold N defined by (8.5) is symplectic. Then the subspace $T_xN \subset T_xM$ is symplectic which is equivalent to the condition $T_xN \cap T_xN^{\perp} = \{0\}$, which in turn is equivalent to the condition, that $T_xN^{\perp} \subset T_xM$ is symplectic, as we saw in section 3. This means that the restriction of the symplectic form ω onto $T_xN^{\perp}$ is non-degenerate, which in view of (8.5) is the case precisely if

(8.6) $$\operatorname{Det} \omega(X_{F_i}, X_{F_k}) = \operatorname{Det}(\{F_i, F_k\}) \neq 0 \text{ on } N .$$

M has to be even in this case. Thus the submanifold N is a symplectic submanifold if and only if (8.6) holds.

The dimension m ...

In order to give an example, we take the symplectic manifold $M = (R^n \times R^{n*}, d\theta)$ and define the submanifold N by

$$N: \quad F_1(\xi,\eta) = |\xi| - 1 = 0$$
$$F_2(\xi,\eta) = \langle \xi,\eta \rangle = 0$$

Clearly dF_1 and dF_2 are independent on N, and

$$\{F_1 , F_2\} = 2\,|\xi|^2 = 2 \quad \text{on } N .$$

Therefore N is a symplectic submanifold of M, and $(N, j^* d\theta)$ is a symplectic manifold. Evidently, $N = T^*S^{(n-1)}$ is the cotangent bundle of the unit sphere and we ask whether its "natural" symplectic structure $d\theta_0$ defined by its 1-form θ_0 agrees with the symplectic structure $j^* d\theta$. This is indeed the case, we even have $j^*\theta = \theta_0$. This follows from our discussion of the example in section 7d.

If N is the n-dimensional submanifold $N = \{(x,g(x)) \in M = R^n \times R^{n*} \mid x \in R^n\}$ then N is a Lagrange submanifold if and only if

$$g(x) = \frac{\partial}{\partial x} G(x) ,$$

for some function G. In fact, if $j: N \to M$ is the inclusion map, thus

$$j^*(d\theta) = d(j^*\theta) = d \langle g,dx \rangle ,$$

hence $j^* d\theta = 0$ if and only if $\langle g,dx \rangle$ is closed, hence exact.

We finally give another characterization of a map to be symplectic. If (M_1, ω_1) and (M_2, ω_2) are two symplectic manifolds of the same dimension we can define the product manifold $M = M_1 \times M_2$. It carries the symplectic structure Ω defined by

$$\Omega = \pi_1^* \omega_1 - \pi_2^* \omega_2 ,$$

where $\pi_i : M_1 \times M_2 \to M_i$ are the projection maps. We claim that a map $f: M_1 \to M_2$ is symplectic if and only if the submanifold $N = \{(p, f(p)) \mid p \in M_1\}$ is a Lagrangian submanifold of M. In fact, let $j: N \to M$ denote the inclusion map, then we have for the tangentspace T_qN at some point $q = (p,f(p)) \in N$

$$T_qN = \{\hat{X} = (X, dfX) \mid X \in T_pM_1\}$$

and therefore

$$j^*\Omega(\hat{X},\hat{Y}) = \omega_1(X,Y) - \omega_2(dfX , dfY) = (\omega_1 - f^*\omega_2)(X,Y) ,$$

therefore $j^*\Omega = 0$ if and only if $\omega_1 = f^*\omega_2$ as we have claimed.

(d) Special canonical coordinates.

The aim is to generalize our considerations in section 4b. We consider on the symplectic manifold (M,ω) with $\dim M = 2n$, s functions $F_1, \ldots, F_s$, $s \le n$ with the properties that

(8.7) (i) $dF_1,\ldots, dF_s$ are linearly independent and

(ii) $\{F_i, F_j\} = 0$, $1 \le i,j \le s$.

There cannot be more than n such functions. Indeed, by (i) the vectors X_{F_j} are linearly independent, and from (ii) we conclude recalling the definitions of the Poissonbracket and theorem 7.4, that $\omega(X_{F_j}, X_{F_i}) = 0$. Therefore span $\{X_{F_1}, \dots, X_{F_s}\}$ is an isotropic subspace of TM of dimension s and therefore must have dimension $\leq n$, as we saw in section 3a. Our aim is to prove the following statement which will be useful later on.

Theorem 8.4

If F_j are $s \leq n$ functions satisfying (i) and (ii), then every point $p \in M$ lies in a symplectic coordinate chart (U, ψ) such that

$$\psi^*\omega = \sum_{k=1}^{n} dy_k \wedge dx_k \quad \text{and } F_j \circ \psi = y_j, \quad 1 \leq j \leq s .$$

Proof: We first prove the theorem for the case s = n. By Darboux's theorem (Theorem 7.1) every point $p \in M$ has symplectic coordinates (U, ϕ) with $\phi^*\omega = \sum_{k=1}^{n} dy_k \wedge dx_k = \omega_0$. and $F_j \circ \phi = f_j$ are n functions with $\{f_j, f_k\} = 0$ and df_j linearly independent. By the results in 4b we can find a canonical transformation χ, i.e. $\chi^*\omega_0 = \omega_0$, such that $f_j \circ \chi = y_j$. Thus $\psi = \phi \circ \chi$ gives the assertion. To prove the general case s < n we shall make use of the following result which is ~~done~~ due to Frobenius.

Theorem 8.5

Let M be any manifold of dim n and let $X_1, \dots, X_k$ be

vectorfields spanning at every point x of M a k-dimensional subspace $S(x) \subset T_xM$ of the tangentspace, i.e.

$$\text{span}\,\{X_1, \dots, X_k\} = S(x) .$$

Assume that for every $x \in M$

$$[X_i, X_j](x) \in S(x) .$$

Then ~~Thus~~ every point x on M has a neighborhood U and local coordinates (U, ϕ), $\phi : (\tau_1, \dots \tau_k, y_1, \dots, y_{n-k}) \to U$ such that

$$\phi^*(X_j) = \sum_{s=1}^{k} a_{js} \frac{\partial}{\partial \tau_s} ,$$

where $a_{js} = a_{js}(\tau, y)$ are (differentiable) functions. In the special case that the vectorfields commute, i.e. $[X_i, X_j] = 0$, we can achieve that

$$\phi^*(X_j) = \frac{\partial}{\partial \tau_j} .$$

Proof: The statement being local we may assume $M = R^n$ and x = 0. We pick a (n-k)-dimensional subspace $E \subset R^n$ through the point x = 0 which is transversal to $S(0) = \text{span}\{X_1, \dots, X_k\}$ at x = 0 ~~such~~ so that $R^n = S(0) \oplus E$, and we denote by $y = (y_1, \dots, y_{n-k})$ the coordinates in E. If $\phi_j^{t_j}$ denotes the flow of the vectorfield X_j we define the map

$$\phi(\tau, y) = \phi_k^{\tau_k} \circ \phi_{k-1}^{\tau_{k-1}} \circ \dots \circ \phi_1^{\tau_1}(0, y)$$

for $(\tau, y) = (\tau_1, \dots, \tau_k, y_1, \dots, y_{n-k}) \in R^n$ in a neighborhood of zero. Clearly $\phi(0,y) = y$ and $d\phi(0)$ maps the vectors

$\frac{\partial}{\partial \tau_j}$ into $X_j(0)$. Therefore $d\phi(0)$ is surjective and hence ϕ defines a local diffeomorphism in a neighborhood of the origin. In order to prove the statement we shall show that

$$d\phi(\tau,y)\left(\frac{\partial}{\partial \tau_j}\right) = \sum_{s=1}^{n} b_{js} \quad (X_s \circ \phi) \quad ,$$

i.e. a linear combination of the vectors $X_1, \dots, X_k$. It then follows that

$$\frac{\partial}{\partial \tau_j} = \sum_{s=1}^{n} b_{js} \, (d\phi)^{-1} X_s \circ \phi = \sum_{s=1}^{n} b_{js} \, \phi^*(X_s)$$

as is claimed in the statement. We observe

$$d\phi(\tau,y)\left(\frac{\partial}{\partial \tau_j}\right) = \frac{\partial}{\partial s} \phi_k^{\tau_k} \circ \dots \circ \phi_j^{s} \circ \dots \circ \phi_1^{\tau_1} (0,y)\Big|_{s=\tau_j} .$$

In the special case where the vectorfields X_j and hence their flows ϕ_j^t commute so we can put ϕ_j^s into the first spot and find

$$d\phi\left(\frac{\partial}{\partial \tau_j}\right) = X_j \circ \phi$$

and therefore

$$\frac{\partial}{\partial \tau_j} = (d\psi)^{-1} X_j \circ \phi = \phi^*(X_j) \quad ,$$

hence the last statement is proved. In the general case, if the flows do not commute we find by the chain rule

$$d\phi\left(\frac{\partial}{\partial \tau_j}\right) = d\phi_k^{\tau_k} \; d\phi_{k-1}^{\tau_{k-1}} \dots d\phi_{j+1}^{\tau_{j+1}} X_j \; (\phi_j^{\tau_j} \circ \dots \circ \phi_1^{\tau_1} (0,y)).$$

Recalling the notation of the pull back, $\psi^{*-1}(X) = d\psi \, X \circ \psi^{-1}$, we can write:

$$d\phi\left(\frac{\partial}{\partial \tau_j}\right) = (\phi_k^{-\tau_k})^* \circ \dots \circ (\phi_{j+1}^{-\tau_{j+1}})^* X_j \circ \phi .$$

In order to prove that this vector is a linear combination of $X_1 \circ \phi, \dots, X_k \circ \phi$ it suffices to prove that the vectors

$$(\phi_j^{\tau_j})^* X_i$$

for τ_j small are such linear combinations. Let ϕ^t denote the flow of the vectorfield $X = X_j$. Then we have by the assumption of the theorem

$$[X, X_i] = \sum_{s=1}^{k} \lambda_{is} X_s$$

with functions λ_{is} . Therefore

$$\text{(8.8)} \quad \frac{d}{dt} (\phi^t)^* X_i = (\phi^t)^* [X, X_i] = (\phi^t)^* \sum_{s=1}^{k} \lambda_{is} X_s = \sum_{s=1}^{k} (\lambda_{is} \circ \phi^t)\, (\phi^t)^* X_s \quad ,$$

where we have used (7.8) . Denoting by Λ the matrix function $\Lambda = (\lambda_{is})$ we define $A(t)$ as the fundamental solution of the linear equation

$$\text{(8.9)} \quad \frac{d}{dt} A(t) = (\Lambda \circ \phi^t)\, A(t) \quad , \quad A(0) = 1 .$$

If $\alpha_{ij}(t)$ are the matrix elements of $A(t)$ we introduce the tangentvectors

$$Y_j(t) = (\phi^t)^* X_j - \sum_{s=1}^{k} \alpha_{js}(t)\, X_s$$

$i \le j \le k$. It then follows from (8.8) and (8.9) that the $Y_j(t)$'s are solutions of the linear equations

$$\frac{d}{dt} Y_j(t) = \sum_{s=1}^{k} \lambda_{js}(t)\, Y_s(t), \quad 0 \le j \le k$$

with initial conditions $Y_j(0) = 0$, and therefore $Y_j(t) \equiv 0$, hence

$$(\phi^t)^* X_j = \sum_{s=1}^{k} a_{js}(t)\, X_s$$

as we wanted to prove.

Lemma 8.2

Let M be a symplectic manifold, dim M = 2n. Assume the functions F_j, $1 \le j \le s$ for some $s < n$ have the properties (8.7) (i) and (ii). Then every point p has a neighborhood U, and there are functions $F_{s+1}, \ldots, F_n$ defined on U, such that the functions $F_1, \ldots, F_n$ satisfy the properties (8.7) (i) and (ii) on U.

Proof:

The statement being local we assume $M = R^{2n}$ and $p = 0$. By the last statement in Frobenius's theorem there are local coordinates $(\tau_1, \ldots, \tau_s, y_1, \ldots, y_{2n-s})$, not necessarily symplectic, such that

$$X_{F_j} = \frac{\partial}{\partial \tau_j}, \quad 1 \le j \le s .$$

We seek a function $F_{s+1} = G$ satisfying

(8.10) $\{F_j, G\} = X_{F_j}(G) = \frac{\partial}{\partial \tau_j} = 0, \; 1 \le j \le s$,

and such that, in addition, $F_1, \ldots, F_s$, G are linearly independent. Let $S = \text{span}(X_{F_1}, \ldots, X_{F_s})$ at the origin. Then $S \subset S^{\perp}$ (where $\perp$ is the orthogonal complement with respect to the symplectic structure) but $S \neq S^{\perp}$ since dim $S = s < n$, and so S is not maximally isotropic. Hence there is a vector $Y \in S^{\perp}$ and $Y \notin S$:

$$Y = \sum_{j=1}^{s} \lambda_j \frac{\partial}{\partial \tau_k} + \sum_{j=1}^{2n-s} \mu_j \frac{\partial}{\partial y_j},$$

where not all the μ's vanish, e.g. $\mu_{j_0} \neq 0$. We then define

$$G(\tau, y) = y_{j_0},$$

such that $\{F_j, G\} = 0$ by (8.10) and moreover F_j, G are linearly independent. In fact at the origin we have $df_j(Y) = \omega(X_{F_j}, Y) = 0$, but $dG(y) = \mu_{j_0} \neq 0$. Proceeding ~~indirectly~~ inductively the Lemma follows.

In order to finish the proof of theorem 7.5 we simply remark that by the Lemma we can assume $s = n$, and in this case we have already proved the statement.

(e) A global symplectic invariant.

We concluded from Darbaux's theorem that locally there are no symplectic invariants other than the dimension. This is globally not the case as we shall show.

We consider compact, connected and oriented manifolds M of dimension two, i.e. surfaces. The orientation will be given by a volume form, say ω. This 2-form vanishes nowhere hence is nondegenerate; it is clearly also a closed form

Therefore (M,ω) is a symplectic manifold with the volume form as symplectic structure.

Assume now that (M_1,ω_1) and (M_2,ω_2) are two such symplectic manifolds of dimension two. If $f: M_1 \to M_2$ is any diffeomorphism which respects the orientations then

$$\int_{M_1} f^*\omega_2 = \int_{M_2} \omega_2 .$$

If f, in addition, is symplectic :

(8.11) $$f^*\omega_2 = \omega_1 ,$$

which in 2 dimensions is the same as volume preserving, we find

(8.12) $$\int_{M_1} \omega_1 = \int_{M_2} \omega_2 .$$

The total volume therefore has to be the same; it is a symplectic invariant.

Our aim is to prove the converse. We shall show that if two compact, connected and oriented surfaces (M_1,ω_1) and (M_2,ω_2) are diffeomorphic (which is the case their Euler characteristics are equal) and if in addition their total volume is the same, i.e. (8.12) holds, then there is a symplectic diffeomorphism $f: M_1 \to M_2$, i.e. $f^*\omega_2 = \omega_1$. This gives a classification of compact, connected two-dimensional symplectic manifolds according to the Euler characteristic and the total volume.

More generally we prove the following statement for volume preserving diffeomorphisms.

Theorem 8.6

Assume M is a compact, connected and orientable manifold of dimension n without boundary. If α and β are two volume forms such that their total volume is the same, i.e.

(8.13) $$\int_M \alpha = \int_M \beta ,$$

then there is a diffeomorphism f of M with $f^*\beta = \alpha$.

Hence the total volume is the only invariant of volume-preserving diffeomorphisms.

Proof :

We proceed as in the proof of Darboux's theorem and use a deformation argument. We define the family of volume forms

$$\alpha_t = (1-t)\alpha + t\beta , \quad 0 \le t \le 1$$

These forms α_t are indeed volume forms, since they are locally represented by $\alpha = a\, dx_1 \wedge \ldots \wedge dx_n$ and $\beta = b\, dx_1 \wedge \ldots \wedge dx_n$ with two nonvanishing functions a and b which by (8.13) must have the same sign.

We construct a family of diffeomorphisms ϕ^t satisfying

(8.14) $$(\phi^t)^* \alpha_t = \alpha , \quad \phi^0 = id , \quad 0 \le t \le 1 .$$

The diffeomorphism $f = \phi^1$ will solve our problem. Since

M is compact, connected and orientable, we conclude from

$$\int_M (\beta - \alpha) = 0 \quad ,$$

that $\beta - \alpha = d\gamma$ for some $(n-1)$ - form γ. This is a special case of the de Rham theorem. Since α_t is a volume form, there is a unique time dependent vectorfield X_t solving the equation

$$(8.15) \qquad i_{X_t} \rfloor \alpha_t = - \gamma \quad , \quad 0 \leq t \leq 1 .$$

In fact, a volume form α is locally represented by

$$\alpha_x = a(x) \ dx_1 \wedge \ldots \wedge dx_n \quad ,$$

for some function $a(x) \neq 0$, and if $X = (X_1, \ldots, X_n)$ is a vector-field one has

$$i_X \alpha = \sum_{i=1}^{n} (-1)^{i-1} a X_i \ dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \quad ,$$

from which it is evident, that (8.15) has a unique solution X_t. Let ϕ^t, with $\phi^0 = id$, be the flow of this vectorfield X_t. It exists for all $0 \leq t \leq 1$ on M since M is compact. Since $d\alpha_t = 0$ we find using formula (7.10)

$$\frac{d}{dt}(\phi^t)^* \alpha_t = (\phi^t)^* \{ L_{X_t} \alpha_t + \dot{\alpha}_t \}$$

$$= (\phi^t)^* \{ d(i_{X_t} \alpha_t) + (\beta - \alpha) \} ,$$

which is equal to 0 since $d(i_{X_t} \alpha_t) + \beta - \alpha = d(i_{X_t} \alpha_t + \gamma) = 0$ by our choice of the vectorfield X_t. Hence (8.14) holds true and the proof is finished.

CHAPTER 2

1. Poincaré's Perturbation Theory of Periodic Orbits.

(a) Floquet Multipliers and Section Maps.

In this section we treat the local perturbation theory of periodic orbits, which goes back to Poincaré. We consider a family of autonomous vector fields on R^n or more generally on an n-dimensional manifold M

$$(1.1) \qquad \dot{x} = f(x;\mu) \ .$$

We assume that if $\mu = 0$, the "unperturbed" system $\dot{x} = f(x,0)$ possesses the periodic solution

$$p(t) = p(t + T) \quad \text{with period} \quad T > 0 ,$$

so that

$$f(p(t),0) \neq 0 , \qquad t \in R .$$

The aim is to construct periodic solutions near $p(t)$ of the system (1.1) for small values of μ. In this section we consider vector fields on manifolds but soon will reduce the problem to a local one. But before this reduction we will use the notation of calculus on manifolds (Chapter I, §8). If $\phi^t(x;\mu)$ denotes the flow of (1.1), the condition for an orbit to be closed is

$$(1.2) \qquad \phi^{\tau}(x;\mu) = x \ , \quad \text{some} \quad \tau > 0 \ .$$

As $\phi^T(p;0) = p$ for $p \in p(t)$, one is tempted to apply the implicit function theorem, in order to solve (1.2) keeping the period $\tau = T$ fixed. However, the relevant linearized at p,

$$d\phi_p^T - I$$

is singular; where d stands for the linearization of ϕ^T at the point p. This is due to the autonomous character of the vector field. In fact, if f is any vector field with flow ϕ^t, we find by differentiating $\phi^t \circ \phi^s = \phi^s \circ \phi^t$ in s

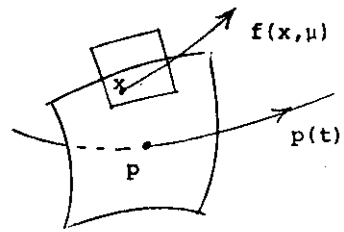
$$d\phi^t f\left(\phi^s(x)\right) = f(\phi^{s+t}(x)),$$

which applied to our situation, setting $\mu = 0$, $s = 0$, $t = T$, gives

$$(1.3) \qquad d\phi^T f(p;0) = f(\phi^T(p,0)) = f(p,0)$$

for $p \in p(t)$. Hence (1) is an eigenvalue of $d\phi_p^T$ with eigenvector $f(p,0)$. In fact, in general such a periodic solution with a fixed period $\tau = T$ does not exist; see Exercise 1. Therefore we allow the period τ to be variable, which gives us an additional independent variable. We describe the geometrical situation effectively by construction of the so-called section map. We intersect the periodic orbit p(t) at p with an (n-1)-dimensional hypersurface, i.e. a submanifold $\Sigma \subset M$ of codimension 1, to which the vector field $f(x;\mu)$ is not tangent i.e. $T_x\Sigma$ and $f = f(x,\mu)$ span T_xM:

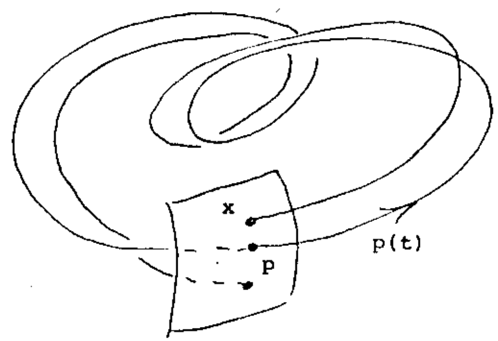
$$(1.4) \qquad T_x\Sigma + \langle f(x;\mu)\rangle = T_xM, \quad x \in \Sigma. \quad (*)$$



If $\mu = 0$, then $\phi^T(p,0) = p \in \Sigma$, hence, as the flow depends differentiably on x and μ, we can define a local map $\psi: \Sigma \to \Sigma$ near p via

$$(1.5) \qquad \psi(x,\mu) = \phi^{\tau(x,\mu)}(x;\mu), \qquad x \in \Sigma,$$

where $\tau = \tau(x;\mu)$ is close to T and uniquely determined so that $\phi^\tau(x,\mu) \in \Sigma$.



(*) $\langle v\rangle$ stands for the one-dimensional vector space spanned by v.

Obviously if $\mu = 0$, then $\tau(p,0) = T$ and $\psi(p,0) = p$, i.e. p is a fixed point of the map for $\mu = 0$. The map ψ, which is for small μ a local diffeomorphism of Σ is called Poincaré's map or a section map. It reduces the global study of the flow in a neighborhood of a periodic orbit to the local study of a map $\psi: \Sigma \to \Sigma$ in a neighborhood of a fixed point, if $\mu = 0$. Clearly fixed points of ψ are initial data for periodic orbits of the system (1.1) having periods close to T, and periodic points of ψ i.e. $\psi^n(q) = q$, $q \in \Sigma$ correspond to periodic orbits of period close to nT.

In order to find fixed points of ψ for small μ by means of the implicit function theorem it is important to know the eigenvalues of the linearized map $d\psi_p$ for $\mu = 0$:

(1.6) $$d\psi_p: T_p\Sigma \to T_p\Sigma ,$$

where $p = \psi(p) \in p(t)$ is the fixed point for $\mu = 0$. These eigenvalues are fundamental for our purposes and they are independent of the particular section map chosen. Indeed, if Σ_1 and Σ_2 are two transversal sections at p and q respectively with corresponding section maps ψ_1 and ψ_2, then there is a local diffeomorphism:

$$\chi: \Sigma_1 \to \Sigma_2 ,$$

such that locally

$$\chi \circ \psi_1 = \psi_2 \circ \chi ,$$

and therefore at p

$$d\chi \cdot d\psi_1 = d\psi_2 \cdot d\chi .$$

If $p \neq q$ the map χ is simply defined by following a point in Σ_1 along the flow to the next intersection with Σ_2.

The relation of these eigenvalues to those of

(1.7) $$\Phi = d\phi_p^T: T_pM \to T_pM ,$$

$\mu = 0$, is given by the following

<u>Lemma</u> 1.1. Φ has 1 as an eigenvalue with eigenvector f(p), and the remaining eigenvalues agree with those of $d\psi_p$, i.e.

$$\det(\lambda - \Phi) = (\lambda - 1) \det(\lambda - d\psi_p) .$$

<u>Proof</u>: We have already proved the first part of the statement, see (1.3). Differentiating (1.5) at p gives for $\xi \in T_p\Sigma$,

$$d\psi\, \xi = (d\phi^T)\xi + \frac{d}{dt}\phi^t\Big|_{t=T} (d\tau)\, \xi$$

$$= \Phi\, \xi - \ell(\xi) \cdot f(p) ,$$

where we have abbreviated $\ell(\xi) = (d\tau)\xi$. Therefore, with respect to the splitting

$$T_pM = \langle f(p)\rangle + T_p\Sigma ,$$

the linear map Φ has the representation

$$\Phi = \left(\begin{array}{c|c} 1 & \ell \\ \hline 0 & d\psi_p \end{array}\right) ,$$

For the computation of the linear map Φ it is not necessary to know the flow ϕ^t but it suffices to integrate a linear system of differential equations, as we show in case $M = R^n$. In this case the linearized system in question is

$$\frac{d}{dt} y = f_x(p(t))y \tag{1.8}$$

which is called the variational system of p(t). The general solution of this system is given by

$$y(t) = \Phi(t)\ y(0)$$

where $\Phi(t) = d\phi^t(p(0))$, as one proves readily by differentiation of

$$\frac{d}{dt} \phi^t = \dot{f} \circ \phi^t .$$

Thus $\Phi(t)$ is the fundamental solution of (8) with $\Phi(0) = I$ and therefore is determined by the solutions of (1.8)

If λ is an eigenvalue of $\Phi = \Phi(T)$ then there exists a solution $y(t) \neq 0$ of (1.8) with

$$y(t + T) = \lambda\ y(t) ,$$

and conversely. Therefore the eigenvalues of Φ are called Floquet multipliers of (1.8). They are associated with the periodic orbit p(t) and we will refer to them as Floquet multipliers of p(t). In later examples we will use this remark to compute these eigenvalues.

Theorem 1.1. Suppose the autonomous system

$$\dot{x} = f(x;\mu)$$

has for $\mu = 0$ the periodic solution p(t) of period $T > 0$. If 1 is a simple Floquet multiplier of this periodic solution, then for small μ there is up to a time-shift a unique periodic solution $p(t,\mu)$ for (1.1) having period

$$T(\mu) \text{ close to } T ,$$

such that $p(t,\mu) \to p(t)$ and $T(\mu) \to T$ as $\mu \to 0$. Moreover $p(t,\mu)$ is as smooth as f.

Proof: The proof is an easy application of the implicit function theorem. Let $\Sigma \subset M$ be a transversal section of f at $p \in p(t)$, with section map $\psi(y,\mu): \Sigma \to \Sigma$. We may choose the local coordinates $y = (y_1,\ldots,y_{n-1})$ such that p corresponds to $y = 0$; hence $\psi(0,\mu) = 0$ if $\mu = 0$. By assumption and Lemma 1,

$$\frac{\partial}{\partial y} \psi(0,0) - I$$

is not singular and therefore there is a unique smooth function $y(\mu)$ such that $y(0) = 0$, and

$$\psi(y(\mu),\mu) = y(\mu) ,$$

hence we have found the initial data of the required periodic solution.

It is clear from the implicit function theorem, that for μ fixed, the periodic solutions $p(t,\mu)$ are isolated among the periodic solutions having periods close to T. There may of course exist other periodic solutions near $p(t,\mu)$ of longer periods, close to nT, $n \geq 2$. They correspond to periodic points of the section map, i.e. to fixed points of ψ^n.

(b) Example.

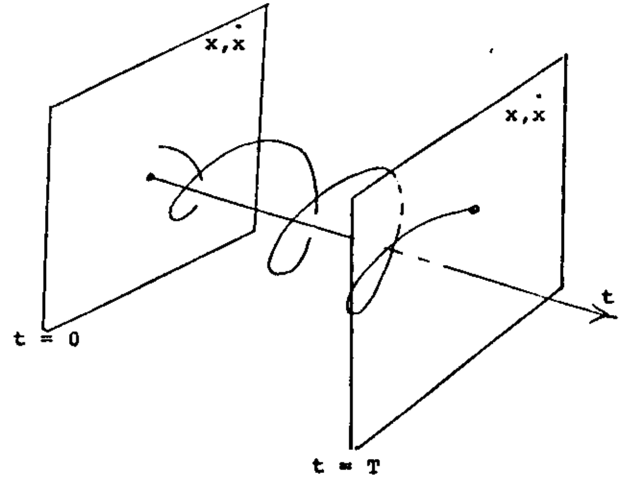
We shall illustrate this method by the following simple example.

$$\ddot{x} + \sin x = \mu\, f(t,x,\dot{x},\mu) ,$$

where $f(t + T, x, \dot{x}) = f(t,x,\dot{x})$ for some $T > 0$. We are looking for T-periodic solutions of this equation near $x = 0$ and $x = \pi$. This equation is nonautonomous and our consideration seems not to be applicable. But if we introduce $x_0 = t + \text{const.}$, $x_1 = x$, $x_2 = \dot{x}$ we can write it as the autonomous system

$$\begin{aligned} \dot{x}_0 &= 1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 + \mu\, f(x_0,x_1,x_2) . \end{aligned}$$

The plane $x_0 = 0 \pmod T$ is a transversal section of the vector field, and if $\mu = 0$ then $x_0 = t$, $x_1(t) = 0 , \pi$, $x_2(t) = 0$ is a periodic solution:



The variational equations for $\mu = 0$ along these reference solutions

$$\dot{y}_0 = 0 , \qquad \dot{y}_1 = y_2 , \qquad \dot{y}_2 = \pm y_1 ,$$

have the fundamental solution

$$\Phi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}$$

and the eigenvalues of $\Phi = \Phi(T)$ are

$$\{1, e^{\pm iT}\} \quad \text{or} \quad \{1, e^{\pm T}\} .$$

Therefore if we exclude the resonances, i.e. if

$$\frac{T}{2\pi} \neq \text{integer} ,$$

in the first case, $T \neq 0$ in the second, there is a unique periodic solution $x = p(t,\mu)$ for small μ having period T, which "continues" the reference solution.

(c) Degenerate Cases, Integrals.

Theorem 1 is not applicable to a system possessing an integral, because as we shall show below such an integral gives rise to a second Floquet multiplier equal to 1. More generally, we shall assume that the vector field $f(x)$, with flow ϕ^t possesses r independent integrals $G_\alpha \in C^1$, i.e.

$$G_\alpha(\phi^t(x)) = G_\alpha(x) \ , \qquad \alpha = 1,2,\dots,r$$

such that

$$dG_\alpha(x) \ , \qquad \alpha = 1,2,\dots,r$$

are linearly independent for x in a neighborhood of the periodic solution p(t) of period T. The integral surfaces

$$S = S_c = \left\{x \in M \mid G_\alpha(x) = c_\alpha \ , \ \alpha = 1,2,\dots,r\right\},$$

with $c = (c_1,\dots,c_r) \in R^r$, are then invariant under the flow and have tangent spaces

$$TS = \{\xi \in TM \mid dG_\alpha(\xi) = 0, \ \alpha = 1,2,\dots,r\}.$$

In particular, $f(x) \in T_xS$, $x \in M$, and if $\Phi = d\phi_p^T$, $p \in p(t)$, we have

$$dG_\alpha(\Phi\xi) = dG_\alpha(\xi) \ ,$$

for every $\xi \in T_pM$ and therefore

$$\Phi(T_pS) = T_pS \ ,$$

S being the integral surface of the periodic orbit p(t). In order to compute the eigenvalues of Φ, we choose in a neighborhood of p(0) local coordinates $x_1,x_2,\dots,x_n$ such that

$$G_\alpha(x) = x_\alpha \ , \qquad \alpha = 1,2,\dots,r \ .$$

If we split $x = (x_1,\dots,x_n)$ into $x' = (x_1,\dots,x_r)$, $x'' = (x_{r+1},\dots,x_n)$ then the map ϕ^T is expressed by

$$\phi^T: \begin{pmatrix} x' \\ x'' \end{pmatrix} \to \begin{pmatrix} \phi'(x) = x' \\ \phi''(x) \end{pmatrix}$$

Thus we find for the linearized map $\Phi = d\phi_p^T$

$$\Phi = \left(\begin{array}{c|c} I_r & 0 \\ \hline * & \hat{\Phi} \end{array}\right) \tag{1.9}$$

where $\hat{\Phi} = \Phi|T_pS$ is the restriction of Φ to the tangent space T_pS. We now proceed as in the proof of Lemma 1.1 considering the flow ϕ^t restricted to S. Clearly,

$$\hat{\Phi} f(p) = f(p)$$

and choosing a transversal section $\sigma \subset S$ in the integral surface S, we find with respect to the splitting

$$T_pS = \langle f(p)\rangle + T_p\sigma \ ,$$

$$\hat{\Phi} = \left(\begin{array}{c|c} 1 & \ell \\ \hline 0 & d\psi_p \end{array}\right) , \tag{1.10}$$

where $\psi: \sigma \to \sigma$ is the section map. Summarizing we have proved by (1.9) and (1.10):

Lemma 1.2. If the system (1.1) possesses an independent integral then r+1 Floquet-multipliers are equal to 1. The remaining Floquet multipliers are the eigenvalues of $d\psi_p$, where ψ is a section map of a section $\sigma \subset S$, i.e.

$$\det (\lambda - \Phi) = (\lambda - 1)^{r+1} \det (\lambda - d\psi_p) .$$

To prove a continuation theorem for such degenerate cases, we assume that the vector fields

$$\dot{x} = f(x;\mu)$$

possess, for *every* μ, r linearly independent integrals

$$G_\alpha(x;\mu) , \qquad \alpha = 1,2,\dots,r,$$

$G_\alpha \in C^1$. We denote the invariant integral surfaces by

$$S_{c,\mu} = \{x \in M \mid G_\alpha(x;\mu) = c_\alpha , \alpha = 1,2,\dots,r\}.$$

We assume that $f(x;0)$ possesses the periodic orbit $p(t)$ of period T which lies on the integral surface for $\mu = 0$, with $c = 0$, say.

Theorem 1.2. If the system (1.1) has r independent integrals near the periodic orbit $p(t)$ and if $p(t)$ has exactly r+1 Floquet multipliers equal to 1, then there exists for small μ a unique smooth r-parameter family $p(t,c,\mu)$ of periodic orbits for the system (1.1), having periods $T(c,\mu)$ close to T and lying on the integral-surfaces $S_{c,\mu}$ i.e.

$$G_\alpha(p(t,c,\mu),\mu) = c_\alpha ,$$

$\alpha = 1,2,\dots,r$. Moreover $p(t,c,\mu) \to p(t)$ and $T(c,\mu) \to T$ as $c \to 0$, $\mu \to 0$, and $p(t,c,\mu)$ is as smooth as f and G_α.

Proof. Clearly we can introduce coordinates $x_1,\dots,x_n$ in the neighborhood of $p(0)$ so that $G_\alpha(x,\mu) = x_\alpha$, $\alpha = 1,2,\dots,r$ holds for all μ near 0 and so that $x_n = 0$ defines the section Σ. In these local coordinates the section map is expressed by

$$\psi: \begin{pmatrix} x' \\ x'' \end{pmatrix} \mapsto \begin{pmatrix} \psi'(x,\mu) = x' \\ \psi''(x,\mu) \end{pmatrix}$$

where $x' = (x_1,x_2,\dots,x_r)$ but here $x'' = (x_{r+1},\dots,x_{n-1})$. In order to find fixed points of ψ we only have to solve the n-r-1 equations

$$\psi''(x,\mu) = x''$$

since the first r equations

$$\psi'(x,\mu) = x'$$

are automatically satisfied. To the first equation the implicit function theorem is applicable, since

$$d\psi_p = \left(\begin{array}{c|c} I_r & 0 \\ \hline * & \frac{\partial}{\partial x''} \psi'' \end{array} \right)$$

and by assumption and Lemma 1

$$\frac{\partial}{\partial x''} \psi'' - I_{n-r-1}$$

is nonsingular, therefore there is a unique map $(x',\mu) \to x'' = x''(x',\mu)$ solving the equation. Thus $x' = c$, $x'' = x''(c,\mu)$ are the initial values of the desired solutions.

Setting $\mu = 0$ in the theorem we observe that the original periodic orbit also lies in an r-parameter family of periodic solutions $p(t,c,0)$ with periods close to T, and hence is not isolated. These periodic solutions correspond to different values of the integrals i.e.

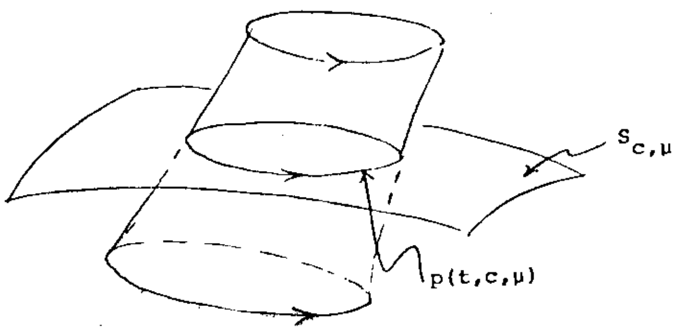
$$G_\alpha(p(t,c,0),0) = c_\alpha ,$$

$\alpha = 1,2,\dots,r$ and on these integral surfaces the periodic solutions are indeed isolated among those with periods close to T.

(d) Remarks. (i) Geometrically the periodic solutions $p(t,c,\mu)$ of period $T(c,\mu)$ fill out for fixed μ an r+1 dimensional embedded cylinder $S^1 \times B_\gamma$, where $B_\gamma = \{c \in R^r \mid |c| < \gamma\}$ for some $\gamma > 0$. In fact, the embedding is simply given by

$$(s,c) \mapsto p(sT,c,\mu) \in M ,$$

where s (mod 1) $\in S^1$, $c \in B$ and $T = T(c,\mu)$.



(ii) Clearly in our proof of Theorem 1.2 the dependence on one or on several parameters μ is irrelevant. We could consider any set of vector fields f with integrals $G_1,\dots G_r$ close to the vector field $f^{(0)}$ with independent integrals $G_1^{(0)},\dots,G_r^{(0)}$, where $f^{(0)}$ possesses the periodic solution $p_0(t)$ of period T_0. With the same argument as above we can show that for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon)$ such that for any vector field of our set satisfying

$$|f - f^{(0)}|_{C^1} + \sum_{\alpha=1}^{r} |G_\alpha - G_\alpha^0|_{C^1} < \delta(\varepsilon)$$

there exists a periodic solution $p(t)$ having period T, such that

$$|p(t)-p_0(t)|_{C^1} + |T - T_0| < \varepsilon.$$

(e) Hamiltonian Case.

We consider a Hamiltonian vector field X_H on a symplectic 2n-dimensional manifold M with symplectic structure ω. The periodic orbit $p(t)$ lies on the invariant energy surface

$$S = \{x \in M \mid H(x) = H(p(t))\}.$$

Theorem 1.3. A transversal section $\Sigma \subset S$ of X_H in the energy surface is a symplectic manifold of dimension $2n-2$ with the symplectic structure $\omega|_\Sigma$. Moreover the section map $\psi: \Sigma \to \Sigma$ is a canonical map.

Proof: If $j: \Sigma \to M$ denotes the injection map, we conclude from the formula

$$d(j^*\omega) = j^*(d\omega)$$

together with $d\omega = 0$ that $j^*\omega = \omega|_\Sigma$ is a closed two form on Σ, and it remains to prove that it is nondegenerate. As $dH(\xi) = -\omega(X_H,\xi)$, the tangent spaces at S are

$$T_xS = \left\{\xi \in T_xM \mid \omega(X_H,\xi) = 0\right\}.$$

The two form ω is nondegenerate, and therefore every $v \in T_xS$ which is orthogonal to T_xS is proportional to X_H, i.e. $v = c\,X_H$. Assume now $\omega(v,u) = 0$ for some $v \in T_x\Sigma$ and for all $u \in T_x\Sigma$ we have to show $v = 0$. Since Σ is a transversal section, we have

$$T_xS = \langle X_H(x)\rangle + T_x\Sigma ,$$

Hence $\omega(v,\xi) = 0$ for all $\xi \in T_xS$, and by the above observation $v = c\,X_H$, and so $v = 0$.

To prove that $\psi: \Sigma \to \Sigma$ is canonical we recall from the proof of Lemma 1.1

(1.11) $$d\phi^\tau\,\xi = d\psi\,\xi + \ell(\xi)\,X_H ,$$

for every $\xi \in T_x\Sigma$, where $\tau = \tau(x)$. We use that ϕ^τ is canonical, hence

$$\omega(\xi,\eta) = \omega(d\phi^\tau\,\xi,\ d\phi^\tau\,\eta) ,$$

for a pair $\xi,\eta \in T_x\Sigma$. Inserting (1.11) into the right-hand side, and observing $\omega(X_H, d\psi\,\xi) = \omega(X_H, d\psi\,\eta) = 0$ as $d\psi(T_x\Sigma) = T_x\Sigma \subset T_xS$, we find

$$\omega(\xi,\eta) = \omega(d\psi\,\xi,\ d\psi\,\eta) ,$$

proving that ψ is canonical with respect to $\omega|_\Sigma$.

(f) Further Degenerations.

As a side remark we observe that for Hamiltonian systems an additional integral of X_H frequently gives rise to two Floquet multipliers equal to 1 not just one. In this case we have no general existence proof for periodic orbits.

Lemma 1.3. Suppose X_{H_1} has r integrals H_α, $\alpha = 1,2,\dots,r$, i.e.

$$\{H_1,H_\alpha\} = 0$$

and dH_α independent. Assume, in addition, that at $p \in p(t)$

(1.12) $$\{H_\alpha,H_\beta\}(p) = 0 , \quad \text{for } \alpha= 1,2,\dots r \text{ and } \beta =1,2,\dots s.$$

Then the periodic orbit $p(t)$ has r+s Floquet multipliers equal to 1.

Proof: Introducing locally in p the integrals as coordaintes, we have in the notation previous to Lemma 1.2

$$\Phi = \left(\begin{array}{c|c} I_r & 0 \\ \hline * & \hat{\Phi} \end{array} \right)$$

where $\hat{\Phi} = \Phi | T_p S$, S being the integral surface of p(t) with tangent space at p:

$$T_p S = \left\{ \xi \in T_p M \mid \omega(X_{H_\alpha}, \xi) = 0, \ \alpha = 1,2,\dots,r \right\}$$

It suffices to prove that $\hat{\Phi}$ has s eigenvalues 1 belonging to the eigenvectors X_{H_α} , $\alpha = 1,2,\dots,s$. Since

$$[X_{H_\alpha}, X_{H_1}] = X_{\{H_1, H_\alpha\}} = 0$$

we have (Exercise 3 I §1)

$$\phi^t \circ \psi^s = \psi^s \circ \phi^t$$

where ϕ^t is the flow of X_{H_1} and ψ^s that of X_{H_α}. Differentiating in s at s = 0, gives

$$d\phi^t \, X_{H_\alpha} = X_{H_\alpha}(\phi^t) \ ,$$

hence for t = T

$$\Phi \, X_{H_\alpha}(p) = X_{H_\alpha}(p) \ , \qquad p \in p(t) \ ,$$

for every $\alpha = 1,2,\dots,r$. It remains to prove that $X_{H_\beta}(p) \in T_p S$, for $\beta = 1,2,\dots,s$. But this is precisely our assumption (1.12), since at p:

$$\omega(X_{H_\alpha}, X_{H_\beta}) = \{H_\alpha, H_\beta\} = 0 \ .$$

We observe that, as

$$T_p S = \langle X_{H_1}, \dots, X_{H_r} \rangle^{\perp} \ ,$$

the case r=s occurs precisely if $T_p S^{\perp} \subset T_p S$ hence $T_p S \subset T_p M$ is a coisotropic subspace.

Exercise 1 Show that the system

$$\dot{x} = (1 + \mu)y + (1 - x^2 - y^2)x$$

$$\dot{y} = -(1 + \mu)x + (1 - x^2 - y^2)y$$

possesses only one periodic solution aside from the equilibrium solution; show that its period depends on μ.

Exercise 2 Let X_j , $j = 1,2$ be vector fields on M_j having the periodic orbits $p_j(t)$. Show that if these vector fields are C^k-equivalent (in the extended sense) locally in open neighborhoods of the periodic orbits, where $1 \le k \le \infty$, or $k = \omega$, then the section maps ψ_j , $j = 1,2$ of $p_j(t)$ are C^k-conjugate locally near their fixed points.

Remark: The converse is also true in the differentiable case.

We claim that every vector $v \in T_xS$ which is orthogonal to T_xS in the symplectic sense, i.e. which is in $T_xS^{\perp}$ must be proportional to X_H. In fact from the property $\dim E + \dim(E^{\perp}) = \dim V$ for a subspace E of a symplectic vector-space V (section 3 of chapter I) we see that

$$\dim T_xS + \dim T_xS^{\perp} = \dim T_xM = 2n$$

and since $\dim T_xS = 2n-1$ we conclude that $\dim T_xS^{\perp} = 1$. Since $T_xS^{\perp}$ contains X_H by the above characterization of T_xS, the claim follows.

to insert: Chapter II section 1
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2. A Theorem by Liapunov

(a) Formulation of the theorem.

We shall use Poincaré's perturbation method in order to find periodic solutions of a C^1-vector field near an equilibrium point. We study in R^n

$$\dot{x} = f(x), \qquad f \in C^1. \tag{2.1}$$

with $f(0) = 0$ and

$$\frac{\partial f}{\partial x}(0) = A \quad \text{nonsingular,}$$

so that

$$\dot{x} = A\,x \tag{2.2}$$

is the linearized system. Clearly the presence of purely imaginary eigenvalues of A is necessary to find periodic solutions of (2.2) in a small neighborhood of 0. If A has the eigenvalues

$$\alpha_1, \alpha_2, \dots, \alpha_n,$$

not necessarily distinct from each other, we shall assume

$$\alpha_1 = i\omega, \quad \alpha_2 = -i\omega$$

is a pair of simple imaginary eigenvalues, with eigenvector

$$A(e_1 + ie_2) = i\omega(e_1 + ie_2) \tag{2.3}$$

such that (2.2) has the family of periodic solutions

$$x(t) = \mathrm{Re}\left\{c(e_1 + ie_2)\, e^{i\omega t}\right\} \text{ of period } \frac{2\pi}{\omega},$$

filling out the plane $E = \mathrm{span}\,(e_1, e_2)$. We want to find a family of periodic solutions for the system (2.1) close to these. However, in general, such periodic solutions need not exist as the system

$$\dot{x}_1 = -x_2 + r^2 x_1$$
$$\dot{x}_2 = x_1 + r^2 x_2 \ , \quad r^2 = x_1^2 + x_2^2$$

shows. Indeed for every solution $x(t)$ we have

$$\frac{d}{dt}\frac{1}{2}|x|^2 = |x|^4$$

and therefore, if $x(t)$ is a periodic solution of period T, one has

$$0 = \int_0^T |x(t)|^4 \, dt \ ;$$

hence $x(t) = 0$ is the only periodic solution.

In order to find periodic orbits the class of vector fields under consideration has therefore to be restricted. We shall assume the existence of an integral $G \in C^2$, i.e.

(2.4) $$\langle G_x, f\rangle = 0 \ ,$$

with

$G|E$ having a nonvanishing Hessian at $x = 0$.

Another possible restriction would be the class of reversible vector fields (see paragraph 5). We first examine the form of such an integral. Differentiating (2.4) we find for every $\xi \in R^n$

(2.5) $$\langle G_{xx}\xi, f(x)\rangle + \langle G_x(x), f_x(x)\xi\rangle = 0 \ ,$$

hence, as $f(0) = 0$, the equation (2.5) implies $\langle G_x(0), A\xi\rangle = 0$ for all ξ, and therefore $G_x(0) = 0$ as A is nonsingular. Next, let

$$(d^2G)(\xi) = \langle Q\xi, \xi\rangle, \quad Q = Q^T = G_{xx}(0)$$

be the Hessian of G at $x = 0$, we can differentiate (2.5) at $x = 0$ and find

$$\langle Q\xi, A\xi\rangle = 0$$

for every $\xi \in R^n$ or $A^TQ + QA = 0$. Restricting the form to E, by setting $\xi = \xi_1 e_1 + \xi_2 e_2$ we compute

$$0 = \langle Q\xi, A\xi\rangle = \omega\{-\xi_1^2\langle Qe_1, e_2\rangle + \xi_1\xi_2[\langle Qe_1, e_1\rangle - \langle Qe_2, e_2\rangle] + \xi_2^2\langle Qe_2, e_1\rangle\} \ ,$$

i.e. $\langle Qe_1, e_2\rangle = 0$ and $\langle Qe_1, e_1\rangle = \langle Qe_2, e_2\rangle = \rho$ and therefore, with $\xi = \xi_1 e_1 + \xi_2 e_2 \in E$,

(2.6) $$\langle Q\xi, \xi\rangle = \rho(\xi_1^2 + \xi_2^2) \ ,$$

so that by assumption $\rho \neq 0$. Thus $d^2G|_E$ is either positive or negative definite.

Theorem 2.1. Let $\alpha_1 = i\omega$, $\alpha_2 = -i\omega \neq 0$ be a pair of imaginary eigenvalues of A and E the real eigenspace belonging to α_1, α_2. If the other eigenvalues of A, α_k, $k = 3, \ldots, n$, satisfy

$$\frac{\alpha_k}{\alpha_1} \neq \text{integer} \ ,$$

and if G is a C^2-integral with $d^2G|_E \neq 0$, e.g. positive definite, then for every small $\varepsilon > 0$ there exists a unique periodic solution $x = p(t,\varepsilon)$ near E of period near $2\pi/\omega$, lying on

$$G(x) - G(0) = \varepsilon^2.$$

We observe that the nonresonance condition $\alpha_k/\alpha_1 \neq$ integer, $k \geq 3$ requires that the plane E contains all periodic solutions of $\dot{x} = Ax$ having period $T = 2\pi/\omega$.

(b) Proof

We write (2.1) in the form

$$\dot{x} = Ax + \hat{f}(x) \; , \quad \frac{\partial}{\partial x} \hat{f}(0) = 0 \; ,$$

and stretch the variables, $x = \varepsilon y$, $\varepsilon > 0$, so that

$$\dot{y} = \varepsilon^{-1} f(\varepsilon y) = Ay + \varepsilon^{-1}\hat{f}(\varepsilon y)$$

(2.7)

$$= Ay + g(y,\varepsilon) \; ,$$

where

$$\max_{|y|<C} \left(|g| + |g_y| \right) \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0 \; .$$

This allows us to consider the vector field in the fixed domain $|y| < C$ for some $C > 0$. We will apply Theorem 2.1 where ε plays the role of μ. The system (2.7) has the integral

$$F(y,\varepsilon) = \varepsilon^{-2}\big(G(\varepsilon y) - G(0)\big)$$

$$= \frac{1}{2} \langle Qy,y \rangle + \hat{F}(y,\varepsilon) \; ,$$

where on $|y| < C$, $|\hat{F}(\varepsilon)|_{C^2}$ tends to zero as $\varepsilon \to 0$, where $\hat{F}(\varepsilon)(y) = \hat{F}(y,\varepsilon)$. We now apply Poincaré's continuation method described by Theorem 1.2 to the fixed integral value

$$F(y,\varepsilon) = 1 \; ,$$

which corresponds to $G(x) - G(0) = \varepsilon^2$. As reference solution for $\varepsilon = 0$ we take

$$y(t) = \text{Re}\,\{c(e_1 + ie_2)\, e^{i\omega t}\}$$

with period $T = 2\pi/\omega$, where $c > 0$ is uniquely determined such that $F(y(t),0) = 1$. Also on E, if $\varepsilon = 0$

$$F_y = Qy = \eta_1 Qe_1 + \eta_2 Qe_2 \neq 0$$

if $y = \eta_1 e_1 + \eta_2 e_2 \neq 0$ as follows from (2.6). Next we check the Floquet multipliers, which by Lemma 1.1 are determined as the eigenvalues of

$$e^{AT} = e^{2\pi A\omega^{-1}}$$

i.e.

$$\lambda_k = e^{2\pi\alpha_k/\omega} = e^{2\pi i\alpha_k/\alpha_1} \; , \quad k = 1,2,\ldots,n.$$

Thus under our assumptions $\lambda_1 = \lambda_2 = 1$, but $\lambda_k \neq 1$ for $k \geq 3$, hence there are precisely two Floquet multipliers equal to 1 and Thereem 1.2 is applicable. It defines a unique differentiable 1-parameter family of periodic orbits $y(t,\varepsilon)$, $\varepsilon > 0$ on $F(y,\varepsilon) = 1$ of periods near $2\pi/\omega$, moreover $y(t,\varepsilon) \to y(t) = y(t,0)$ as $\varepsilon \to 0$,. We therefore find the periodic solutions $x(t,\varepsilon) = \varepsilon y(t,\varepsilon)$ on $G(x) - G(0) = \varepsilon^2$ as we have claimed.

(c) Remarks.

In case there are s distinct pairs of purely imaginary eigenvalues

$$\alpha_j = \pm\, i\, \omega_j \; , \qquad j = 1,2,\ldots,s,$$

we find s distinct families of periodic solutions having periods close to $2\pi/\omega_j$ provided the nonresonance conditions

$$\frac{\alpha_k}{\alpha_j} \neq \text{integer}$$

hold for all $k \neq j$ and $j = 1,2,\ldots,s$. These families are indeed distinct as follows from the local uniqueness statement of Theorem 1.2. Also, if the purely imaginary eigenvalues of A are merely distinct from each other, the theorem gives always *one* family of periodic orbits, namely the one corresponding to $\max\{|\alpha_j|, \alpha_j + \bar{\alpha}_j = 0\}$, i.e. the one with shortest period. The other families need not exist as the following example for n = 4 shows:

$$\dot{z}_1 = i(jz_1 + \bar{z}_2^j)$$
$$\dot{z}_2 = i(-z_2 + j\,\bar{z}_1\,\bar{z}_2^{j-1}), \qquad j = 2,3,\ldots,$$

where in complex notation $z_k = x_k + iy_k$. The function

$$G = j|z_1|^2 - |z_2|^2$$

is an integral with d^2G nondegenerate. Moreover

$$\frac{d}{dt}\,\mathrm{Im}\,(z_1\,z_2^j) = \left(|z_2|^2 + j^2|z_1|^2\right)|z_2|^{2j-2} \geq 0,$$

so that periodic solutions have $z_2 = 0$, and therefore $z_1 = c\,e^{ijt}$, $z_2 = 0$ gives the short period solutions.

If the nonresonance conditions are violated, no periodic solutions except the equilibrium point need exist. An example is given by the Hamiltonian system, with the complex notation $z_k = x_k + iy_k$:

$$H(z_1,z_2,\bar{z}_1,\bar{z}_2) = \frac{1}{2}\left(|z_2|^2 - |z_1|^2\right) + \left(|z_1|^2 + |z_2|^2\right)\mathrm{Re}(z_1 z_2).$$

For the corresponding Hamiltonian equations $\dot{x}_k = H_{y_k}$, $\dot{y}_k = -H_{x_k}$, or in complex notation

$$\dot{z}_k = -2i\,\frac{\partial H}{\partial \bar{z}_k}$$

one computes readily

$$-\frac{d}{dt}\,\mathrm{Im}\,(z_1\,z_2) = 2\left(\mathrm{Re}\,(z_1\,z_2)\right)^2 + 2|z_1|^2|z_2|^2 + \left(|z_1|^2 + |z_2|^2\right)^2$$
$$\geq 4\left(\mathrm{Re}\,(z_1\,z_2)\right)^2 + \left(|z_1|^2 + |z_2|^2\right)^2,$$

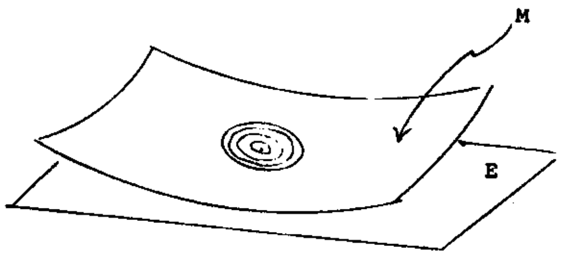
and since the right-hand side is positive for $(z_1,z_2) \neq (0,0)$, we conclude that $z_1 = z_2 = 0$ is the only periodic solution of this system. We remark that the Hessian of H at 0 is indefinite in this example. In contrast, a recent theorem due to A. Weinstein(*) states that if the Hamiltonian function on R^{2n} is definite at the equilibrium point, e.g. positive definite, then on every energy surface $H(z) - H(0) = \varepsilon^2 > 0$ there are n periodic solutions with periods close to the periods of the linearized system. No nonresonance conditions are required, but instead $H_{xx}(0)$ is required to be definite.

The proof of Theorem 2.1 shows that the periodic solutions $x = p(t,\varepsilon)$ having periods $T = T(\varepsilon)$ fill out a two dimensional continuously embedded disc, where $0 \leq \varepsilon \leq \varepsilon_0$ plays the role of a radius, and where $p(t,0) = 0$ corresponds to the equilibrium point. This embedding, given by

$$(s,\varepsilon) \rightarrow p(sT,\varepsilon) \in R^n,$$

(*) A. Weinstein, Normal Modes for nonlinear Hamiltonian systems, Inv. Math. 20, 1973, 47-57.

s (mod 1) $\in S^1$, $0 \le \varepsilon \le \varepsilon_0$ and $T = T(\varepsilon)$ is C^1 for any annulus $\varepsilon_1 \le \varepsilon \le \varepsilon_0$, if $\varepsilon_1 > 0$. But the differentiability at $\varepsilon = 0$ is not assured since s,ε play the role of polar coordinates. Actually, if $f \in C^{r+1}$ one can show that the periodic solutions found fit together into a two dimensional embedded invariant manifold M which is C^r also at the origin $x = 0$, and which is tangent to E at $x = 0$, if $r \ge 1$.



In case f and the integral G are analytic, the embedding of M is even analytic, represented by convergent power series, as was shown by C. L. Siegel. (1)

(1) C. L. Siegel and J. K. Moser: Lectures on Celestial Mechanics, Springer, 1971, p. 104-1..

3. A Theorem by E. Hopf.

(a) Statement of the Theorem.

We consider a family of vector fields in R^n

$$\dot{x} = f(x,\mu) , \qquad f \in C^2 ,$$

depending on a parameter μ, $|\mu| \le \mu_0$, and defined on an open neighborhood of a common equilibrium point, $x = 0$, such that

$$\dot{x} = A(\mu)x + \hat{f}(x,\mu) , \tag{3.1}$$

with $\hat{f}(0,\mu) = 0$ and $\frac{\partial}{\partial x}\hat{f}(0,\mu) = 0$.

We shall assume $R^n = E + F$, with dim $E = 2$, such that $x = x_E + x_F$, and require that for $\mu = 0$, $A(0)$ leaves this splitting invariant, i.e.

$$A(0) = \begin{pmatrix} A_E(0) & 0 \\ 0 & A_F(0) \end{pmatrix} .$$

We assume that $A_E(0)$ has a pair of imaginary eigenvalues $\alpha_1(0), \alpha_2(0) = -\alpha_1(0)$,

$$\alpha_1(0) = i\,\beta(0) , \qquad \beta(0) > 0 ,$$

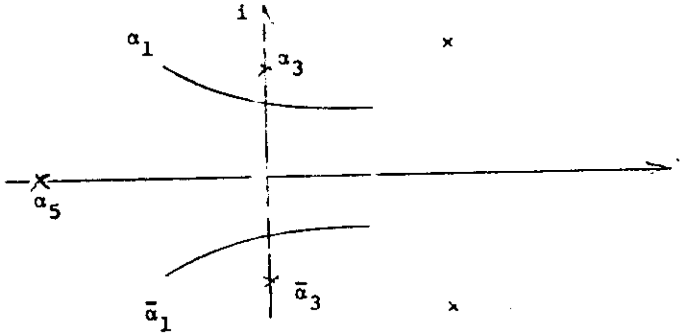
such that the plane E is filled with periodic solutions having period $T = \frac{2\pi}{\beta(0)}$ for the system $\dot{x} = A(0)x$. For the eigenvalues $\alpha_3(0),\ldots,\alpha_n(0)$ of $A_F(0)$ we require

$$\frac{\alpha_k(0)}{\alpha_1(0)} \ne \text{integer}, \quad k \ge 3. \tag{3.2}$$

The eigenvalues $\alpha_1(\mu), \alpha_2(\mu) = \overline{\alpha_1(\mu)}$ being simple depend differentiably on μ, and we require in addition

$$\mathrm{Re}\,\frac{d\alpha_1}{d\mu}(0) \neq 0 \ ,$$

i.e. the pair $\alpha_1, \bar{\alpha}_1$ of complex conjugate eigenvalues of $A(\mu)$ crosses the imaginary axis at $\mu = 0$ from the right or the left.



Summarizing the conditions on

$$\alpha_1(\mu) = \alpha(\mu) + i\beta(\mu) \ , \qquad \alpha, \beta \text{ real},$$

we require

(3.3) $$\begin{cases} \alpha(0) = 0 \ , & \beta(0) \neq 0 \\ \alpha'(0) \neq 0 \ , & \end{cases}$$

Theorem 3.1 Under the assumptions (3.2) and (3.3) there exists a continuous family of periodic solutions in the (x,μ) space near E

$$x = u(t,\varepsilon) \ , \quad \mu = v(\varepsilon) \ ,$$

$0 \leq \varepsilon \leq \varepsilon_0$ having periods $T = T(\varepsilon)$ near $2\pi/\beta(0)$ and satisfying $u(t,\varepsilon) \to u(t,0) = 0$, $v(\varepsilon) \to v(0) = 0$ and $T(\varepsilon) \to T(0) = 2\pi/\beta(0)$ as $\varepsilon \to 0$. This family of periodic solutions is C^1 for $0 < \varepsilon \leq \varepsilon_0$ and fills out an embedded two dimensional disc through $(0,0)$ in the (x,μ)-space.

We explain the role of the auxiliary parameter ε. Since $\alpha_1, \alpha_2 = \bar{\alpha}_1$ are simple eigenvalues, there is a differentiable splitting $R^n = E(\mu) + F(\mu)$ which is left invariant by $A(\mu)$, and we can achieve by a linear coordinate transformation depending on the parameter μ, that $E(\mu) = E(0) = E$ and $F(\mu) = F(0) = F$, such that

(3.4) $$A(\mu) = \begin{pmatrix} A_E(\mu) & 0 \\ 0 & A_F(\mu) \end{pmatrix} .$$

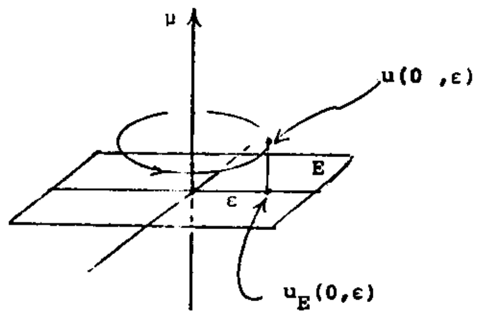
Moreover, by another change of coordinates in E, depending differentiably on μ, we can achieve

(3.5) $$A_E(\mu) = \begin{pmatrix} \alpha(\mu) & -\beta(\mu) \\ \beta(\mu) & \alpha(\mu) \end{pmatrix} ,$$

where $\alpha(\mu) + i\beta(\mu) = \alpha_1(\mu)$. With respect to these coordinates in $R^n = E + F$ we shall show that the auxiliary parameter ε agrees with

(3.6) $$u_E(0,\varepsilon) = (\varepsilon, 0) ,$$

and it will follow from the proof that with this normalization of the parameter ε the solution is unique.



Postponing the proof we first look at the simple model case in R^2

$$\dot{z} = f(z,\mu) = (i + \mu - |z|^2)\, z ,$$

where $\alpha_1(\mu) = i + \mu$, or in real notation $z = x + iy$,

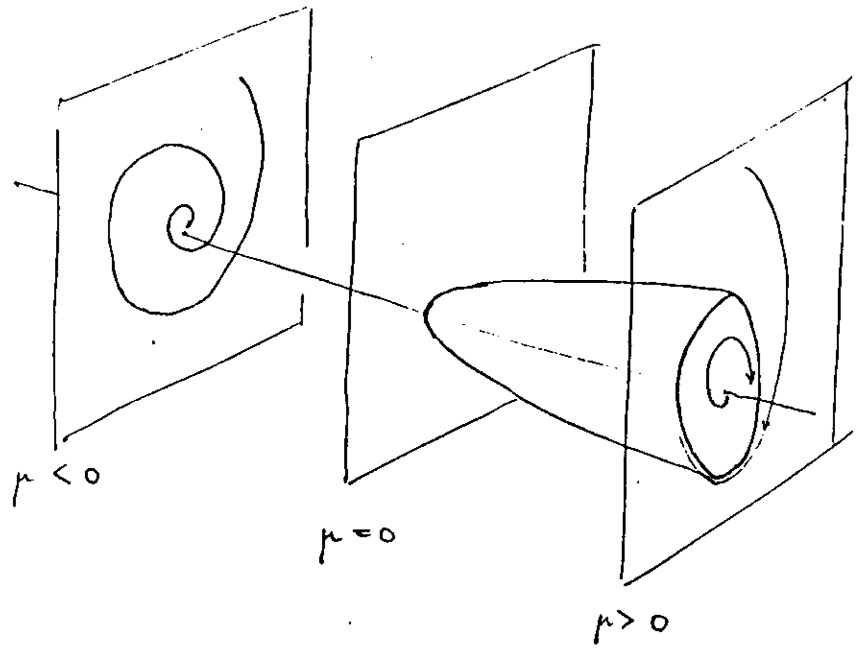
$$\dot{x} = \mu x - y - (x^2 + y^2)x$$

$$\dot{y} = x + \mu y - (x^2 + y^2)y .$$

For $\mu > 0$ we find the periodic solution on $|z| = \sqrt{\mu}$ with period 2π given by

$$z(t) = \sqrt{\mu}\, e^{it} .$$

For $\mu < 0$ the equilibrium point $z = 0$ is an attractor and there are no periodic solutions, while as μ increases past 0, an attracting periodic orbit $|z| = \sqrt{\mu}$ appears. The periodic orbits fill out a smooth two dimensional surface M in the (x,μ)-space tangent to the $\mu = 0$ plane at $z = 0$.



Changing the sign in front of $|z|^2$ in the equation leads to an invariant surface in $\mu < 0$.

(b) Proof of Theorem 3.1.

We assume that in $R^n = E + F$, $A(\mu)$ is already in the form (3.4) and (3.5). Setting $x = \varepsilon y$, $\varepsilon > 0$, the system becomes

$$\dot{y} = A(\mu)y + g(y,\varepsilon,\mu) ,$$

where $g_\varepsilon(y,\mu) = g(y,\varepsilon,\mu)$ satisfies

$$|g_\varepsilon|_{C^1} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

on $|y| \leq 2$ and $|\mu| \leq \mu_0$. From now on we will denote any such function of y, ε and μ as $\omega_1(\varepsilon)$. Next we introduce polar coordinates r, θ in E and abbreviate $w = y_F$. On the domain D given by $\frac{1}{2} < r < \frac{3}{2}$, $0 \leq \theta \leq 2\pi$ and $|w| < \frac{1}{2}$ the system is then given by

$$\begin{aligned} \dot{r} &= \alpha r + \omega_1(\varepsilon) \\ \dot{\theta} &= \beta \ + \omega_1(\varepsilon) \\ \dot{w} &= Bw + \omega_1(\varepsilon) , \end{aligned}$$

with $\alpha = \alpha(\mu)$, $\beta = \beta(\mu)$ and $B = A_F(\mu)$. Since $\beta(0) \neq 0$, the plane $\theta = 0$ is a transversal section of the vector field in D for μ and ε sufficiently small. The aim is to find fixed points of the section map. Introducing the angle variable θ as independent variable and integrating the nonautonomous system

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{\alpha}{\beta} r + \omega_1(\varepsilon) \\ \frac{dw}{d\theta} &= \frac{1}{\beta} Bw + \omega_1(\varepsilon) \end{aligned}$$

from $\theta = 0$ to $\theta = 2\pi$, we find for the section map:

$$\begin{pmatrix} r \\ w \end{pmatrix} \to \begin{pmatrix} e^{2\pi\frac{\alpha}{\beta}} r + \omega_1(\varepsilon) \\ e^{\frac{2\pi}{\beta}B} w + \omega_1(\varepsilon) \end{pmatrix} ,$$

and it remains to solve the following two equations:

$$\begin{aligned} F_1(r,w,\varepsilon,\mu) &= e^{2\pi\frac{\alpha}{\beta}} r - r + \omega_1(\varepsilon) = 0 , \\ F_2(r,w,\varepsilon,\mu) &= e^{\frac{2\pi}{\beta} B} w - w_1 + \omega_1(\varepsilon) = 0 . \end{aligned}$$

We shall first solve the second equation and observe that the eigenvalues of

$$e^{\frac{2\pi}{\beta} B}$$

are, for $\mu = 0$,

$$\lambda_k = e^{\frac{2\pi}{\beta(0)} \alpha_k(0)} = e^{2\pi i \frac{\alpha_k(0)}{\alpha_1(0)}} \neq 1 ,$$

$k \geq 3$, due to the nonresonance assumption (3.2). Therefore $\frac{\partial}{\partial w} F_2$ is nonsingular and there is a unique solution $w = h(r,\varepsilon,\mu)$ of the second equation, such that $h = 0$ for $\varepsilon = 0$. We now set

$$r = 1$$

and insert $w = h(1,\varepsilon,\mu)$ into F_1, giving $F_1(1,h,\varepsilon,\mu) = \Phi(\varepsilon,\mu)$. Since

$$\frac{\partial}{\partial\mu}\,\Phi = \frac{\partial}{\partial\mu}\,F_1 = 2\pi\,\frac{\alpha'(0)}{\beta(0)} \neq 0$$

if $\varepsilon = 0$ and $\mu = 0$, due to the assumption (3.3), we can solve the first equation uniquely for $\mu = v(\varepsilon)$ with $v(0) = 0$.

From

$$\frac{d\theta}{dt} = \beta + \omega_1(\varepsilon)$$

we finally find the periods

$$T(\varepsilon) = \frac{2\pi}{\beta(0)} + o(1) ,$$

as we wanted to prove.

(c) Liapunov's Theorem.

It may happen that the whole family of periodic solutions in Theorem 3.1 is contained in the $\mu = 0$ plane. Indeed, if $\dot{x} = f(x)$ is a system satisfying the assumptions of Liapunov's theorem with the integral $G \in C^2$, we define the family

$$\dot{x} = f(x) + \mu\, G_x(x) . \tag{3.7}$$

One verifies readily that this system meets the assumptions (3.2) and (3.3) of Theorem 3.1. We therefore have a family of periodic solutions $x = u(t,\varepsilon)$ and $\mu = v(\varepsilon)$ for (3.7). We claim that $v(\varepsilon) = 0$, such that the family belongs to $\dot{x} = f(x)$. In fact, along a periodic solution of period T of (3.7),

$$\frac{d}{dt}\, G = \langle G_x, f\rangle + \mu |G_x|^2 = \mu |G_x|^2 ,$$

since $\langle G_x, f\rangle = 0$ as G is an integral of f. Thus

$$0 = \mu \int_0^T |G_x|^2\, dt$$

and therefore $\mu = 0$, since $G_x \neq 0$ along a nontrivial periodic solution. This shows that Liapunov's theorem can be viewed as a special case of Hopf's theorem.

4. The restricted three body problem

(a) The three body problem.

We apply the ideas of the previous sections to construct periodic solutions for the three body problem. Historically these methods were developed by Poincaré for just this application.

If m_k, $k = 1,2,3$ are the masses and $x_k \in R^3$ the positions of the three mass points, then the differential equations are given by

$$m_k \frac{d^2 x_k}{dt^2} = - \frac{\partial U}{\partial x_k}, \qquad -U(x) = \sum_{1 \le i < j \le 3} \frac{m_i m_j}{|x_i - x_j|}$$

or, dividing by m_k

$$\frac{d^2 x_k}{dt^2} = \sum_{j \ne k} \frac{m_j (x_j - x_k)}{|x_j - x_k|^3}, \quad k = 1,2,3. \tag{4.1}$$

In the planar three body problem we require $x_k \in R^2$, i.e. x_k to be two vectors.

The above system is a Hamiltonian system with

$$H = \sum_{j=1}^{3} \frac{|y_j|^2}{2m_j} + U(x), \quad y_j = m_j \dot{x}_j$$

and the phase space has 18 or 12 dimensions, in the spacial or planar case.

(b) The restricted three body problem is a limit case of the planar three body problem in which we assume one of the masses to be zero, e.g. $m_3 = 0$ in (4.1). Then the first two equations become independent of x_3:

$$(4.2) \quad \begin{cases} \dfrac{d^2 x_1}{dt^2} = m_2 \dfrac{x_2 - x_1}{|x_2 - x_1|^3} \\[2ex] \dfrac{d^2 x_2}{dt^2} = m_1 \dfrac{x_1 - x_2}{|x_1 - x_2|^3} \end{cases}$$

and describe the motion of the two body problem. Its orbits are the well known conic sections; we require, however, that x_1, x_2 move on *circles* about their center of gravity. Thus the restricted three body problem is characterized by the two requirements (i) $m_3 = 0$ and (ii) the mass points x_1, x_2 move on circular orbits. The problem is to describe the motion of x_3.

To derive the equation it is handy to use complex notation: If $x_k = (x_k^1, x_k^2) \in R^2$ we will set

$$z_k = x_k^1 + i\, x_k^2$$

to describe the position of the k^{th} mass point. Moreover, we normalize the total mass of the system to be one and set

$$m_1 = 1 - \mu, \quad m_2 = \mu \quad \text{with} \quad 0 \le \mu \le 1 .$$

From (4.2) we find for the center of mass

$$z_0 = m_1 z_1 + m_2 z_2$$

the equation $\ddot{z}_0 = 0$ and we may assume that

$$z_0 = m_1 z_1 + m_2 z_2 = 0 .$$

Then the motion of z_1, z_2 is described by $\zeta = z_2 - z_1$ which satisfies

$$\ddot{\zeta} = - \frac{\zeta}{|\zeta|^3} .$$

The assumption that x_1, x_2 move on circles amounts to taking the solutions

$$\zeta = a\, e^{i\alpha t}, \qquad \alpha^2 a^3 = 1$$

of this equation, or equivalently

$$z_1 = w_1\, e^{i\alpha t}, \qquad z_2 = w_2\, e^{i\alpha t},$$

where

(4.3) $$w_1 = -\mu a, \qquad w_2 = (1-\mu)a .$$

Here $\alpha \neq 0$ and $a = \alpha^{-2/3} > 0$. Usually one normalizes also $a = |z_2 - z_1|$ to be one so that $\alpha = \pm 1$, but we prefer to show the dependence on α and therefore forego this normalization.

The differential equation for the third mass point described by z_3 is t-dependent since it involves the distance of z_3 from the circling $z_1 = z_1(t)$, $z_2 = z_2(t)$. But in a rotating coordinate system given by

$$z_3 = w\, e^{i\alpha t}$$

these mass points will be at rest and the differential equations become

(4.4) $$\ddot{w} + 2\, i\alpha\, \dot{w} - \alpha^2 w = \mu \frac{w_2 - w}{|w_2 - w|^3} + (1-\mu) \frac{w_1 - w}{|w_1 - w|^3},$$

where w_1, w_2 are given by (4.3).

In real notation

$$w = u + i\, v$$

these equations become

(4.5) $$\begin{cases} \ddot{u} - 2\alpha\, \dot{v} - \alpha^2 u = V_u \\ \ddot{v} + 2\alpha\, \dot{u} - \alpha^2 v = V_v \\ V = \dfrac{\mu}{|w_2 - w|} + \dfrac{1-\mu}{|w_1 - w|}, \end{cases}$$

which are the differential equations of the restricted three body problem. The phase space is 4 dimensional and the system is independent of t. The system can also be written in Hamiltonian form

$$\dot{x}_j = H_{y_j}, \qquad \dot{y}_j = -H_{x_j},$$

(4.6) $$H(x,y,\mu) = \frac{1}{2}|y|^2 + \alpha(x_2 y_1 - x_1 y_2) - V(x,\mu),$$

where

$$x_1 = u, \qquad y_1 = \dot{u} - \alpha v,$$
$$x_2 = v, \qquad y_2 = \dot{v} + \alpha u.$$

(c) Periodic Solutions of the First Kind.

We will construct periodic solutions of the system (4.4) for small values of μ which we write in the form

(4.7) $$\ddot{w} + 2i\alpha\dot{w} - \alpha^2 w + \frac{w}{|w|^3} = F(w, \bar{w}, \mu),$$

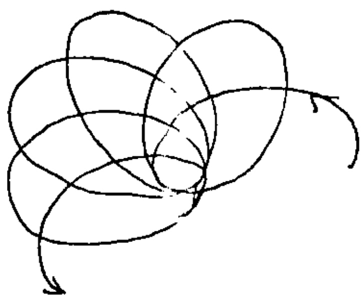
where F is a complex valued function vanishing for $\mu = 0$:

$$F(w, \bar{w}, 0) = 0 .$$

Moreover F is an analytic function of u,v except at the points $u + i\,v = w_1, w_2$.

In the application in astronomy one usually takes the mass point (m_1) to be the sun, (m_2) Jupiter and (m_3) an asteroid whose mass can be neglected. Moreover, the mass parameter μ is for this application 0.954×10^{-3}, and can be viewed as small.

For $\mu = 0$ the system (4.7) describes the Kepler problem in a rotating coordinate system. Therefore elliptical orbits generally give rise to nonperiodic orbits but precessing ellipses:



On the other hand circular orbits always give rise to periodic orbits

(4.8) $$w = r\,e^{i\beta t}$$

of (4.5) if

(4.9) $$(\beta + \alpha)^2\, r^3 = 1\,, \qquad \beta \neq 0\,.$$

Their period is $T = 2\pi\beta^{-1}$. Since $w_1 = 0$, $w_2 = \alpha^{-2/3}$ for $\mu = 0$ we avoid collisions if we require

$$(\beta + \alpha)^2 \neq \alpha^2$$

or

$$\beta \neq 0,\ -2\alpha\ .$$

From (4.9) we also have $\beta \neq -\alpha$, so that

$$\beta \neq 0,\ -\alpha,\ -2\alpha\ .$$

We shall apply the results of Section 1 to find periodic solutions of (4.7) for small values of μ. The resulting solutions correspond to closed orbits in the rotating coordinate system only and not in the original system. In that system we can assert only that the triangle formed by the three mass points will be congruent for time values t_1, t_2 differing by a period.

We also observe that the perturbation theory will not succeed for arbitrary perturbations $F = F(w,\bar{w},\dot{w},\dot{\bar{w}},\mu)$ in (4.7) vanishing for $\mu = 0$. Indeed if $F = \mu\,\dot{w}$ then one finds

$$\frac{d}{dt}\left(\frac{1}{2}\,|\dot{w}|^2 - \frac{\alpha^2}{2}\,|w|^2 - |w|^{-1}\right) = \mu\,|\dot{w}|^2\ .$$

Hence for a periodic orbit of period T we would have

$$0 = \mu \int_0^T |\dot{w}|^2\ .$$

This implies w = const. and leads only to uninteresting stationary solutions which by (4.7) lie on the circle $|w|^{-3} = \alpha^2$ and correspond to the excluded value $\beta = 0$ in (4.8).

Thus we need additional properties of F to ensure the existence of periodic solutions. In our case such a property is given in the existence of an integral. Since our system can be written in Hamiltonian form we have an integral H given by (4.6). It can be rewritten as

$$H = \frac{1}{2} |\dot{w}|^2 - \frac{\alpha^2}{2} |w|^2 - V$$

and becomes for $\mu = 0$

$$H_0 = \frac{1}{2} |\dot{w}|^2 - \frac{\alpha^2}{2} |w|^2 - |w|^{-1} . \tag{4.10}$$

The gradient of this function is certainly not zero if $\dot{w} \neq 0$.

Let us consider the integral surface $H_0 = -\frac{1}{2} C$ (C is called the Jacobi constant) and determine the reference solutions (4.8) on it. For this purpose we compute the value of C for the solution (4.8). By (4.9) we have

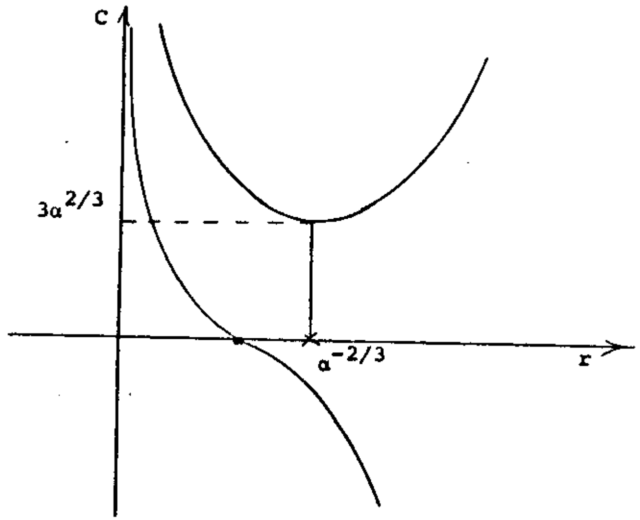
$$\beta + \alpha = \pm r^{-3/2}$$

and hence

$$\begin{aligned} C = -2H_0 &= -\beta^2 r^2 + \alpha^2 r^2 + 2r^{-1} \\ &= r^2\left(-\beta^2 + \alpha^2 + 2(\beta+\alpha)^2\right) \\ &= r^2(\beta + \alpha)(\beta + 3\alpha) \\ &= r^2\left[(\beta + \alpha)^2 + 2(\beta + \alpha)\alpha\right] \\ &= r^2(r^{-3} \pm 2\alpha r^{-3/2}) \\ &= r^{-1} \pm 2\alpha r^{1/2}. \end{aligned}$$

The function $r^{-1} - 2\alpha r^{1/2}$ is monotone decreasing for $\alpha > 0$

and $r^{-1} + 2\alpha r^{1/2}$ is decreasing for $0 < r < \alpha^{-2/3}$ and increasing for $r > \alpha^{-2/3}$.



Hence for all C there exists at least one periodic solution (4.8) on the energy surface and for $C > 3\alpha^{2/3}$ there are three. For $C = 3\alpha^{2/3}$ we have two periodic orbits one of which is excluded by (4.10) since it corresponds to a collision orbit.

The solutions so obtained are called periodic solutions of the first kind. They are characterized by the property that they tend to circular orbits for $\mu = 0$, while periodic solutions of the second kind issue from elliptic orbits.

Theorem 4.1. Let

$$w = p(t,C_0)$$

be the periodic solution (4.3) with

$$\beta \neq 0, -\alpha, -2\alpha$$

and

$$(4.11) \qquad \frac{\alpha}{\beta} \neq \text{integer} .$$

Then for sufficiently small μ and C near C_0 there exists a unique periodic solution

$$w = p(t,C,\mu)$$

of the restricted three body problem on the energy surface $H = -\frac{1}{2} C$ continuously depending on the parameters and tending to $p(t,C_0)$ as $\mu \to 0$ and $C \to C_0$.

Proof: We apply Theorem 1.2 and use that $dH \neq 0$ for $\mu = 0$ and on the reference orbits $w = p(t,C_0)$. For this purpose, we have to compute the Floquet multipliers of the reference solution. The variational equation of (4.4) for $\mu = 0$ along the solution (4.8) are given by

$$(4.12) \qquad \ddot{\eta} + 2\alpha i \dot{\eta} - \alpha^2 \eta = \frac{1}{2} (\alpha + \beta)^2 (\eta + 3e^{2i\beta t} \bar{\eta}) .$$

in complex notation. The eigenvalues of the fundamental matrix solution at $t = T = 2\pi\beta^{-1}$ can be computed to be

$$1 , 1 , e^{2\pi i\alpha/\beta} , e^{-2\pi i\alpha/\beta} .$$

Hence 1 is a double eigenvalue precisely if (4.11) holds.

Under this assumption Theorem 1.2 yields the statement.

(d) Discussion.

The above solutions are periodic only in the rotating coordinate system but not in the original coordinate system if $\mu \neq 0$. For $\mu = 0$ this is the case only if

$$\frac{\alpha}{\beta} \text{ is rational.}$$

To get a clearer picture of the motion in the resting coordinate system we set

$$\alpha_2 = \alpha \ , \qquad \alpha_3 = \alpha + \beta ,$$

so that the unperturbed orbits are

$$(4.13) \qquad z_1 = w_1 e^{i\alpha_2 t}, \quad z_2 = w_2 e^{i\alpha_2 t}, \quad z_3 = r e^{i\alpha_3 t},$$

and α_2, α_3 are the frequencies of the two "planets" z_2, z_3 in resting coordinates.

The condition (4.11) reads

$$(4.14) \qquad \frac{\alpha_2}{\alpha_3 - \alpha_2} \neq \text{integer.}$$

It can be interpreted as the condition that the solution (4.13) in the resting coordinate system should not have the period

$$T = \frac{2\pi}{\beta} = \frac{2\pi}{\alpha_3 - \alpha_2}$$

of the solution in the rotating system.

We constructed a one parameter family of periodic orbits, depending on the parameter C. The period $T = T(C,\mu)$ of this orbit depends on C and it is possible to prescribe the period of the orbit, provided

(4.15) $$\frac{dT}{dC} \neq 0 \quad \text{for } \mu = 0 \ .$$

We can then solve the equation

$$T(C,\mu) = T(C_0,0) = T_0$$

for $C = C(\mu)$ near C_0 and obtain a periodic solution with fixed period T_0. From the dependence of C on r and from (4.9) the inequality (4.15) is easily read off under our restrictions on β.

We give a short list of terms used in astronomy: The fixed coordinate system is called the "sidereal" system and the rotating one the "synodic" system. Usually α_2 is taken positive and then the orbit of z_3 is called "direct" if $\alpha_3 > 0$ and "retrograde" if $\alpha_3 < 0$. For retrograde orbits we have $\alpha_3/\alpha_2 < 0$ and hence the quotient

$$\frac{\alpha_2}{\alpha_3-\alpha_2} = \left(\alpha_3/\alpha_2 - 1\right)^{-1}$$

lies between −1 and 0 so that (4.14) is automatically satisfied for retrograde orbits. But in astronomy only the direct orbits are of interest and for these orbits (4.14) constitutes a real condition.

This approach is not applicable to the elliptical orbits of (4.7) for $\mu = 0$, since in that case all four Floquet multipliers are equal to 1. This is due to the fact that for $\mu = 0$ the system (4.7) has not only the energy integral H_0 given by (4.10) but also the angular momentum $u\dot{v} - \dot{u}v = \operatorname{Im}(\bar{w},\dot{w})$.

On the circular orbits the gradients of these two integrals are linearly dependent but on the elliptical one these integrals are independent. (See Exercise 1.) Even though this approach fails to establish periodic solutions near the elliptic orbits they do exist and will be discussed later; they are called periodic orbits of the second kind.

Exercise 1 Show that for $w \neq 0$, the gradients of H_0 given by (4.10) and of Im $(\bar{w}\,\dot{w})$ are linearly dependent if and only if

$$\dot{w} = i\lambda w\ , \qquad \lambda = |w|^{-3} - \alpha^2\ .$$

Interpret these points as initial conditions for circular orbits.

Exercise 2 Determine Floquet multipliers for the system (4.12)

Hint: The substitution

$$\eta = e^{-i\beta t}\,\zeta$$

transforms (4.12) into a system with constant coefficients:

$$\ddot{\zeta} + 2i(\alpha+\beta)\dot{\zeta} - (\alpha+\beta)^2\zeta = \frac{1}{2}(\alpha+\beta)^2(\zeta + 3\bar{\zeta})\ .$$

Determine the four characteristic exponents of this system as $0,\ 0,\ \pm(\alpha+\beta)$.

5. Reversible Systems

(a) Systems admitting a reflection.

In this section we shall describe a similar perturbation method for periodic solutions possessing a certain symmetry property. This method which also goes back to Poincaré may be applicable in a situation when that of Section 2 fails.

The systems

(5.1) $$\dot{x} = f(x)\ , \qquad x \in R^n$$

considered are required to admit a reflection ρ, which is a linear map of R^n satisfying

$$\rho^2 = I\ .$$

Thus ρ has the eigenvalues +1 and −1 and we require that they have the same multiplicity. Thus, if E_+ , E_- are the eigenspaces of the eigenvalues +1, −1 of ρ we can represent any $x \in R^n$ in the form

$$x = x_+ + x_-\ , \quad x_+ \in E_+\ ,\ x_- \in E_-\ .$$

and

$$\rho x = x_+ - x_-\ .$$

Since $\dim E_+ = \dim E_-$ the dimension n must be even.

We call the system (5.1) reversible with respect to ρ if

(5.2) $$f(\rho x) = -\rho\, f(x)\ .$$

In other words the system is invariant under the transformation

$$(t,x) \to (t^* = -t,\ x^* = \rho x) .$$

Condition (5.2) is equivalent to

$$\phi^t \circ \rho = \rho\, \phi^{-t} \tag{5.3}$$

for the flow ϕ^t of (5.1).

Any second order system $\ddot{u} = F(u)$ which does not contain $\dot{u}$ is an example of a reversible system. Indeed, if we write

$$x = \begin{pmatrix} u \\ \dot{u} \end{pmatrix}, \quad f = \begin{pmatrix} \dot{u} \\ F(u) \end{pmatrix}, \quad \rho x = \begin{pmatrix} u \\ -\dot{u} \end{pmatrix}$$

This system takes the form (5.1) which satisfies (5.2).

Also the restricted three body problem (4.5) admits such a symmetry. If we set

$$\rho(u,v,\dot{u},\dot{v}) = (u,-v,-\dot{u},\dot{v}) \tag{5.4}$$

then the system (4.5), if converted into a first order system, is reversible with respect to ρ. Here it is important that the points w_1, w_2 (i.e. the positions of the first two bodies) are on the real axis.

(b) Symmetric Solutions.

From (5.3) it is clear that with $x = x(t) = \phi^t(x(0))$ also $\rho x(-t)$ is a solution of our system. We call a solution symmetric (with respect to ρ) if

$$x(t) = \rho\, x(-t) . \tag{5.5}$$

In view of (5.3) symmetric solutions are characterized by the condition

$$x(0) = \rho\, x(0) \quad \text{or} \quad x(0) \in E_+ . \tag{5.6}$$

Lemma 5.1. A symmetric solution $x(t)$ has the period 2τ if and only if

$$y(t) = x(t + \tau)$$

is also a symmetric solution.

Proof: If y is symmetric, i.e. $y(t) = \rho\, y(-t)$ then

$$x(t + \tau) = \rho\, x(-t + \tau) .$$

On the other hand, by the symmetry of $x(t)$ we have

$$x(t - \tau) = \rho\, x(-t + \tau) ,$$

hence

$$x(t + \tau) = x(t - \tau) ,$$

i.e. $x(t)$ has period 2τ. Conversely if the symmetric solution has period 2τ then the last equation holds hence by the symmetry of $x(t)$

$$x(t + \tau) = x(t - \tau) = \rho\, x(-t + \tau) ,$$

which shows that $y(t) = x(t + \tau)$ is symmetric.

In other words a symmetric solution (of period 2τ) is characterized by

$$x(0) \in E_+ \quad \text{and} \quad x(\tau) \in E_+ . \tag{5.7}$$

We interpret this condition for the example of a second order system in the plane

$$\begin{aligned} \ddot{u} &= f(u,v,\dot{u},\dot{v}) \\ \ddot{v} &= g(u,v,\dot{u},\dot{v}) , \end{aligned} \tag{5.8}$$

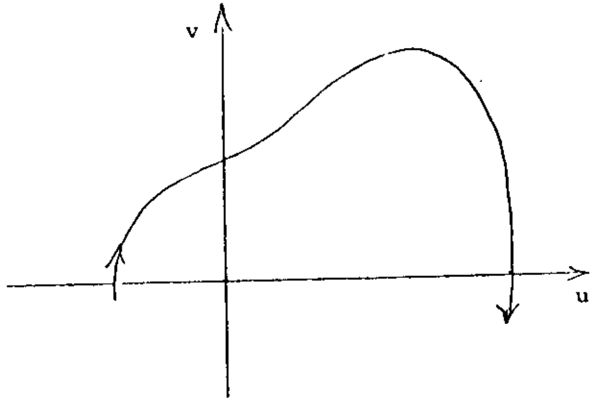
which is reversible with respect to ρ given by (5.4) if

$$f(u,-v,-\dot{u},\dot{v}) = f(u,v,\dot{u},\dot{v})$$

$$g(u,-v,-\dot{u},\dot{v}) = -g(u,v,\dot{u},\dot{v}) .$$

The condition (5.7) means in this example that for $t = 0$ and $t = \tau$ the orbit intersects the u-axis at a right angle, since E_+ is given by points $(u,v,\dot{u},\dot{v})$ satisfying

$$v = 0 , \quad \dot{u} = 0 .$$



It is clear from the figure that reflection gives rise to a periodic orbit, which is symmetric.

The restricted three body problem is an example of (5.8).

(c) Perturbation theory for symmetric periodic orbits.

We consider a reversible system depending on a parameter μ:

(5.9) $$\dot{x} = f(x,\mu) ,$$

where we assume for simplicity that ρ is independent of μ. We assume, moreover, that $x = x^*(t)$ is a symmetric periodic orbit of (5.9) for $\mu = 0$ and ask for a symmetric periodic solution of (5.9) for small values of μ.

For this purpose we have to determine the initial values so that (5.7) holds. Let $\phi^t(x,\mu)$ be the flow of $f = f(x,\mu)$ and

$$\Phi = d\phi^\tau .$$

Then Φ maps the tangent space at $x(0)$ into that at $\phi^\tau(x(0))$. The splitting

$$R^n = E_+ + E_-$$

gives rise to the representation

$$\Phi = \begin{pmatrix} \Phi_{++} & \Phi_{+-} \\ \Phi_{-+} & \Phi_{--} \end{pmatrix} .$$

The condition (5.7) requires finding $x = x_+(0) \in E_+$ so that

$$\phi^\tau(x_+,\mu) \in E_+$$

or

$$\left(\phi^\tau(x_+,\mu)\right)_- = 0 .$$

For $\mu = 0$ this condition holds for $x_+ = x^*(0)$. The implicit function theorem is applicable if

(5.10) $$\det(\Phi_{-+}) \neq 0$$

and gives a symmetric periodic solution of period 2τ for small $|\mu|$. At this point we have to require that $\dim E_- = \dim E_+$ so that the number of unknowns and equations match.

We can interpret condition (5.10): The two subspaces E_+ and $d\phi^\tau E_+$ must be transversal at the point $x^*(\tau)$ for $\mu = 0$, i.e. they must span R^n. Thus we have the following

Theorem 5.1. If $x = x^*(t) = x^*(t + 2\tau)$ is a symmetric periodic solution of (5.9) for $\mu = 0$ and if

$$E_+ \text{ and } d\phi^\tau E_+ \text{ are transversal}$$

at $x^*(\tau)$ for $\mu = 0$ then the system (5.9) possesses a unique symmetric periodic solution of the same period 2τ, which depends continuously on μ and agrees with $x^*(\tau)$ for $\mu = 0$.

(d) Symmetric Orbits of the Restricted Three-Body Problem.

We apply this theorem to the restricted three body problem (4.5) with the reflection ρ given by (5.4). The circular orbits (4.8) are clearly symmetric solutions for $\mu = 0$ and thus the above theorem is applicable provided E_+ and $d\phi^\tau E_+$ are transversal, or equivalently if

$$\det (\Phi_{-+}) \neq 0 \quad \text{for} \quad \tau = \pi \beta^{-1}.$$

The transversality condition has to be checked for $\mu = 0$, in which case the system reduces to the Kepler problem in a rotating coordinate system. The solutions are known explicitly and therefore it is in principle trivial to determine when the transversality condition holds. Nevertheless we will carry out the calculation. One may expect that $\det(\Phi_{-+})$ can be determined from the Floquet multipliers but this is not the case since Φ_{-+} represents a map from E_+ at $w(0), \dot{w}(0)$ to E_- at a different point, and therefore $\det(\Phi_{-+})$ is not invariantly defined. Only the vanishing

of this determinant has an invariant meaning. One can introduce different coordinates at these two points and thus replace Φ_{-+} by

$$T_1 \Phi_{-+} T_2 ,$$

where T_1, T_2 are the Jacobians of these coordinate transformations. We make use of this remark and introduce in E_+ instead of the coordinates $u, \dot{v}$ the semimajor axis a and the eccentricity ε which parametrize the relevant ellipses. Namely using exercise 4 of I §1 we can write the symmetric solutions in terms of the eccentric anomaly ψ as

$$u + iv = w = a\, e^{-i\alpha t}(\varepsilon + \cos\psi + i\sqrt{1-\varepsilon^2}\sin\psi)$$

$$|w| = a(1 + \varepsilon\cos\psi)$$

$$t = a^{3/2}(\psi + \varepsilon \sin\psi) .$$

The reference solution (4.8) with the half period $\tau = \pi\beta^{-1}$ corresponds to

$$a = r, \qquad \varepsilon = 0 .$$

One verifies easily that for $t = 0$, if $\varepsilon = 0$

$$\frac{\partial(u,\dot{v})}{\partial(a,\varepsilon)} = \frac{1}{2\sqrt{a}} \neq 0 , \tag{5.10}$$

and thus a, ε can indeed be used as coordinates in E_+. In the image space of ϕ^τ, $\tau = \pi\beta^{-1}$, we shall use instead of $(v,\dot{u})$ different coordinates for E_-, namely $A = \arg w$ and $B = \dot{R}$. This is motivated by the fact that for a point in E_+

one has orthogonal crossing, given by $\dot{R} = 0$, on the negative u-axis, given by $A = \pi$. From

$$v = R \sin A$$
$$\dot{u} = \dot{R} \cos A - \dot{A}R \sin A$$

one sees that, if $A = \pi$,

$$(5.11) \quad \frac{\partial(v,\dot{u})}{\partial(A,B)} = \det \begin{pmatrix} R \cos A & 0 \\ * & \cos A \end{pmatrix} = R \cos^2 A = |w| > 0 \ .$$

Thus A, B can be used as coordinates in E_-. It remains to determine

$$\frac{\partial(A,B)}{\partial(a,\varepsilon)} \ .$$

Since

$$A = -\alpha t + \psi + O(\varepsilon)$$

$$B = \frac{d}{dt} |w| = \frac{d}{d\psi} |w| \cdot \left(\frac{dt}{d\psi}\right)^{-1} = \frac{-a \ \varepsilon \sin \psi}{a^{3/2}(1+\varepsilon \cos \psi)}$$

we have $\frac{\partial}{\partial a} B = 0$ for $\varepsilon = 0$, hence if $\varepsilon = 0$

$$\frac{\partial(A,B)}{\partial(a,\varepsilon)} = A_a B_\varepsilon = -\psi_a \cdot a \sin \psi \ .$$

In order to determine ψ_a we observe that for $t = \tau = \pi\beta^{-1}$ the Kepler equation

$$\tau = a^{3/2}(\psi + \varepsilon \sin \psi)$$

determines

$$\psi = \psi(a,\varepsilon) = \tau a^{-3/2} + O(\varepsilon) \ ,$$

from which we get

$$\psi_a = -\frac{3}{2} a^{-5/2}\tau$$

for $\varepsilon = 0$.

Summarizing we find for $\varepsilon = 0$ and $a = r$

$$(5.12) \quad \frac{\partial(A,B)}{\partial(a,\varepsilon)} = \frac{3}{2}\tau r^{-3/2} \sin \psi = \frac{3}{2} \psi \sin \psi \ ,$$

where on account of (4.9)

$$\psi = \psi(r,0) = \tau \ r^{-3/2} = \pi \ \frac{\alpha+\beta}{\beta} \ .$$

We conclude that E_+ and $d\phi^\tau E_+$ are transversal precisely if

$$\frac{\alpha}{\beta} \neq \text{integer} \ ,$$

which agrees with (4.11).

In this case Theorem 5.1 is applicable and system (4.5) possesses a unique symmetric periodic solution $p(t,\mu)$ of fixed period $T = 2\tau = 2\pi\beta^{-1}$ which depends continuously on μ and which agrees with the reference solution (4.8) for $\mu = 0$.

These periodic solutions having the fixed period T_0 of the circular reference solution belong to the family $p(t,C,\mu)$ of Theorem 4.1 by local uniqueness. We saw at the end of the previous paragraph how to choose the integral surface $H = -\frac{1}{2} C$ carrying the periodic solutions having a fixed period for all small μ. By local uniqueness we conclude that the solutions of the restricted three body problem obtained in Section 4 are automatically symmetric.

Although it is irrelevant we can compute $\det \Phi_{-+}$, combining (5.10), (5.11) and (5.12):

$$\det \Phi_{-+} = \frac{\partial(v(\tau),\dot{u}(\tau))}{\partial(u(0),\dot{v}(0))}$$
$$= |w| \ \frac{\partial(A,B)}{\partial(a,\varepsilon)} \ (2\sqrt{a}) \ ,$$

and at $a = r$, $\varepsilon = 0$

(5.13) $$\det \Phi_{-+} = 3\tau \sin \psi, \qquad \psi = \tau r^{-3/2} ,$$

which is, expressed in the frequencies $\alpha_2 = \alpha$, $\alpha_3 = \alpha+\beta$ of the two planets in the resting coordinate system, equal to

$$3\tau \sin \pi \frac{\alpha_3}{\alpha_3-\alpha_2} .$$

(e) Periodic solutions near a stationary point in the reversible case

We consider again, as in Section 2, the system in R^n:

$$\dot{x} = f(x) = Ax + \hat{f}(x) ,$$

with $\hat{f}(0) = \hat{f}_x(0) = 0$, A having a pair of imaginary eigenvalues

(5.14) $$\alpha_1 = i\omega, \quad \alpha_2 = -i\omega, \quad \omega > 0 ,$$

which the other eigenvalues $\alpha_3,\dots,\alpha_n$ of A satisfy

(5.15) $$\frac{\alpha_j}{\alpha_1} \neq \text{integer} , \qquad j \geq 3 .$$

But instead of assuming the existence of an integral we require that $\dot{x} = f(x)$ is reversible with respect to ρ, i.e.

(5.16) $$\rho f(x) = -f(\rho x) ,$$

where ρ is a linear map with $\rho^2 = id$. We shall show that also in this case a family of periodic solutions exists. We shall apply the same trick as in Section 3 and consider the modified system

(5.17) $$\dot{x} = f(x) + \mu x = A(\mu)x + \hat{f}(x) ,$$

where the vector field $g(x) = x$ is invariant under ρ, i.e.

$$\rho\, g(x) = g(\rho x) .$$

We choose in the eigenspace E of the eigenvalues $\alpha_1, \alpha_2 = \bar{\alpha}_1$ the basis e_1, e_2 such that

(5.18) $$A(e_1 + i\, e_2) = i\omega\, (e_1 + i\, e_2) .$$

Thus we can write with respect to the splitting $R^n = E + F$

$$A(\mu) = \begin{pmatrix} A_E(\mu) & 0 \\ 0 & A_F(\mu) \end{pmatrix}$$

where

$$A_E(\mu) = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} .$$

It is clear that the system (5.17) satisfies the assumptions of Theorem 3.1, and we conclude that there is a *unique* family of periodic solutions

$$x = u(t,\varepsilon) , \quad \mu = v(\varepsilon) , \quad 0 \leq \varepsilon \leq \varepsilon_0 ,$$

of (5.17) having periods close to $2\pi/\omega$ and satisfying the normalization condition:

$$u_E(0,\varepsilon) = \varepsilon e_1 .$$

Now we shall prove that $v(\varepsilon) = 0$. Since our system (5.17) goes into itself under the transformation $(t,x,\mu) \to (-t,\rho x,-\mu)$ we see that

$$x^* = \rho u(-t,\varepsilon) , \quad \mu = -v(\varepsilon)$$

is also a family of periodic solutions of (5.17), which in addition satisfies

(5.19) $$(\rho u)_E(0,\varepsilon) = \varepsilon e_1$$

as we shall show below. Therefore, by the uniqueness of Theorem 3.1 we conclude

$$u(t,\varepsilon) = \rho u(-t,\varepsilon) \quad \text{and} \quad v(\varepsilon) = -v(\varepsilon) ,$$

and thus $v(\varepsilon) = 0$. Therefore $= u(t \quad)$ are solutions of $\dot{x} = f(x)$ and the modification was not necessary at all. Moreover these solutions are symmetric.

In oder to prove (5.19) we note that from (5.16) we have

$$\rho A + A\rho = 0 ,$$

which implies

$$\rho E = E .$$

Moreover one verifies easily (Exercise 1) that

(5.20) $$\rho e_1 = e_1 \quad \text{and} \quad \rho e_2 = -e_2 .$$

Therefore

$$(\rho u)_E(0,\varepsilon) = \rho_E u_E(0,\varepsilon) = \rho_E(\varepsilon,0) = (\varepsilon,0) = \varepsilon e_1 ,$$

as we wanted to show. We thus have proved the following modification of Liapunov's theorem:

<u>Theorem 5.2</u> Under the assumptions (5.14)-(5.16), the system $\dot{x} = f(x)$ has a unique family of symmetric periodic solutions

$$x = u(t,\varepsilon) , \qquad 0 \le \varepsilon \le \varepsilon_0 ,$$

near E having periods near $2\pi/\omega$ and satisfying

$$u_E(0,\varepsilon) = \varepsilon e_1 .$$

The family is continuous for $0 \le \varepsilon \le \varepsilon_0$ and differentiable for $0 < \varepsilon \le \varepsilon_0$.

Exercise 1

Show that under the assumptions of Theorem 5.2 the basis e_1, e_2 of A defined by (5.18) satisfies $\rho\, e_1 = e_1$ and $\rho\, e_2 = -e_2$.

Exercise 2

Show that all solutions near the origin of the planar vector field

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

are periodic, provided

$$f(x,-y) = -f(x,y)\ , \quad g(x,-y) = g(x,y)$$
$$f(0,0) = g(0,0) = 0$$

and

$$\frac{\partial(f,g)}{\partial(x,y)} > 0 \quad \text{at} \quad x = y = 0\ .$$

H. Poincaré: "Sur les courbes définis par les équations différentielles", Journal de Mathématique pures et appliquées, Ser. 4, Vol. 1, 1893, p. 167-244; in particular see p. 193.

Exercise 3

Let $\rho\colon R^{2n} \to R^{2n}$ be a diffeomorphism. Show that

$$\{F \circ \rho,\ G \circ \rho\} = -\{F,G\} \circ \rho$$

for all functions F, G, if and only if

$$\rho^{-1} \circ \exp(tX_H) \circ \rho = \exp(-tX_{H\circ\rho})$$

for all functions H, i.e. a Hamiltonian system with Hamiltonian $H = H(x)$ is transformed by $x \to x^* = \rho x$ into a system with Hamiltonian

$$H^*(x^*) = -H(\rho x)\ .$$

(The transformation ρ is canonical in the generalized sense with multiplier -1.) The reflection $\rho(x_1,x_2,y_1,y_2) = (x_1,-x_2,-y_1,y_2)$ is an example.

6. The Plane Three and Four Body Problem.

(a) Elimination of the Center of Mass.

We shall apply the continuation method described in the previous paragraph to the plane 3-body problem:

$$\frac{d^2x_k}{dt^2} = \sum_{j\neq k} \frac{m_j(x_j - x_k)}{|x_j - x_k|^2}, \quad k = 1,2,3. \tag{6.1}$$

where $x_j \in R^2$ describes the position of the particle with mass m_k. In contrast to the restricted three body problem we do not assume that one of the particles has zero mass, but assume instead that the masses of the two bodies m_2 and m_3 are small compared to the first m_1. The interaction between m_2 and m_3 is then small and vanishes in the limit $m_2 = m_3 = 0$, so that we obtain two decoupled two body problems. We shall normalize

$$m_1 = 1$$

with

$$M = \sum_{k=1}^{3} m_k ,$$

the center of mass is

$$X_0 = \frac{1}{M} \sum_{k=1}^{3} m_k x_k ,$$

and we introduce the relative coordinates of the second and third particle with respect to the center of mass setting

$$X_2 = x_2 - X_0 , \qquad X_3 = x_3 - X_0 ,$$

so that with $m_1 = 1$,

$$x_1 = X_0 - m_2X_2 - m_3X_3$$
$$x_2 = X_0 + X_2$$
$$x_3 = X_0 + X_3 ,$$

and we then obtain the equations:

$$\ddot{X}_0 = 0$$

$$\ddot{X}_2 = \ddot{x}_2 = \frac{x_1 - x_2}{|x_1 - x_2|^3} + m_3 \frac{x_3 - x_2}{|x_3 - x_2|^2}$$

$$\ddot{X}_3 = \ddot{x}_3 = \frac{x_1 - x_3}{|x_1 - x_3|^3} + m_2 \frac{x_2 - x_3}{|x_2 - x_3|^2}$$

The first equation expresses that the center of mass X_0 moves on a straight line and can be ignored. The other two equations can with

$$x_2 - x_3 = X_2 - X_3$$
$$x_2 - x_1 = X_2 + m_2X_2 + m_3X_3$$
$$x_3 - x_1 = X_3 + m_2X_2 + m_3X_3$$

be written as

$$\ddot{X}_2 = - |X_2|^{-3}X_2 + O(m)$$
$$\ddot{X}_3 = - |X_3|^{-3}X_3 + O(m)$$

where $O(m)$ stands for functions which vanish for $m_2 = m_3 = 0$.

We turn to complex notation and describe X_j for $j = 2,3$ by the complex numbers z_j. Then our system can be written in the form

$$\ddot{z}_2 = - \frac{z_2}{|z_2|^3} + F_1(z,\bar{z},m)$$
$$\ddot{z}_3 = - \frac{z_3}{|z_3|^3} + F_2(z,\bar{z},m) \tag{6.2}$$

where $z = (z_2, z_3)$, $m = (m_2, m_3)$. The functions $F_j = F_j(z, \bar{z}, m)$ $(j = 2,3)$ have the following properties

$$(6.3)\quad \begin{array}{ll} \text{(i)} & F_j(z,\bar{z},0) = 0 \\ \text{(ii)} & F_j(\eta z, \overline{\eta z}, m) = \eta F_j(z,\bar{z},m) \quad \text{for } |\eta| = 1 \\ \text{(iii)} & F_j(\bar{z},z,m) = \overline{F_j(z,\bar{z},m)} . \end{array}$$

Property (i), which was established already, expresses that the equations (6.2) reduce to two uncoupled Kepler problems for $m_2 = m_3 = 0$. Properties (ii) and (iii) express that the (6.2) is invariant under rotations $z_j \to e^{i\psi} z_j$, and under the reflections $z_j \to \bar{z}_j$. These properties are verified most easily for the original system (6.1) and that way deduced for (6.2).

(b) Periodic Solutions of the First Kind.

We search for orbits of (6.2) for small m_2, m_3 close to the circular orbits

$$z_j = r_j e^{i\alpha_j t}, \quad \alpha_j^2 r_j^3 = 1, \quad \text{for } j = 2,3.$$

These are clearly solutions of (6.2) for $m = 0$, but generally *not* periodic! But in rotating coordinates w_j introduced by

$$z_j = w_j e^{i\alpha t}, \qquad \alpha = \alpha_2$$

the above orbits become

$$(6.4)\qquad w_2 = r_2, \quad w_3 = r_3 e^{i(\alpha_3 - \alpha_2)t}$$

which are always periodic of period $T = 2\pi(\alpha_3 - \alpha_2)^{-1}$. The differential equations become

$$(6.5)\quad \left\{ \begin{array}{l} \ddot{w}_j + 2i\alpha\dot{w}_j - \alpha^2 w_j + \dfrac{w_j}{|w_j|^3} = e^{-i\alpha t} F_j'(e^{i\alpha t}w, e^{-i\alpha t}\bar{w}, m) \\ \qquad\qquad = F_j(w, \bar{w}, m) . \end{array} \right.$$

In the last equation we used (6.3) (ii).

<u>Theorem 6.1</u>. If $\alpha_3 \neq \pm \alpha_2$ and

$$\frac{\alpha_2}{\alpha_3 - \alpha_2} \neq \text{integer}$$

then for small m_2, m_3 the system (6.5) possesses a unique periodic solution $w_j = p_j(t,m)$ of period $2\pi(\alpha_3 - \alpha_2)^{-1}$ satisfying the symmetry condition

$$p_j(-t,m) = \overline{p_j(t,m)}$$

and tending to the circular solution (6.4) for $m \to 0$.

<u>Remark</u>. This statement holds for any system (6.5) where the perturbation forces satisfy (6.3) and not only for those derived for the three body problem. For this result the Hamiltonian character of the system is unessential. As a matter of fact the perturbation method of Section 1 does not apply since $\lambda = 1$ is a Floquet multiplier of multiplicity four; and we do not have three integrals of the system to make that method applicable.

Proof: We write the system (6.5) as a first order system for w_j, $\dot{w}_j$ $(j = 2,3)$ and observe that the resulting system is reversible with respect to the reflection

$$\rho: (w_j, \dot{w}_j) \mapsto (\bar{w}_j, -\dot{\bar{w}}_j)$$

or using real coordinates by $w_j = u_j + i\, v_j$

$$\rho: (u_j, v_j, \dot{u}_j, \dot{v}_j) \mapsto (u_j, -v_j, -\dot{u}_j, \dot{v}) \quad \text{for} \quad j = 2,3 .$$

The reference solution (6.4) is clearly symmetric with respect to ρ with half-period

(6.6) $$\tau = \frac{\pi}{\alpha_3 - \alpha_2} .$$

Thus we can apply Theorem 5.1 provided the nondegeneracy condition holds. Since E_+ is given by $v_j = \dot{u}_j = 0$ we have to determine the initial condition

$$(u_j(0), 0, 0, \dot{v}_j(0))$$

so that $v_j(\tau) = \dot{u}_j(\tau) = 0$. This requires the nonvanishing of the 4 by 4 determinant

$$\det \Phi_{-+} = \frac{\partial(v(\tau), \dot{u}(\tau))}{\partial(u(0), \dot{v}(0))} \quad \text{for} \quad m = 0$$

where $u = (u_1, u_2)$, $v = (v_1, v_2)$. Since for $m = 0$ the system (6.5) decouples into two Kepler problems in rotating coordinates we have

$$\det \Phi_{-+} = \det \Phi_{-+}^{(2)} \det \Phi_{-+}^{(3)}$$

where

$$\det \Phi_{-+}^{(j)} = \frac{\partial(v_j(\tau), \dot{u}_j(\tau))}{\partial(u_j(0), \dot{v}_j(\tau))} .$$

We can use the computation (5.13) of the previous section and find

$$\det \Phi_{-+} = (3\tau)^2 \sin \tau\alpha_2 \sin \tau\alpha_3$$

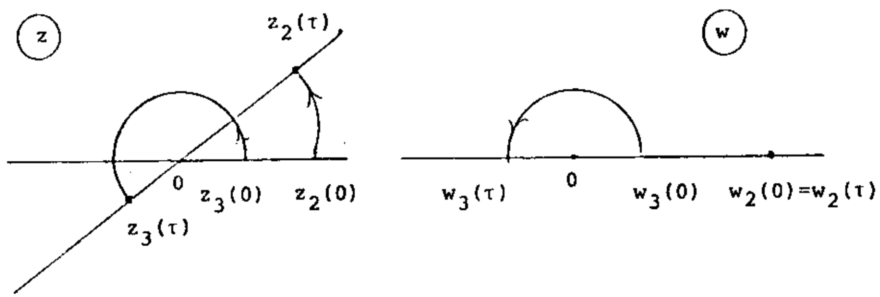
with τ given by (6.6). The condition $\det \Phi_{-+} \neq 0$ is therefore equivalent to the assumption

$$\frac{\sigma_2}{\alpha_3 - \alpha_2} \neq \text{integer}$$

of the theorem, since $\alpha_3\tau = \alpha_2\tau + \pi$.

Thus Theorem 5.1 yields the desired symmetric periodic solutions $w_j = p_j(t,m)$ for which one has $p_j(-t,m) = \overline{p_j(t,m)}$. Conversely this condition characterizes symmetric solutions, so that also their uniqueness follows from Theorem 5.1.

We repeat again that the solutions obtained are periodic only in a rotating coordinate system. We illustrate the configuration for $m = 0$ in the two figures below: For $t = 0$ both $z_2(0)$, $z_3(0)$ lie on the positive real axis (the planets are in conjunction) and for $t=\tau$ the points $z_2(\tau)$, $z_3(\tau)$ lie on opposite sides of a line through the origin (the planets are in opposition).



For m_2, m_3 small not zero the orbits will again start with two points on the same side of a line through 0 and will for $t = \tau$ have $z_2(\tau)$, $z_3(\tau)$ on the opposite sides of another line. The orbits will intersect these lines at right angles. These statements are merely geometrical interpretations of the symmetry of the orbits.

According to Poincaré one speaks of periodic solutions of the first kind of the 3- or n-body problem if they issue from circular and not elliptic orbits.

(c) The four body problem

For the four body problem there is a similar periodic solution. If m_1, m_2, m_3, m_4 are the four masses we take again $m_1 = 1$ and consider the other masses small. After eliminating the center of mass we get again a system of differential equations for z_2, z_3, z_4 which in rotating coordinates w_2, w_3, w_4,

$$z_j = w_j e^{i\alpha t}, \quad \alpha = \alpha_2, \quad j = 2,3,4$$

again take the form (6.5) where $j = 2,3,4$. The reference solutions become

$$(6.7) \qquad w_j = r_j e^{i(\alpha_j - \alpha_2)t}, \ldots, \alpha_j^2 r_j^3 = 1, \quad j = 2,3,4$$

which are periodic if and only if $\alpha_3 - \alpha_2$ and $\alpha_4 - \alpha_2$ are rationally dependent. This is equivalent to the requirement that there exists three integers j_2, j_3, j_4, not all zero, such that

$$(6.8) \qquad \sum_{k=2}^{4} j_k \alpha_k = 0, \quad \sum_{k=2}^{4} j_k = 0.$$

We may and will assume that the j_2, j_3, j_4 have no common factor and that $\alpha_2, \alpha_3, \alpha_4$ are distinct. The half period τ for these orbits is calculated as

$$(6.9) \qquad \tau = \frac{\pi j_2}{\alpha_3 - \alpha_4} = \frac{\pi j_3}{\alpha_4 - \alpha_2} = \frac{\pi j_4}{\alpha_2 - \alpha_3}.$$

The equality of these expressions follows from the relations (6.8). The same argument as before succeeds if

$$\det \Phi_{-+} = \prod_{j=2}^{4} \det \Phi_{-+}^{(j)} = (3\tau)^3 \prod_{j=2}^{4} \sin(\alpha_k \tau) \neq 0$$

which requires that none of the $\alpha_k \tau \pi^{-1}$ is an integer. But since, by (6.9), the differences

$$(\alpha_3 - \alpha_2)\tau\pi^{-1} = -j_4, \qquad (\alpha_4 - \alpha_2)\tau\pi^{-1} = j_3$$

are integers it suffices to require only one of the numbers not be an integer. This leads to

Theorem 6.2. Let α_2^2, α_3^2, α_4^2 be distinct numbers satisfying (6.8) and the conditions

$$\alpha_2 \tau\, \pi^{-1} \neq \text{integer},$$

where τ is defined by (6.9). Then for small m_2, m_3, m_4 the system (6.5) for $j = 2,3,4$ with F_j satisfying (6.3) has a unique periodic solution $w_j = p_j(t,m) = \overline{p_j(-t,m)}$ of period 2τ which for $m \to 0$ tend to the circular reference solution (6.7).

This result is of interest in astronomy. When Laplace [(1)] developed his perturbation theory he observed that the frequencies $\alpha_2, \alpha_3, \alpha_4$ of three of the Galilean moons of Jupiter (namely Io, Europe and Ganymede) satisfy the relation (6.8) with $j_2 = 1$, $j_3 = -3$, $j_4 = +2$ to a high degree of accuracy. Poincaré [(2)] used this example to derive the periodic solutions of the four body problem which are given by Theorem 6.2. Later de Sitter [(3)] determined the values of $\alpha_2, \alpha_3, \alpha_4$ to high accuracy and gave the values

$$\alpha_2 = 203°.\ 488\ 955\ 28$$

$$\alpha_3 = 101°.\ 374\ 723\ 96$$

$$\alpha_4 = 50°.\ 317\ 608\ 33$$

which are the angle of advance measured in degrees per day. From these values one finds

$$\alpha_2 - 3\alpha_3 + 2\alpha_4 = (6\cdot 10^{-8})°,$$

(1) Marquis de la Laplace: Traité de mécanique célèste, tome 4, livres huitième (théorie des satellites de Jupiter de Saturne et d'Uranus) Imprimerie Royale, Paris, 1845.

(2) H. Poincaré, Les Methodes Nouvelles de la méchanique célèste, vol. 1, Art. 50, Gauthiers Villars, Paris, 1892.

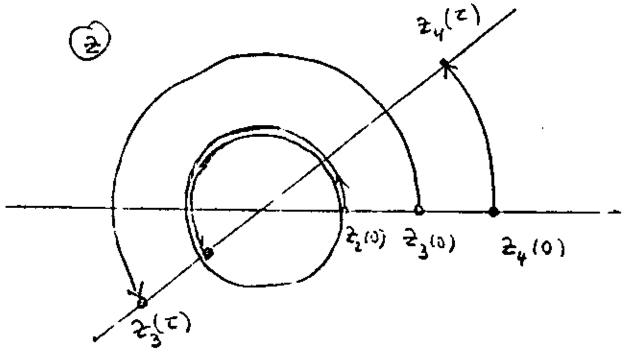
(3) W. de Sitter, Proc. Acad. Sci., Amsterdam, Vols. 10, 11, 1908, 1909.

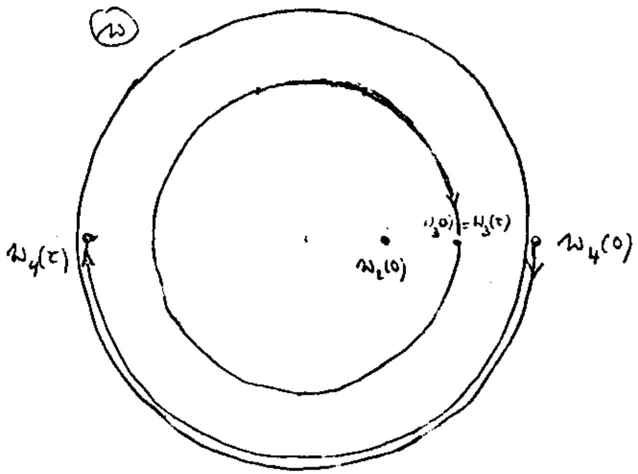
i.e. (6.8) holds to remarkable accuracy. On the other hand, if we check the nondegeneracy condition we find that

$$\frac{\alpha_2\tau}{\pi} = \frac{\alpha_2}{\alpha_3-\alpha_4} = 4 - 0.0145$$

is rather close to an integer, but compared to the high approximation of $\alpha_2-3\alpha_3+2\alpha_4 = 0$, it is markedly different from an integer. Clearly to see whether these orbits are of significance for this astronomical situation requires quantitative estimates and numerical calculations. In any event one would not expect these three moons to perform a periodic motion but possibly oscillate about such an orbit.

We illustrate the circular motion of the reference orbits (6.7) for $j_2 = 1$, $j_3 = -3$, $j_4 = 2$ between $t = 0$ and $t = \tau$ in the resting and the rotating coordinate systems in the figure below.





7. Poincaré-Birkhoff Fixed Point Theorem

(a) Formulation

While our previous results on periodic solutions were based on perturbation methods and ultimately depended on the implicit function theorem we turn now to a topological result. We will state and prove an old theorem to which Poincaré was led in his studies of the restricted three body problem. In 1913 G. D. Birkhoff gave a surprising but simple proof. The theorem asserts the existence of fixed points of an area-preserving mapping of an annulus in the plane into itself. Such a fixed point theorem can be used to establish the existence of periodic solutions of differential equations, as we shall see in the following sections. This connection is similar to that of periodic orbits of a vector field and the fixed points of a section map as it was discussed in Section 1.

After formulating the theorem we present essentially Birkhoff's original proof (c) and then discuss the proofs under relaxed assumptions (d). In Section 8 we shall give much simpler proofs but under additional hypotheses; they have the advantage that they can be generalized to higher dimensions.

Consider an annulus

$$A : \quad a \le u^2 + v^2 \le b$$

in the u,v-plane and a homeomorphism ψ of A into itself.

We require that ψ maps the inner boundary into itself as well as the outer boundary into itself. This does not mean that the individual points are fixed under the map ψ. We will require that the two boundary circles are turned in opposite directions under ψ and refer to this as the "twist condition". We will give a precise formulation below. Finally we require that the Lebesgue measure

$$\iint_E du \wedge dv = \iint_{\psi(E)} du \wedge dv$$

is preserved for any open set $E \subset A$. Under these assumptions the Poincare-Birkhoff fixed point theorem asserts the existence of two fixed points of ψ in the interior of A.

To formulate this condition more precisely we introduce "polar coordinates" by $p: (x,y) \rightarrow (u,v)$ by

(7.1) $$u = \sqrt{y} \cos x, \quad v = \sqrt{y} \sin x, \quad a \le y \le b .$$

Here we choose $y = u^2 + v^2$ as the radial variable so that

$$du \wedge dv = \frac{1}{2} dy \wedge dx \quad \text{or} \quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2} .$$

Therefore the mapping corresponding to ψ in the x,y-variables will again preserve the Lebesgue measure

$$\iint_E dy \wedge dx .$$

Now it is important to note that the homeomorphism $\psi: (u,v) \rightarrow (u_1,v_1)$ of A does *not* determine uniquely a mapping

$$\phi: (x,y) \rightarrow (x_1,y_1)$$

of the strip

(7.2) $$S = \{x,y \mid a \le y \le b,\ x \in R\} ,$$

since u, v determines x only up to an integer multiple of 2π. Denoting the map (7.1) by p we require that ϕ be a homeomorphism of S into itself such that

(7.3) $$p \circ \phi = \psi \circ p .$$

Clearly such a map exists. Moreover, if

$$s: (x,y) \rightarrow (x + 2\pi,\ y)$$

then $s^n \circ \phi$ also is a map satisfying (7.2), since $p \circ s = p$. As a matter of fact, the most general homeomorphism of (7.2) is of this form. It is clear also that a fixed point of ϕ gives rise to a fixed point of ψ but not conversely. In the following we will make assertions about fixed points of ϕ in S, which are stronger than assertions about ψ.

We write the mapping ϕ in the form

(7.4) $$\phi: (x,y) \rightarrow \big(f(x,y), g(x,y)\big) ,$$

where f,g are continuous functions in the strip (7.2). Since the points (x,y) and $s(x,y) = (x + 2\pi,\ y)$ have equivalent image points we conclude that

$$f(x+2\pi,y) - f(x,y) = 2\pi k$$
$$g(x+2\pi,y) - g(x,y) = 0$$

for some integer k. In other words, $\phi \circ s = s^k \circ \phi$, or

$$f(x,y) - kx \; , \quad g(x,y)$$

have period 2π in x. For the inverse map ϕ^{-1} we find a similar integer k' and the identity $\phi \circ \phi^{-1} = id$ shows that $k \cdot k' = 1$, i.e. $k = \pm 1$. We will require that ψ is also orientation preserving so that $k = +1$ and

$$f(x,y) - x \; , \quad g(x,y)$$

are continuous functions in S of period 2π in x, i.e.

(7.5) $$\phi \circ s = s \circ \phi \; .$$

A homeomorphism ϕ of S which belongs to a homeomorphism ψ of A is characterized by the following properties: The functions $f - x$, g defined by (7.4) are continuous and of period 2π and the corresponding functions for ϕ^{-1} have the same properties. If ϕ is one such homeomorphism belonging to ψ the most general one is given by $s^n \circ \phi$.

The requirement that the two boundary circles are preserved under ϕ amounts to

(7.6) $$g(x,a) = a \; , \quad g(x,b) = b \; ,$$

and the "twist condition" becomes

(7.7) $$\left(f(x,a) - x\right)\left(f(x,b) - x\right) < 0 \; .$$

Finally we require that for any open set E the Lebesgue measure

$$m(E) = \iint_E dy \wedge dx$$

is preserved. We will call such a mapping "area-preserving" if in addition it preserves the orientation. For a diffeomorphism this would mean that its Jacobian is +1.

<u>Theorem</u> 7.1. An area-preserving homeomorphism ϕ of S into itself satisfying (7.5), (7.6) and (7.7) possesses at least two fixed points F_1, F_2 which are not equivalent, i.e. $s^j F_1 \neq F_2$ for all integers j.

(b) <u>Infinitely many periodic orbits of ψ.</u>

As a consequence of Theorem 7.1 one can construct infinitely many periodic points for an annulus mapping ψ. Here we will call a point $P \in A$ periodic if $\psi^N(P) = P$ for some integer $N \geq 1$; the smallest such N is called its primitive period. For any point $P \in A$ we call the set

$$\left\{\psi^j(P), \; j = 0, \pm 1, \pm 2, \cdots\right\}$$

the orbit through P. For a periodic point its orbit consists of only finitely many points.

Assume that the map $\phi: S \to S$ belongs to ψ and is represented by (7.4). Let c_1, c_2 be two real constants such that

(7.8) $$f(x,b) - x \leq c_1 < c_2 \leq f(x,a) - x \; , \qquad x \in R \; ,$$

and let p/q be any rational number such that

(7.9) $$c_1 < 2\pi \frac{p}{q} < c_2 \; .$$

Denoting ϕ^q by $(x,y) \to (f_q, g_q)$ it is clear that the map $s^{-p}\phi^q$ satisfies the "twist condition" since

$$f_q(x,b) - 2\pi p - x \le qc_1 - 2\pi p < 0 < qc_2 - 2\pi p \le f_q(x,a) - 2\pi p - x.$$

Therefore for any p/q satisfying (7.9) there exist fixed points F with

$$s^{-p}\phi^q(F) = F .$$

If p'/q' is another rational number in the interval (7.9) and F' a fixed point with

$$s^{-p'}\phi^{q'}(F') = F' ,$$

then F' and F are not equivalent, i.e. $F' \neq s^j(F)$ for all integers j. Indeed since s commutes with ϕ we would conclude from $F' = s^j(F)$ and the last relation

$$\phi^{q'}(F) = s^{-j}\circ\phi^{q'}(F') = s^{-j+p'}(F') = s^{p'}(F) ,$$

hence

$$\phi^{qq'}(F) = s^{qp'}(F) = s^{q'p}(F) ,$$

by symmetry in p/q, p'/q'. Therefore we would have

$$qp' = q'p \quad \text{or} \quad \frac{p'}{q'} = \frac{p}{q} .$$

We have proven

Theorem 7.2. If an area-preserving map ψ of the annulus A gives rise to a mapping ϕ of the strip S satisfying conditions (7.5), (7.6) and (7.8) then ψ possesses infinitely many different periodic orbits in the interior of A.

In the above fixed point theorem the area-preserving property cannot be dropped, as the following simple example shows: If $\alpha = \alpha(y)$, $\beta = \beta(y)$ are continuous in $a \le y \le b$ and $\beta(y)$ strictly monotonically increasing with $\beta(a) = a$, $\beta(b) = b$ then

$$\phi: (x,y) \to \big(x + \alpha(y), \beta(y)\big)$$

defines a homeomorphism of S onto itself. If $\alpha(a) < 0$, $\alpha(b) > 0$ then the twist condition holds, but if $\beta(y) \neq 0$ in $a < y < b$ then ϕ has no fixed point.

On the other hand, one can replace the condition that ϕ is area-preserving by a weaker topological assumption: We require that for no open set E of the annulus A one has proper containment $\psi(E) \subset E$ or $E \subset \psi(E)$. For an area-preserving mapping this assumption is obviously true. We will not present the proof under this more general assumption but refer to the original paper by G. D. Birkhoff. (*)

Incidentally, it is, of course, inessential that the annulus A is bounded by concentric circles and it is also valid for an annulus bounded by two continuous curves which are "starlike" with respect to some point. If we take this point as the origin then it means that each ray issuing from the origin meets each of these curves in precisely one point, i.e. these curves can be represented in the polar coordinates of (7.1) by

$$y = p_j(x) , \quad 0 < p_1(x) < p_2(x),$$

(*) G. D. Birkhoff, An extension of Poincaré's last geometric theorem, Acta Math. 47, 1925, 297-311.

where $p_j(x)$ are continuous functions of period 2π. It is easily shown (see Exercise 1) that the annulus

$$p_1(x) \leq y \leq p_2(x)$$

can be mapped by an area-preserving homeomorphism into an annulus $a \leq y \leq b$ bounded by concentric circles. Therefore Theorems 7.1 and 7.2 are valid also for the above annulus.

(c) Proof of Theorem 1.

First we prove the existence of a single fixed point of ϕ. We assume ϕ to have no fixed point in S and will bring this assumption to a contradiction. We replace the twist condition (7.7) by

$$(7.10) \qquad f(x,a) - x > 0 \, , \quad f(x,b) - x < 0 \, .$$

If this condition is violated it will hold after replacing (x,y) by $(-x,y)$.

We introduce the concept of an index. Let

$$\alpha(P) = \arg\big(P,\phi(P)\big)$$

denote the angle between the vector from P to $\phi(P)$ and the positive x-axis. It is defined only modulo 2π but, since ϕ has no fixed point it can be defined as a continuous function in S. It is well defined if we normalize it by requiring that $\alpha(P) = 0$ for P on $y = a$. Then the required index is

$$j(\phi) = \alpha(P)\Big|_{y=b}$$

i.e. the value of α at the upper boundary. One could also define

$$j_C(\phi) = \int_C d\alpha$$

as the increase of α along a curve C in S. This number is the same for two homotopic curves with the same end points Therefore the index defined above agrees with $j_C(\phi)$ where C is any curve going from $y = a$ to $y = b$. In other words $j_C(\alpha)$ is independent of the choice of the curve C, connecting $y = a$ and $y = b$. This expression $j_C(\phi)$ is meaningful also without the normalization (7.10) of the twist.

Proposition 1. For any fixed point free area-preserving map satisfying our conditions including (7.10) $j(\phi)$ is independent of ϕ, in fact,

$$j(\phi) = \pi \, .$$

The verification of this claim is the main burden of the proof and we postpone it. First we show that it leads to a contradiction — and hence to the existence of a fixed point of ϕ.

Define the reflection

$$\rho: (x,y) \rightarrow (-x,y) \, .$$

Then $\rho^{-1}\phi\,\rho$ is also a mapping satisfying all conditions of Theorem 7.1 and it is fixed point free if ϕ is.

Proposition 2. For a fixed point free ϕ we have

$$j(\rho^{-1}\phi\,\rho) = -\,j(\phi)$$
$$j(\phi^{-1}) = j(\phi) \, .$$

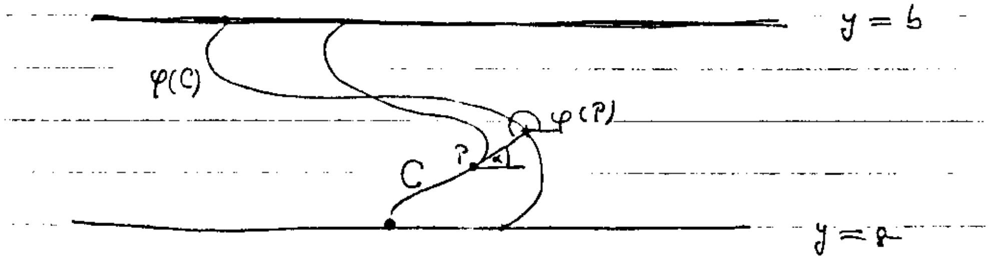
Proof: Let C be the curve $t \to (x = 0, y = t)$ for $a \le t \le b$, so that $\rho C = C$. If we replace ϕ by $\rho^{-1}\phi\,\rho$ then the angle α is replaced by $\pi - \alpha$ and

$$j_C(\rho^{-1}\phi\,\rho) = \int_C d(\pi - \alpha) = -\int_C d\alpha = -\,j_C(\phi)\ .$$

Similarly, if C is any curve going from $y = a$ to $y = b$ so is $\phi(C)$ such a curve and

$$j_{\phi(C)}(\phi^{-1}) = j_C(\phi)$$

since the angle α at $P \in C$ is to be replaced $\pi + \alpha$ at $\phi(P) \in \phi(C)$ (see Figure).



This proves Proposition 2.

Note if ϕ satisfies (7.10) then so does $\rho^{-1}\phi^{-1}\rho$, but by Proposition 2 we have

$$j(\rho^{-1}\phi^{-1}\rho) = -\,j(\phi)$$

which contradicts Proposition 1.

It remains to prove Proposition 1. For this purpose we extend ϕ to a homeomorphism of the plane by setting

$$f(x,y) = f(x,a) \quad \text{for} \quad y \le a$$
$$= f(x,b) \quad \text{for} \quad y \ge b\ ,$$

$$g(x,y) = y \quad \text{for} \quad y \le a \text{ and } y \ge b\ .$$

With the area-preserving shift

(7.11) $$\tau_\varepsilon\colon (x,y) \to (x, y+\varepsilon)$$

we define

$$\phi_\varepsilon = \tau_\varepsilon \circ \phi\ .$$

If $0 < \varepsilon < \operatorname{dist}(P, \phi(P))$ for all P then ϕ_ε is also fixed point free and we will define $j(\phi_\varepsilon) = \pi \pmod{2\pi}$ by the requirement

(7.12) $$\left| \int_C d\alpha_\varepsilon - j(\phi_\varepsilon) \right| < \frac{\pi}{2}\ ,$$

where

$$\alpha_\varepsilon(P) = \arg\big(P, \phi_\varepsilon(P)\big)$$

and where C is any curve going from $y \le a$ to $y \ge b$. By the same argument as before $j(\phi_\varepsilon)$ is independent of C and ε. It suffices to show that $j(\phi_\varepsilon) = \pi$ and for this purpose we select a particular curve Γ going from $y = a$ to $y > b$ and prove

(7.13) $$\left| \int_\Gamma d\alpha_\varepsilon - \pi \right| < \frac{\pi}{2}\ .$$

which by (7.12) proves $j(\phi_\varepsilon) = \pi$ and hence the Proposition 1.

We define

$$D_0 = \{(x,y) \mid a \le y < a + \varepsilon\} ,$$

and define

$$D_j = \phi_\varepsilon^j(D_0)$$

for all integers j. For $j \le 0$ we have

$$D_j = \{(x,y) \mid a + j\varepsilon \le y < a + (j+1)\varepsilon\}$$

which are disjoint strips. As images under a homeomorphism all the D_j are disjoint. Since ϕ commutes with the shift s, also the D_j are invariant under the shift s and can be regarded as annuli, mod 2π. Their areas are equal as ϕ and τ_ε, hence ϕ_ε, preserves the area. Therefore

$$m(D_j) = m(D_0) = 2\pi\varepsilon > 0 .$$

The D_j for $j \ge 0$ lie in the half plane $y \ge a$ and not all can lie in the strip S since the strip, modulo 2π, has finite area $2\pi(b - a)$. Let N be the integer such that

$$D_0, D_1, \dots, D_{N-1} \subset S , \quad D_N \not\subset S ,$$

and pick Q as the point of maximal y-component in $\bar{D}_{N+1}$. Then $Q \in \partial D_{N+1}$ as ϕ_ε is a homeomorphism. One verifies that

$$Q = \phi_\varepsilon^{N+2}(P_0) ,$$

where P_0 lies in $y = a$. We define

$$P_j = \phi_\varepsilon^j(P_0) , \quad P_{N+2} = Q , \quad 0 \le j \le N+2,$$

and construct a continuous curve $t \to \gamma(t)$ by taking

$$\gamma(t) = \frac{t}{\varepsilon} P_1 + (1 - \frac{t}{\varepsilon}) P_0 \quad \text{for} \quad 0 \le t \le \varepsilon$$

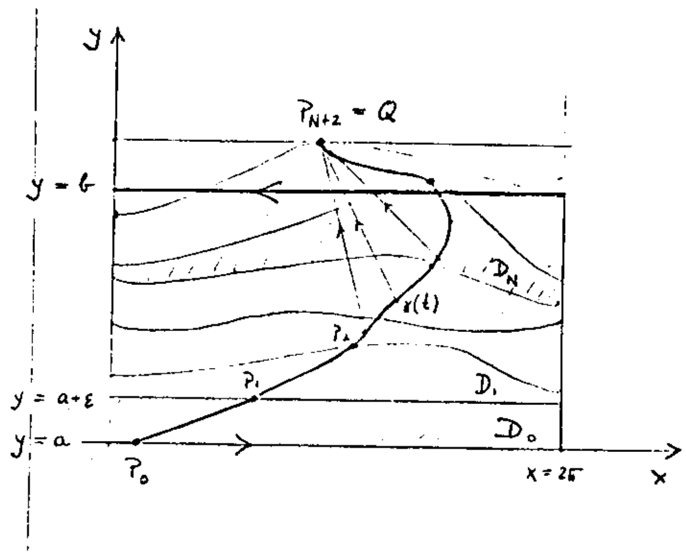
and setting

$$\gamma(t + \varepsilon j) = \phi_\varepsilon^j \circ \gamma(t) \quad \text{for} \quad 0 \le t \le \varepsilon ,$$

and $j = 1,2,\dots,N+1$. Thus $\gamma(t)$ is defined for $0 \le t \le T = (N+2)\varepsilon$ and connects the points

$$\gamma(0) = P_0, \qquad \gamma(T) = P_{N+2} = Q .$$

Moreover, by construction of Q the entire curve lies in the strip between the parallels to the x-axis through P_0 and $P_{N+2} = Q$.



Proposition 3. The curve γ has no selfintersection for $0 \le t \le T$, i.e.

$$\gamma(t) \neq \gamma(s) \quad \text{for} \quad 0 \le t < s \le T .$$

Proof: We observe that

$$\{\gamma(t) \mid j\varepsilon \le t < (j+1)\varepsilon\}$$

lies in D_j and since the $D_0, D_1, \dots, D_N$ are disjoint we can have $\gamma(t) = \gamma(s)$ only if t,s both lie in an interval $j\varepsilon \le t < s \le (j+1)\varepsilon$. But this part of $\gamma(t)$ is the image of the straight line from P_0 to P_1 under the homeomorphism ϕ_ε and therefore certinaly free from selfintersections, proving Proposition 3.

Finally, taking for Γ the curve $\gamma = \gamma(t)$, $0 \le t \le T-\varepsilon$, just constructed we verify (7.13) using an idea which goes back to H. Hopf [(*)] in a similar situation. For this purpose we define

$$\beta(t,s) = \arg\big(\gamma(t),\gamma(s)\big) \quad \text{for} \quad 0 \le t < s \le T ,$$

which is defined mod 2π, as γ is free of selfintersection. Moreover

$$\alpha_\varepsilon(\gamma(t)) = \beta(t,t+\varepsilon) \quad (\text{mod } 2\pi)$$

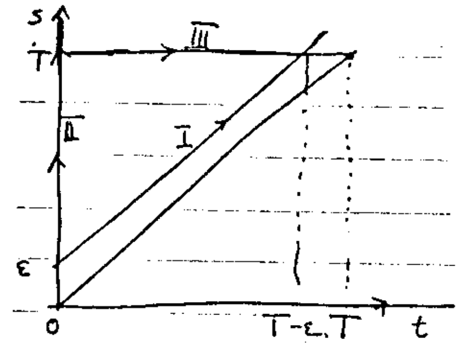
and

(*) H. Hopf. Über die Drehung der Tangenten und Sehnen ebener Kurven, Comp. Math. 2, 1935, 50-62.

$$\int_\Gamma d\alpha_\varepsilon = \int_I d\beta$$

where I is the line $s = t + \varepsilon$, $0 \le t \le T-\varepsilon$, in the t,s-plane. Since the triangle bounded by I, II, III is simply connected we have

$$\int_I d\beta = \int_{II+III} d\beta .$$



To estimate the last integral we observe that for any $(t,s) \in II + III$ the vector from $\gamma(t)$ to $\gamma(s)$ lies in the upper half plane. In fact either $t = 0$ and the vector from $\gamma(0)$ to $\gamma(s)$ is in the upper half plane, since $\gamma(0)$ has minimal y coordinate, or $s = T$ and the vector from $\gamma(t)$ to $\gamma(T) = Q$ is again in the upper half plane, since the y-coordinate of $Q = \gamma(T)$ is by our construction larger than that of $\gamma(t)$ for $0 \le t < T$. Hence $0 < \beta < \pi$ on II + III, or

$$0 < \int_{II+III} d\beta < \pi .$$

Since by assumption $\alpha(P) = \pi \pmod{2\pi}$ for P on the boundary $y = b$, also $\alpha_\varepsilon(P) = (2k+1)\pi + \omega(\varepsilon)$, where $\omega(\varepsilon)$ denotes any function tending to zero as ε tends to zero. Therefore

$$(2k+1)\pi + \omega(\varepsilon) = \int_\Gamma d\alpha_\varepsilon = \int_I d\beta = \int_{II+III} d\beta \, ,$$

which by the previous estimate of the right-hand side implies $k = 0$. This proves (7.13) and thus Proposition 1, and therefore the existence of one fixed point in Theorem 7.1 is established.

(d) Modifications

We still have to establish the existence of the second fixed point. This can be done by a slight modification of the above argument which we indicate below under more relaxed assumptions. They will be important for the application in the next section. It is not necessary that the map ϕ maps S into itself but it suffices that only one of the boundary circles, say $y = a$ is preserved. Then the iterates of ϕ need not be defined in S and we will assume that ϕ and ϕ^2 are defined as homeomorphisms in a neighborhood U(S) of S.

For example, in the application ϕ will be a homeomorphism in a wider strip

$$\tilde{S}: a \le y < c \, , \quad c > b \, ,$$

and we assume that

$$(7.14) \qquad \phi: S \to \tilde{S} \, .$$

We still require that the boundary $y = a$ is preserved:

$$(7.15) \qquad g(x,a) = a$$

and the twist condition is replaced by

$$(7.16) \quad f(x,a)-x > 0, \quad f(x,y)-x < 0 \quad \text{for} \quad (x,y) \in \tilde{S} - S \, .$$

Theorem 7.3: An area preserving homeomorphism ϕ satisfying the above conditions (7.14-16) possess at least two nonequivalent fixed points in S.

The proof proceeds along the same lines as before and we indicate the necessary modification. We assume that the mapping has at most one fixed point F_0 in the rectangle

$$-\pi \le x < \pi \, , \quad a \le y \le b$$

and we can translate it so that it lies on the line $x = -\pi$. We will replace the translation τ_ε of (7.11) by

$$\tau_\varepsilon: (x,y) \to (x, y+\varepsilon p(x))$$

where p(x) is a continuous function of period 2π satisfying

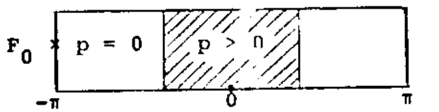
$$0 \leq p(x) \leq 1 \; ; \qquad \int_0^{2\pi} p(x)\, dx > 0$$

and

$$p(x) = 0 \quad \text{for} \quad \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} .$$

For definitions we take

$$p(x) > 0 \text{ in } |x| < \frac{\pi}{2} .$$



Since ϕ has no fixed point in the rectangle

$$|x| \leq \frac{\pi}{2} , \quad a \leq y \leq b$$

we can again choose ε so small that $\phi_\varepsilon = \tau_\varepsilon \circ \phi$ has no other fixed point than ϕ does.

Next we define the change of the argument

$$\int_C d\alpha_\varepsilon$$

where

$$\alpha_\varepsilon(P) = \arg\left(\Gamma, \phi_\varepsilon(P)\right)$$

and C is any nonselfintersecting curve in $\tilde{S}$ starting from a point on $y = a$ to a point in $\tilde{S} - S$ which avoids the fixed points $s^j F_0$. By assumption (7.16) there exists an odd multiple of π which we call the index $j_C(\phi_\varepsilon)$ such that

$$\left| \int_C d\alpha_\varepsilon - j_C(\phi_\varepsilon) \right| < \frac{\pi}{2} .$$

This index remains again unchanged under deformation of the curves C avoiding the fixed points and connecting $y = a$ and $b \leq y < c$. One shows that any such curve can be deformed to one of the equivalent lines

$$x = 0 \pmod{2\pi} , \quad y = t , \quad a \leq t \leq b ,$$

and therefore

$$j(\phi_\varepsilon) = j_C(\phi_\varepsilon)$$

is independent of the curve. (This would, of course, not be true any longer if we had two inequivalent fixed points.) Also

$$j(\phi) = j(\phi_\varepsilon)$$

for ε sufficiently small and it remains to show that $j(\phi)$ is independent of the choice of ϕ, i.e. proposition 1 holds true. Then the previous argument gives a contradiction.

To prove Proposition 1 in this case we construct again a a nonselfintersecting curve Γ connecting the bottom to the top, avoiding the fixed point and along which the index can be evaluated as π. We define the domain

$$D_0 = \{x, y \mid 0 \leq y < \varepsilon p(x)\}$$

and

$$D_j = \phi_\varepsilon^j(D_0) \quad \text{for} \quad j = 1, 2, \ldots, N+1 , \tag{7.17}$$

where N is the maximal index such that $D_{N-1} \subset \dot{S}$.

To show that $N < \infty$ we observe that τ_ε, hence ϕ_ε, is area-preserving and therefore the D_j have the same positive area (mod 2π), namely $\varepsilon \int_0^{2\pi} p(x)\, dx$. Next we verify that the interiors of D_j are disjoint which implies $N < \infty$ as before. We define D_j also for $j \le 0$ by (7.17) using the extension

$$\phi: (x,y) \to \big(f(x,a), y\big) \quad \text{for} \quad y \le a.$$

To show that the interiors $\mathring{D}_j$ of D_j are disjoint, or

$$\mathring{D}_j \cap \mathring{D}_k = \emptyset, \qquad 0 \le j < k \le N+1,$$

it suffices to verify

$$\mathring{D}_{-i} \cap \mathring{D}_0 = \emptyset \quad \text{for} \quad i = k - j > 0.$$

Since $\mathring{D}_{-1}$, and similarly D_{-i} for $i \ge 1$, lies in $y < a$ the last assertion is evident.

Having established $N < \infty$ we note that $D_N = \phi_\varepsilon(D_{N-1})$, $D_{N+1} = \phi_\varepsilon^2(D_{N-1})$ are well defined if $\varepsilon > 0$ is sufficiently small since $D_{N-1} \subset S$, hence $\phi(D_{N-1}) \subset \tilde{S}$ where, by assumption, ϕ is still defined.

Now we choose a point of $Q \subset \bar{D}_{N+1}$ of maximal y-coordinate and construct the curve $\gamma(t)$ just as previously, by constructing preimages $P_0, P_1, \dots$ of Q with $P_0 \subset D_0 \cap \{y = 0\}$, connecting P_0 and $\phi_\varepsilon(P_0) = P_1$ by a straight line segment $L_0: \gamma(t)$, $0 \le t < \varepsilon$ and extending this curve by $\phi_\varepsilon^j(L_0) = L_j$. Now the L_j will generally no longer lie in D_j and we have to verify that $L_j \cap L_k = \emptyset$ for $0 \le j < k \le N+1$. Again this is done by

defining L_j for $j \le -1$. We observe that for $j \le -1$ the arc L_j lies in

$$x_j \le x < x_{j+1}, \quad y \le a, \tag{7.18}$$

where the x_j are defined implicitly by

$$x_{j+1} = f(x_j, a) \quad \text{for} \quad j \le -1$$

and $(x_0, 0) = P_0$. Since the above regions (7.18) are disjoint the same holds for the L_j and the curve $\gamma(t)$ is free from selfintersections. From the fact that ϕ_ε has the same fixed point $s^j F_0$ as ϕ it is clear that the curve $\gamma(t)$ avoids these fixed points. The rest of the argument is unchanged. The use of the modified mapping τ_ε is due to M. Brown and W. D. Neumann.(*)

We shall apply Theorems 7.1-7.3 in Sections 9, 10 and 13 in order to establish the existence of periodic solutions.

(*) M. Brown and W.D. Neumann, Proof of the Poincaré-Birkhoff fixed point theorem, Michigan Math. Journ. 24, 1977, 21-31.

CHAPTER 2 - SECTION 7 - EXERCISE

1. Let $p_j(x)$ be two continuous functions of period 2π satisfying

$$0 < p_1(x) < p_2(x) .$$

With the "polar coordinates" x,y of (7.1) consider the starlike annulus

$$p_1(x) \le y \le p_2(x) .$$

Show that there exists an area-preserving homeomorphism $h: (x,y) \to (x',y')$ taking the above annulus into the concentric one

$$a_1 \le y' \le a_2 .$$

Hint: The mapping can be found in the form

$$x' = A(x)$$
$$y' = B(x) + \big(A'(x)\big)^{-1} y .$$

Setting

$$a_j = \frac{1}{2\pi}\int_0^{2\pi} p_j(x)\,dx , \quad p_j(x) - a_j = \frac{d}{dx} P_j(x)$$

with periodic $P_j(x)$ one finds

$$\begin{cases} x' = A(x) = 1 + \dfrac{P_2(x)-P_1(x)}{a_2-a_1} , \quad A' = \dfrac{p_2-p_1}{a_2-a_1} > 0 , \\ y' = \dfrac{(p_2(x)-y)a_1 + (y-p_1(x))a_2}{p_2(x)-p_1(x)} . \end{cases}$$

§8. Variations on the fixed point theorems

There are related fixed point theorems with much simpler proofs. From a mathematical point of view they are less interesting but they are frequently easier to be applied. More importantly they can be generalized to higher dimensional canonical maps which has not been possible for theorem 7.1. For this reason we discuss these results here.

a) Simple fixed point theorems

We use the same notation as in section 7 for a measure preserving map ψ of the annulus A and its lifted map ϕ on the strip S :

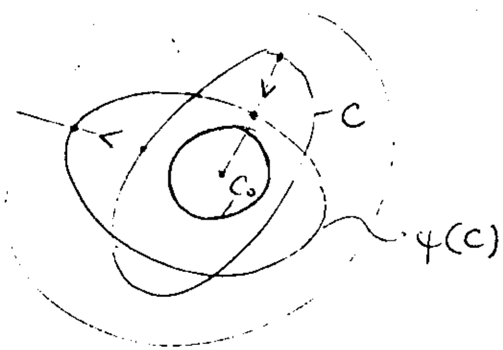
$$\phi : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_1 = f(x,y) \\ y_1 = g(x,y) \end{pmatrix} ,$$

$x \pmod{2\pi}$, $a \le y \le b$ and $f-x$, g periodic in x. We shall assume that ϕ satisfies the assumptions (7.5), (7.6) and also the twist condition (7.7), i.e.

(8.1) $(f(x,a) - x)\,(f(x,b) - x) < 0$.

First we provide simpler proofs of theorem 7.1 under the additional assumption that the function $f(x,y)$ is strictly monotone (i) either in y or (ii) in x.

(i) If $y \to f(x,y)$ is strictly monotone then also the function $y \to f(x,y) - x$ has this property and, by the twist-condition (8.1) has opposite signs for y=a and y=b. Therefore for every x there exists a unique value $y = q(x)$ for which $f(x,y) - x = 0$. Hence $q(x + 2\pi) = q(x)$ and $y = q(x)$ defines a continuous closed curve C in the annulus A, which under the map ϕ is mapped "radially". The inner boundary of this annulus, denoted by C_0: $y=a$, $0 \leq x < 2\pi$, is by assumption invariant under ψ. Since ψ is measure preserving the image curve $\psi(C)$ and the boundary C_0 enclose the same area as the curves C and C_0, and therefore C and $\psi(C)$ intersect in at least two points which are the desired fixed points of ψ.



(ii) Assume now ϕ to be differentiable and $\frac{\partial f}{\partial x}(x,y) > 0$ for (x,y) S. Then $x \to f(x,y)$ is strictly monotone in x, and we can solve the equation $x_1 = f(x,y)$ uniquely for x and write the mapping ϕ implicitly in the form

$$x = u(x_1, y), \quad y_1 = v(x_1, y)$$

for two functions u, v. The area preserving property of ϕ implies the existence of a generating function (see Chap. I,

section 4) $W(x_1, y)$ such that

$$x = W_y, \quad y_1 = W_{x_1},$$

and one verifies that

$$w(x_1, y) = W(x_n, y) - x_1 y$$

has period 2π in x_1. Therefore a critical point of w gives rise to a fixed point of ϕ. The twist condition implies that

$$x - x_1 = w_y(x_1, y)$$

has opposite signs for $y = a$ and $y = b$, hence that the inner normal derivation of w has the same sign on the two bounding circles $y = a$ and $y = b$. If this sign, say, is positive then w cannot have a maximum on the boundary and its maximum lies in the interior of S proving the existence of a fixed point of ϕ. We remark that one would find a second fixed point of ϕ corresponding to a saddle point of w.

It turns out that in these arguments it is quite unessential that the boundaries are preserved provided one requires that the line integral

$$\int_C y dx$$

over every closed curve $y = p(x) > 0$, $0 \leq x < 2\pi$ is preserved. Notice, if the inner boundary is denoted by C_0 : $y = a$, $0 \leq x < 2\pi$, then

$$\int_C y\,dx - \int_{C_0} y\,dx$$

equals the area enclosed between C and C_0 (mod 2π) and therefore for a mapping ϕ which preserves C_0 and the area also the above line integral is preserved. We have defined a differentiable mapping ϕ as "exact symplectic" if for the 1-form $\alpha = y\,dx$

$$\phi^* \alpha - \alpha = dV \ ,$$

is exact, with $V = V(x,y)$ a function on S i.e. periodic in x. This then implies

$$\int_{\phi^{-1}(C)} \alpha = \int_C \phi^* \alpha = \int_C \alpha$$

for any closed curve (mod 2π). Every exact symplectic map is, of course, area preserving since

$$\phi^*(d\alpha) - d\alpha = d(\phi^*\alpha - \alpha) = d(dV) = 0$$

and $d\alpha = dy \wedge dx$. But the converse does not hold. The mapping

$$T_\varepsilon : \quad (x,y) \to (x, y+\varepsilon)$$

is a typical counter-example. In any simply connected domain the concepts area-preserving and exact symplectic are identical.

b) Generalizations to higher dimensions

We generalize the discussion to a $2n$-dimensional space and consider a symplectic map ψ defined on $T^n \times D$ mapping this domain into $T^n \times R^n$ where T^n is the n-dimensional torus R^n/Z^n and where $D \subset R^n$ is a compact region which we assume to be convex with smooth boundary. We parameterize T^n by vectors $x \in R^n$ and identify $x, x' \in R^n$ if $(x-x')/2\pi$ is an integer vector. With $y \in D$, the lifted map ϕ takes the form

$$(8.2) \qquad \phi : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} .$$

Since ϕ maps equivalent points into equivalent points one can show (Exercise 1) that there is an n by n matrix A having integer coefficients and $\det A = \pm 1$, such that the functions

$$f(x,y) - Ax \text{ and } g(x,y)$$

have period 2π in x_j, $1 \le j \le n$. We shall study only the case where $A = I$.

We then call the map ϕ exact symplectic if

$$\phi^*\alpha - \alpha = dV \ ,$$

for a function V defined on $T^n \times D$, where

$$\alpha = \sum_{j=1}^{n} y_j \, dx_j .$$

<u>Theorem 8.1</u>: Let (8.2) define an exact symplectic diffeo-

morphism with $A = I$ mapping $T^n \times D$ into $T^n \times R^n$. Assume that ϕ satisfies

(8.3) $\langle f(x,y) - x, y-c\rangle > 0$, $(x,y) \in T^n \times \partial D$,

for some point $c \in \text{int } D$. If, in addition,

either (i) $\frac{\partial f}{\partial y} + \left(\frac{\partial f}{\partial y}\right)^T$

or (ii) $\frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial x}\right)^T$

is positive definite at all points $(x,y) \in T^n \times D$, then the map ϕ has at least one fixed point in $T^n \times D$.

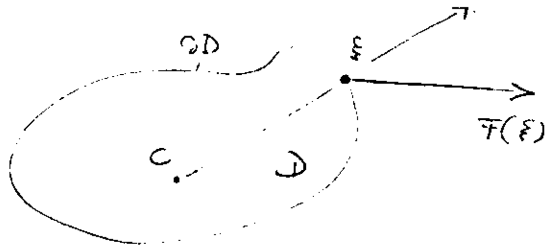
The condition (8.3) on the boundary of D generalizes the twist condition. The other condition requires $f = f(x,y)$ to be monotone with respect to y or with respect to x.

Proof. (i) We shall use the following fact from topology, which is proved with the help of the degree of a map (see N.G. Lloyd "Degree theory" Cambridge University Press, p. 25).

If $D \subset R^n$ is a compact region and $F: D \to R^n$ a continuous map which "points outwards at the boundary ∂D of D", more precisely, which satisfies:

$$\langle F(\xi), \xi - c\rangle > 0 \ , \quad \text{for all } \xi \in \partial D \ ,$$

with some $c \in \text{int } D$, then there is a point $\xi^* \in \text{int } D$ where F vanishes, i.e. $F(\xi^*) = 0$.



By assumption (8.3) we can apply this result to the map $y \to F(y) = f(x,y) - x$ for every x to find a solution y of the equation $f(x,y) = x$. Due to the monotonicity assumption we have

$$\langle f(x,y) - f(x,y'), y - y'\rangle > 0 \ , \ y,y' \in D \ , \ y \neq y' \ ,$$

from which we conclude that this solution is unique; we denote it by $y = q(x)$:

$$f(x,y) = x \ , \quad y = q(x) \ .$$

By the implicit function theorem the map $x \to (x,q(x))$ is differentiable and defines an embedded torus T^*: $y = q(x)$ in $T^n \times D$. Since ϕ is exact symplectic there exists a scalar function $V = V(x,y)$ on $T^n \times D$ with

$$\phi^*\alpha - \alpha = \sum_{j=1}^{n} (y_j^* \, dx_j^* - y_j \, dx_j) = dV \ ,$$

where $x^* = f(x,y)$ and $y^* = g(x,y)$. We restrict this 1-form to the torus T^* : $y = q(x)$ on which we have $x^* - x = f(x,q(x)) - x = 0$ and hence

$$\sum_{j=1}^{n} (y_j^* - y_j) \, dx_j = dF \ , \quad F(x) = V(x,q(x)) \ ,$$

or

$$y_j^* - y_j = \frac{\partial}{\partial x_j} F(x) .$$

Therefore, the maximum and the minimum of the function F defined on T^n provide two distinct fixed points proving the theorem under the hypothesis (i).

We observe that as in the two-dimensional case, we have found the fixed points as the points of intersection of two manifolds namely T^* and $\phi(T^*)$, i.e. as the points

$$T^* \cap \phi(T^*) ,$$

which agree with the critical points of the function $F(x) = V(x,q(x$ on T^*. It is known that a function defined on an n-dimensional torus has at least n+1 distinct critical points and actually we obtain at least n+1 fixed points of ϕ.

(ii) Since $f(x,y) = x + \hat{f}(x,y)$ with $\sup_x |\hat{f}| < \infty$, the function $x \to F(x) = f(x,y) - x^*$, for $x^* \in R^n$, $y \in D$ fixed, satisfies

$$\langle F(x), x\rangle \geq |x|^2 - (|x^*| + \sup_x |\hat{f}|)\, |x| ,$$

which is > 0, if $|x| = R$ is sufficiently large. We can therefore apply the above topological result to large spheres to find a solution x of the equations $f(x,y) = x^*$ for given $x^* \in R^n$ and $y \in D$. Therefore the map

$$x \to f(x,y)$$

is, for $y \in D$ fixed, a surjective map from R^n onto R^n. By the monotonicity assumption we have

$$\langle f(x,y) - f(x_1,y) , x - x_1\rangle > 0 , \ x, x_1 \in R^n , \quad x \neq x_1$$

and the map is one to one and hence a diffeomorphism by the implicit function theorem. Solving the equation $x^* = f(x,y)$ for x we can represent the mapping ϕ with a generating function $W(x^*,y) = \langle x^*,y\rangle + w(x^*,y)$ as

$$x_j = W_{y_j} = x_j^* + w_{y_j}$$
$$x_j^* = W_{x_j^*} = y_j + w_{x_j} ,$$

where $w(x^*,y)$ is a function defined on $T^n \times D$. It suffices to find critical points of w. Notice that by assumption (8.3) for $(x,y) \in T^n \times \partial D$

$$\langle f(x,y) - x, y-c\rangle = \langle x^*-x, y - c\rangle$$
$$= - \langle w_y , y-c\rangle > 0 ,$$

i.e. the derivative of w in the direction of y-c is negative. Since y-c points outwards w cannot have a maximum at the boundary $T^n \times \partial D$. Its maximum lies in the interior of $T^n \times D$ and gives rise to a fixed point of ϕ. This finishes the proof.

The advantage of this theorem 7.1 is that it does not require that the boundary $T^n \times \partial D$ is preserved, a condition which is hard to verify in applications. On the other hand the disadvantage of this last theorem is that it imposes conditions (monotonicity) in the interior, which would not be valid for higher iterates ϕ^j. Therefore we cannot use an argument as in theorem 7.2 in order to obtain infinitely many periodic points without checking these conditions for higher iterates.

(c) Remark 1

1) The idea to construct an auxiliary function whose critical points give rise to fixed points of a map goes back to Poincaré[1]. We shall briefly recall his idea for a symplectic map ϕ on R^{2n} given by

$$\phi : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_1 = f(x,y) \\ y_1 = g(x,y) \end{pmatrix} ,$$

which satisfies

$$\phi^*\alpha - \alpha = \langle y_1, dx_1\rangle - \langle y, dx\rangle = dF ,$$

for a function $F(x,y)$. Here $\langle y,dx\rangle$ is an abbreviation for $\sum_{j=1}^{n} y_j\, dx_j = \alpha$. Poincaré constructed in place of $\langle y_1 , dx_1\rangle - \langle y, dx\rangle$ the 1-form

(8.4) $$\beta = \langle y_1 - y, dx_1\rangle - \langle x_1 - x, dy\rangle ,$$

which is equal to $\langle y_1 , dx_1\rangle - \langle y, dx\rangle + d(\langle x - x_1, y\rangle)$, hence also

$$\beta = dG$$

is an exact 1-form, with $G(x,y) = F + \langle x - x_1, y\rangle$. It has the added advantage that at the critical points of the function G, i.e. at the points where $\beta = dG = 0$ one has $(x,y) = (x_1 , y_1)$, i.e.

1) H. Poincaré : Méthodes nouvelles de la Mécanique céleste, volume 3, Gauthiers Villars, Paris 1899 Chap. 28

$$(x,y) = \phi(x,y)$$

is a fixed point of ϕ , provided that the differentials dx_1 and dy are linearly independed there. Since $dx_1 = f_x dx + f_y\, dy$ this happens precisely if

$$\det(f_x) \neq 0$$

which is for instance the case if the given map ϕ is close to the identity map in the C^1-sense.

Similarly one can consider the alternate 1-form

(8.5) $$2\gamma = \langle y_1 - y, d(x_1 + x)\rangle - \langle x_1 - x, d(y_1 + y)\rangle ,$$

which is equal to $\langle y_1 , dx_1\rangle - \langle y, dx\rangle + \frac{1}{2}d\,(\langle x-x_1 , y+y_1\rangle)$, such that also

$$\gamma = dS$$

is an exact form, with the function $S(x,y) = F + \frac{1}{2}\langle x-x_1, y+y_1\rangle$. Again the critical points of S correspond to fixed points of the map ϕ if at these points $d(x_1 + x)$ and $d(y_1 + y)$ are linearly independent, which is the case if there

(8.6) $$\det(d\phi + 1) \neq 0 ,$$

i.e. if -1 is not an eigenvalue of $d\phi$ at these points. In fact, with the symplectic structure

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} ,$$

and abbreviating $z = (x,y)$ and $z_1 = \phi(z)$ we can write (8.5) as

$$\gamma(z) = \frac{1}{2} \langle J(\phi(z) - z) \quad , d(\phi(z) + z) \rangle \quad ,$$

hence, applied to a vector $v \in \mathbb{R}^{2n}$:

$$\gamma(z)(v) = \frac{1}{2} \langle J(\phi(z) - z) \ , \ (d\phi(z) - 1) v \rangle \ ,$$

and the claim follows.

2) Fixed points of symplectic maps can geometrically be viewed as intersection points of Lagrangian manifolds as we shall explain.

If $(M, \omega$ is a symplectic manifold, then also $(M \times M, \Omega)$ is a symplectic manifold with the 2-form Ω given by

$$\Omega = \pi_2^* \omega - \pi_1^* \omega \quad ,$$

π_1 and π_2 being the projections $M \times M \to M$. To every map $\phi : M \to M$ we can associate its "graph map" $\Phi : M \to M \times M$ by setting

$$\Phi(z) = (z, \phi(z)) \quad , \quad z \in M .$$

If

$$G_\phi = \{ \Phi(z) \mid z \in M \} \quad ,$$

we know from section 8, chapter 1 that ϕ is symplectic if and only if G_ϕ is a Lagrangian manifold of $(M \times M, \Omega)$. In particular for $\phi = id$, the "diagonal Δ in $M \times M$, i.e.

$$\Delta = \{ (z,z) \mid z \in M \}$$

is a Lagrange manifold. We notice that the fixed points $\phi(z) = z$ of ϕ are the points of the intersection of G_ϕ with Δ , i.e. $\Delta \cap G_\phi$.

We shall show in the special case $M = \mathbb{R}^{2n}$ that these points are critical points of a function on M. We shall define a symplectic diffeomorphism

$$\psi : M \times M \to T^* \Delta$$

onto the cotangent bundle of the diagonal Δ. By means of the map $(z,z) \to z$ we can identify Δ with M, such that $T^*\Delta = \mathbb{R}^n$. Denoting its coordinates by $(\xi,\eta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ the symplectic structure on $T^*\Delta$ is $d\theta$ with

$$\theta = \langle \eta, d\xi \rangle = \sum_{j=1}^{2n} \eta_j \, d\xi_j \ .$$

The map ψ is then defined by

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \xi_1 = \frac{1}{2} (x_1 + x_2) \\ \xi_2 = \frac{1}{2} (y_1 + y_2) \\ \eta_1 = y_2 - y_1 \\ \eta_2 = x_1 - x_2 \end{pmatrix} .$$

Abbreviating $z_1 = (x_1 , y_1)$ and $z_2 = (x_2 , y_2)$ one verifies readily that

$$\psi^* \theta = \frac{1}{2} \langle J(z_2 - z_1) \ , dz_1 + dz_2 \rangle \quad ,$$

hence

$$\psi^*(d\theta) = \pi_2^* \omega - \pi_1^* \omega = \Omega \quad ,$$

and ψ is indeed symplectic. The diffeomorphism ψ maps the diagonal $\Delta \subset M \times M$ onto the zero sections of $T^*\Delta$:

$$\psi : \quad (z,z) \to (z,0) \in T^*\Delta ,$$

and $\psi^*\theta$ vanishes on Δ.

If L is any Lagrange manifold in $M \times M$, then its image $\psi(L) \subset T^*\Delta$ is also a Lagrange manifold since ψ is symplectic, and the points $\Delta \cap L$ are mapped onto the points where $\psi(L)$ intersects the zero section in $T^*\Delta$. We denote by

$$p: \quad T^*\Delta \to \Delta \quad , \quad (\xi,\eta) \to \xi$$

the projection map. If the restricted map $p : \psi(L) \to \Delta$ is injective we can represent the manifold $\psi(L)$ as a section in $T^*\Delta$, i.e. as

$$\psi(L) = \{ (\xi \, , u(\xi) \,) \in T^*\Delta \, , \quad \xi \in \Delta \} \, .$$

Since $\psi(L)$ is a Lagrange manifold we have

$$u(\xi) \quad = \quad \frac{\partial}{\partial \xi} S(\xi)$$

for some function S defined on $\Delta \cong M$. In fact if $j: \psi(L) \to T^*\Delta$ is the inclusion map we conclude from $j^* d\theta = d(j^*\theta) = 0$ that $j^*\theta = dS$, and with $j^*\theta = \langle u(\xi), d\xi \rangle$ the claim follows. We therefore find that the critical points of S are precisely the points where $\psi(L)$ intersects the zero section of $T^*\Delta$, and working backwards these points correspond to $\Delta \cap L$ under the diffeomorphism ψ.

If the Lagrange manifold $L = G_\phi$ is the graph of a symplectic map ϕ, then $p : \psi(G_\phi) \to \Delta$ is injective precisely if the condition (8.6) holds true, as one readily verifies, and the points $G_\phi \cap \Delta$ correspond to the critical points of the function S on M.

(d) Remark 2

Although there is no genuine generalization of Poincaré-Birkhoff's fixed point theorem to higher dimensions, the above formal ideas have frequently been used to establish fixed point theorems for symplectic maps in higher dimensions. We mention without proof the following perturbation theorem due to A. Weinstein[1].

If (M,ω) is a compact and simply connected symplectic manifold, then every symplectic diffeomorphism ϕ of M has at least two fixed points provided it is sufficiently close to the identity map in the C^1-sense.

In the special case of a 2-dimensional sphere this statement holds true globally, i.e. without the above proviso, for every measure preserving diffeomorphism[3] although an arbitrary orientation preserving diffeomorphism may have only one single fixed point. As for a generalization of the above perturbation theorem with various applications in mechanics we refer to the paper[2].

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1) A. Weinstein: "Lectures on symplectic manifolds" Regional Conference Series in Mathematics, vol. 29 A.M.S. Providence, R.I. (1977).

2) J. Moser: "A fixed point theorem in symplectic geometry Acta mathematica, vol. 141 (1978), 17-34

3) See C.P. Simon: " A bound for the fixed point index of an area preserving map with applications to mechanics" Inventiones Mathematicae 26 (1974), 187-200, and 32 (1976), 101. and Nikishin, N: "Fixed points of diffeomorphisms on the two-sphere that preserve area" Funkcional Anal.: Prelozen 8, 84-85 (1979)

Exercise 1. Show: If

$$\phi : (x,y) \to (f(x,y)\ ,\ g(x,y)\ )$$

is a homeomorphism of $T^n \times D$ into $T^n \times D_1$ then there exists a matrix A (i.e. a matrix with integer coefficients and $\det A = \pm 1$, such that

$$f(x,y) - Ax_1\ ,\ g(x,y)$$

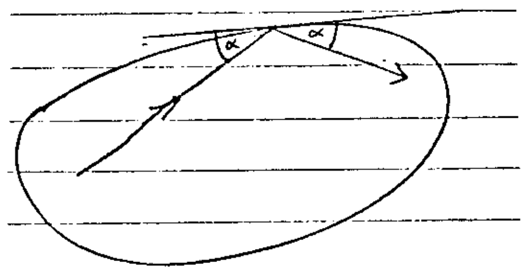
have period 2π in the $x_1, x_2, \ldots, x_n$.

Remark: The matrix A defines the mapping on the first homology group of $T^n \times D$ induced by ϕ.

§9. The Billiard Ball Problem

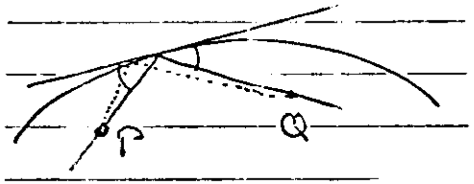
(a) The Problem

We apply the Poincaré - Birkhoff fixed point theorem to two simple geometrical problems. The first is the "billiard ball problem" which can be stated as follows: Consider a bounded, strictly convex region D in the plane, with closed C^1 - boundary curve. We study the motion of a point which moves along straight lines inside and is reflected at the boundary under equal angles.



The orbits of this motion can be very complicated for most regions D although the case of a rectangle corresponding to a conventional billiard table has an easily understood motion. But this case is excluded since the boundary of D is assumed to be continuously differentiable.

One can view this problem as the limit case of the geodesic flow on a convex two-dimensional surface which is flattened into the doubled plane domain D. The geodesics become straight lines in this limit and when the orbit passes from the upper sheet to the lower one it obeys the above reflection law. This follows from the well-known fact of optics, that the shortest path connecting two points P,Q close to each other and to the boundary and passing the boundary D is a broken line forming equal angles with the tangent at the boundary point.



(b) Result

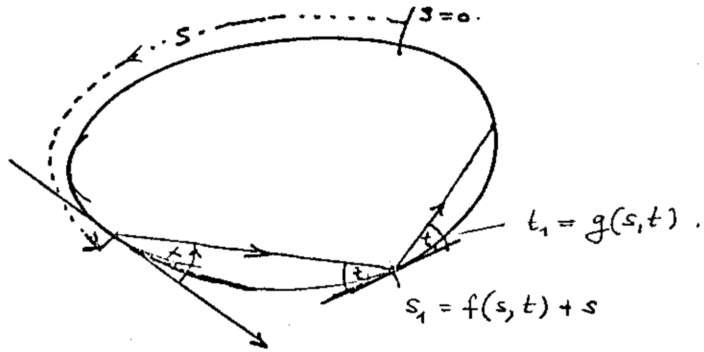
We are interested only in closed orbits, which are represented by polygons which may have self-intersections. It may be considered as a limit case of the problem of closed geodesics This application is due to G. D. Birkhoff[*]. He showed

Theorem 9.1 On a strictly convex billiard table D there exist infinitely many distinct periodic orbits.

To prove this result we associate with this billiard problem an area-preserving annulus mapping whose periodic points correspond to the periodic orbits of the billiard problem. This mapping will take a given oriented line segment joining two points on the boundary of D into the

one obtained by reflection at its end point.

To describe this mapping more precisely we introduce coordinates: Let s be a parameter along the oriented boundary of D, proportionally to the arc length, the factor being so chosen that one revolution corresponds to 2π. The angle which a line segment forms with the positively oriented tangent to the boundary at the initial point of the segment (see figure) will be denoted by t.



The initial states form an open strip

$$R = \{ s,t \mid 0 < t < \pi \}$$

which becomes an annulus if (s,t) is identified with (s',t) for $(s' - s)/2\pi \in \mathbb{Z}$. On this strip we define the mapping $\phi : (s,t) \to (s_1,t_1)$ or

$$s_1 = s + f(s,t) \quad , \quad t_1 = g(s,t) \qquad ,$$

where s_1 , t_1 correspond to the coordinates of the reflected line segment issuing from the endpoint (see figure).

In this mapping $f(s,t)$ is defined only up to an integer multiple of 2π which we shall now fix. The above mapping can be extended to a homeomorphism of R. As $t \to 0$ the line segments from (s,t) to (s_1,t_1) become shorter and we can set

$$f(s,0) = 0 \quad , \quad g(s,0) = 0.$$

This choice fixes $f(s,t)$. We observe that for $t \to \pi$ one has also $f(s,t)$ tend to an integer multiple of 2π but this integer is not zero. In fact, if the orientation of the curve is chosen as above one has

$$f(s,\pi) = 2\pi \quad , \quad g(s,\pi) = 0 .$$

This is readily proven by keeping s fixed and letting t increase from 0 to π. Then the image print (s_1, t_1) will travel once around the boundary of 2π.

Secondly, we observe that this mapping ϕ preserves the area-element

$$\sin t_1 \, dt_1 \wedge ds_1 = \sin t \, dt \wedge ds$$

(see exercise 9.1). Hence if we introduce polar coordinates r, θ by

$$\frac{1}{2}r^2 = c - \cos t \quad , \quad \theta = s \quad \text{with } c > 1$$

the annulus becomes

$$\sqrt{2(c-1)} < r < \sqrt{2(c+1)} \quad ,$$

and

$$\sin t \, dt \wedge ds = r dr \wedge d\theta$$

is the standard area - element. Evidently the two boundaries of this annulus are invariant under the map and we can apply the fixed point Theorem 7.2 : Given any rational number p/q in

$$0 < \frac{p}{q} < 1$$

there exists a fixed point of $\rho^{-p}\phi^q$ where $\rho: (s,t) \to (s + 2\pi, t)$.

Clearly these fixed points correspond to closed orbits of the billiard problem with q bounces per period. Notice that these orbits are oriented and the orbit of opposite orientation obtained by the reflection $(s,t) \to (s, \pi-t)$ and corresponding to p/q goes into one corresponding to $1 - p/q$.

In fact, introducing the map $\tau: R \to R$ by

$$\tau: (s,t) \to (s, \pi-t)$$

we notice that

$$\tau \circ \phi = \rho \circ \phi^{-1} \circ \tau \quad ,$$

hence, since $\tau^2 = \text{id}$,

$$\phi^j = \tau \circ \rho^j \circ \phi^{-j} \circ \tau .$$

If now $\phi^q(m) = \rho^p(m)$ i.e. m is a periodic point with rotation number $\frac{p}{q}$, we have to show that τm is also a periodic point of the same period q with rotation number $1 - \frac{p}{q}$. For this purpose we multiply $\phi^q(m) = \rho^p(m)$ by $\phi^{-q} \circ \rho^{-p}$ from the left to get

$$\phi^{-q}(m) = \rho^{-p}(m) \quad ,$$

and since the left hand side is equal to $\tau \circ \rho^{-q} \circ \phi^q \circ \tau$

we find

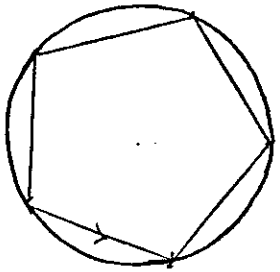
$$\phi^q(\tau m) = \rho^{q-p}(\tau m).$$

Therefore in order to avoid counting orbits with different orientation doubly we restrict ourselves to

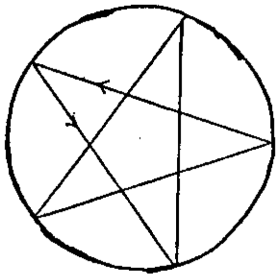
$$0 < \frac{p}{q} \le \frac{1}{2} \quad .$$

Since orbits with different values of q are different we have indeed infinitely many closed polygons which are billiard orbits, proving the theorem 9.1.

It is instructive to interpret the geometrical meaning of the numbers p,q where we assume that p,q are positive and relatively prime. In the figure below we illustrate the cases $p/q = 1/5$, $p/q = 2/5$ for a circular billiard D.



$p = 1,\ q = 5$



$p = 2,\ q = 5$

Actually Theorem 7.2 gives a stronger statement: Given any rational number p/q in $(0, \frac{1}{2})$ there exist at least 2 closed polygons representing billiard orbits with q bounces. In particular for $p = 1$, $q = 2$ there exist two different two-gons which are billiard orbits. Clearly these are given by a line segment which is perpendicular to the boundary curves at both its end points. An example are the two major axis of an elliptical billiard table D.

These orbits of period 2 can be found also in a different way: Let $B(\alpha)$ be the distance of two parallel tangent lines of D which form an angle α with a fixed direction. Thus $B(\alpha)$, the breadth in this direction, is a periodic function of α and, unless $B(\alpha)$ is a constant, it has at least one maximum and one minimum. For these values of α the segment connecting the points of contact of the two tangents provides the desired two-gon, as one easily shows.

Also, the other closed billiard paths are extrema of a function, namely just the length $\ell(\Gamma)$ of a polygon $\Gamma = \Gamma(P_1, P_2, \ldots, P_q)$ connecting q arbitrary points $P_1, P_2, \ldots P_q$ on the boundary of D. Thus ℓ is defined on a torus T^n. However, this approach can not so easily be turned into an existence proof since we have to avoid that two of the points coalesce and we do not follow this thought.

(c) <u>Elliptical billiard</u>

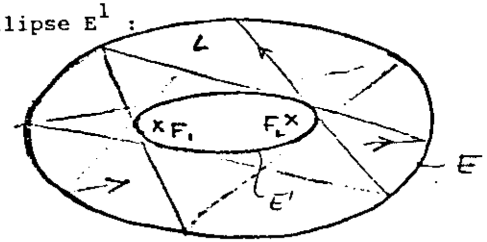
We discuss the special case where the boundary of D is an ellipse E which is especially simple. In particular, we will show that for $q > 2$ the closed billiard paths of period q form a one parameter family. This corresponds to the "integrable case" which we will discuss in the next chapter. The paths of period 2 are the two major axis; they do not belong to a family.

We begin with an orbit through one of the focal points, which we assume to be distinct, i.e. we exclude the case of a circle. We use a wellknown geometrical result: For any point

P on the ellipse we draw the two lines to the focal points F_1, F_2. Then the angle formed by the two lines is halfed by the normal to the ellipse at P. This fact is closely related to the string construction of an ellipse. In our context it implies that a line segment through one of the focal point F_1 is reflected to a line segment through F_2 giving rise to a billiard path going alternately through F_1 and F_2. If we exclude the major-axis these orbits are not closed and one convinces oneself easily that they approach asymptotically the major axis as one follows these orbits forward or backwards to infinity. This makes the major axis an unstable periodic orbit, since we can find arbirarily close line segments through which the orbits move far away. We will see that the minor-axis of the ellipse represents a stable periodic orbit.

There is a generalization of the above geometrical theorem: Let E' be a confocal ellipse of E inside of E and P a point on E. Then the angle formed by the two tangents from P to the confocal ellipse E' is halfed by the normal to the ellipse at P. This fact, which can be reduced to the earlier result by elementary geometric, though non-obvious arguments, will not be proven here. It is related to another string construction of the ellipse: If on loops a closed string about E' and pulls it taught with a pencil then the pencil will describe a confocal ellipse of E' outside of E'.

In our context it means that billiard orbits inside E can be constructed by drawing successive tangents to a confocal ellipse E^1:



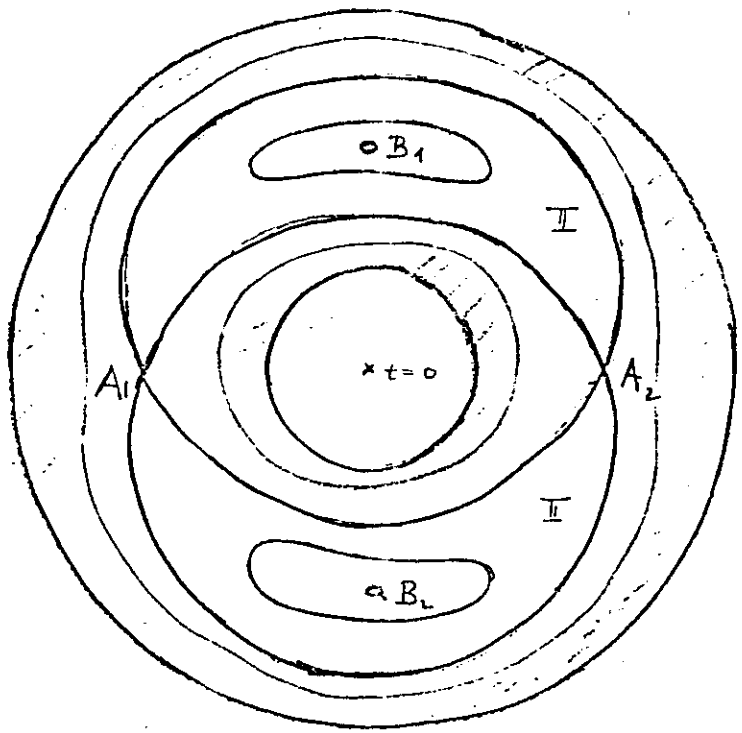
These orbits are in general not periodic. However, if E' is chosen so that one of its tangents gives rise to a closed billiard path then every of its tangents will be closed with the same period. We may call such E' "periodic". For such periodic E' there is a one parameter family of closed billiard paths - a very exceptional property of the elliptical region D.

The above geometrical theorem allows the construction of those orbits starting with a line segment outside the line segment from F_1 to F_2, since it is tangent to a confocal ellipse inside E. How about the orbits crossing this line segment $\overline{F_1F_2}$? They are tangent to the two branches of confocal hyperbolae. If we draw two tangents from $P \in E$ to a confocal hyperbola then the angle by these tangents at P is again halfed by the normal at P.

Thus there are three types of orbits. Those tangent to confocal ellipses, those tangent to confocal hyperbolas and those passing through the focal points. We describe the mapping in the annulus

$$\{s,t \mid 0 < t < \pi \ , \ (s \bmod 2\pi)\}$$

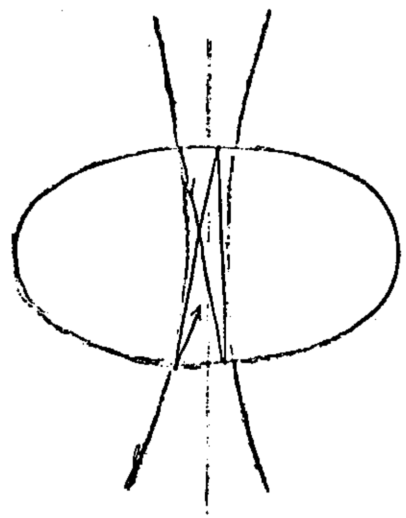
where those s,t corresponding to tangents to a confocal ellipse E^1 form curves invariant under the mapping ϕ. Similarly those segments tangent to a confocal hyperbola form a closed invariant curve.



The fixed points A_1, A_2 of ϕ^2 correspond to the major axis and B_1, B_2 to the minor axis. The heavily drawn curves through

A_1 , A_2 represent the orbits through the focal points. The region II between these curves correspond to orbits tangent to confocal hyperpolae.

The points B_1 , B_2 appear as stable fixed points which can also be seen in the other representation. Orbits starting with line segments close to the minor axis remain close to the minor axis since they are squeezed between two close confocal hyperbolae.



This presents a rather clear picture of the special situation where the billiard table is an ellipse. If it is an oval the orbit structure is by no means as simple even if the oval is close to an ellipse. In this sense, the above example is misleading.

(d) A second geometrical problem

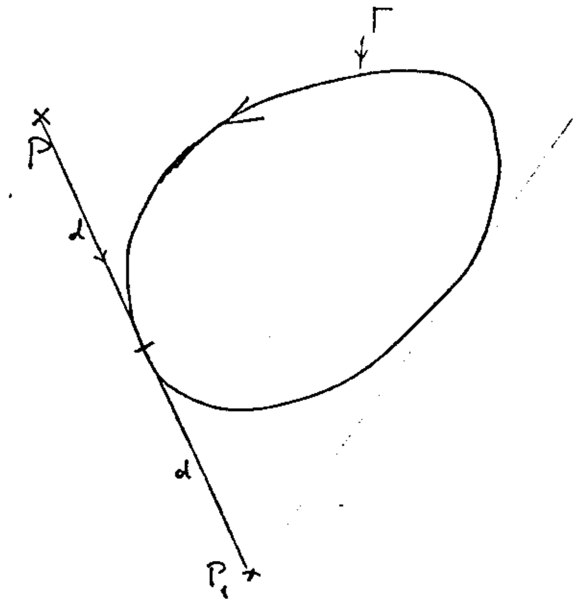
We begin again with a strictly convex domain D in the plane with a closed C^1 - boundary curve Γ. In the outside region D_0 = complement of D we define the following mapping: We consider Γ oriented and draw from a point $P \in D_0$ the positively oriented tangent and denote by P_1 the other point on this tangent which has the same distance d from the point of contact with Γ as P has. Then $P \to P_1 = \phi(P)$ defines a homeomorphism of D_0 into itself which takes Γ pointwise into itself. This mapping has clearly no fixed points in $D_0 - \Gamma$ but we claim

Theorem 9.2: The above mapping has infinitely many different periodic orbits in $D_0 - \Gamma$.

Proof: We sketch the argument. First it is easy to prove that this mapping preserves the area-element $dx \wedge dy$ in the plane. The difficulty in applying the fixed point theorem is that we have no second invariant curve. However, for a sufficiently large circle the mapping is close to a rotation by 180° . Therefore we can apply the generalized form of the theorem 7.2 to the annulus bounded by Γ and a large circle. The large circle is not invariant but rotated by approximately 180° while Γ is kept pointwise fixed. Therefore for any rational number p/q in

$$0 < \frac{p}{q} < \frac{1}{2}$$

there is a periodic orbit of period q for which the orbit circles p times the curve Γ. Since different values of p/q give different orbits the theorem is proven.



Exercise 9.1 Prove that the mapping ϕ of the billiard problem preserves $\sin t \; dt \wedge ds$.

Hint: Let $\tau(s)$ denote the angle between the positively oriented tangent to the boundary curve at the point s and a fixed direction, say the horizontal and let $\alpha = t_\theta + \tau(s)$.

From the figure below show

$$t + \tau(s) = -t_1 + \tau(s_1) = \alpha$$

and, for fixed $\alpha = t + \tau(s)$, derive from the sin - law for triangles

$$-\frac{ds_1}{ds} = \frac{\sin t}{\sin t_1}$$

or

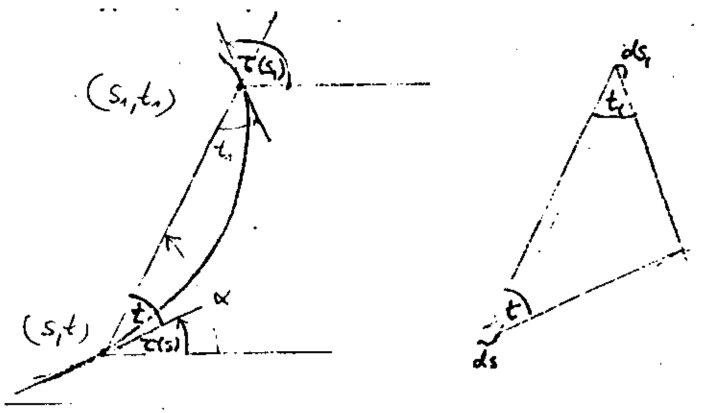
$$\sin t \; ds = - \sin t_1 \; ds_1 \; .$$

For variable α get

$$\sin t \; ds \wedge d\alpha = -\sin t_1 \; ds_1 \wedge d\alpha$$

The statement follows from

$$d\alpha = dt + \tau'(s) \; ds = - dt_1 + \tau'(s_1) ds_1 \quad .$$



Exercise 9.2: Let ϕ be an area preserving mapping in a planar region D and G a function satisfying

$$G \circ \phi = G \quad , \quad dG \neq 0 \quad .$$

Show: If a level line $\{G = c\}$ is connected and contains one fixed point of ϕ then every point of $\{G = c\}$ is a fixed point.

Hint: Use that $\dfrac{ds}{|\text{grad}\, G|}$ is an invariant length element on each level curve $G = \text{const}$.

Exercise 9.3: Let $N \subset R^{2n}$ be a connected and compact Lagrange manifold defined by $F_j = 0$, $1 \leq j \leq n$ with dF linearly independent on N. Assume ϕ is a symplectic diffeomorphism satisfying $F_j \circ \phi = F_j$, $1 \leq j \leq n$.

Show: If N contains a fixed point of ϕ, then every point of N is a fixed point.

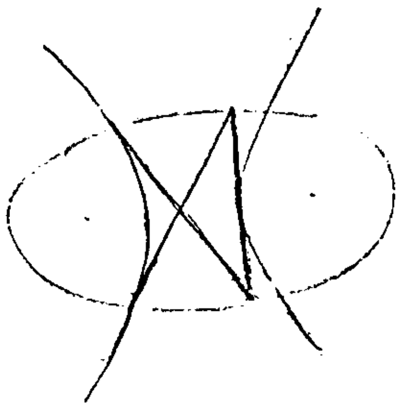
Hint: If ϕ_j^t denotes the flow of X_{F_j}, the action of R^n

$$(t_1,\dots,t_n) \to \phi_1^{t_n} \circ \dots \circ \phi_n^{t_n}(z), \quad z \quad N$$

is transitive on N and commutes with ϕ .

Exercise 9.4: Show that on the ellipse there are periodic billiard orbits of period 6 of the type indicated by the figure below.

Hint: Use that confocal hyperbolae of E intersect E orthogonally



What is its rotation number?

10. A Theorem by Jacobowitz and Hartman.

(a) Result

As an application of the Birkhoff fixed point theorem we consider a nonlinear second order differential equation

$$\frac{d^2x}{dt^2} + f(t,x) = 0 \,, \tag{10.1}$$

where f is periodic in the t-variable, and we establish the existence of infinitely many periodic solutions. More precisely we shall prove:

Theorem 10.1. Let $f(t,x) \in C^1(R^2)$ satisfy the three conditions

(i) $f(t+1,x) = f(t,x)$

(ii) $f(t,0) = 0$

(iii) $\frac{f(t,x)}{x} \to \infty$ as $x \to \pm\infty$ uniformly in t.

Then the differential equation (10.1) has infinitely many periodic solutions of period 1. More precisely, there exists an integer N_0 such that for any integer $N \geq N_0$ equation (10.1) has a solution of period 1 and with exactly $2N$ zeroes in $[0,1)$. If n_0 is the number of zeroes of a nontrivial solution $y(t)$ of $\ddot{y} + f_x(t,0)y = 0$ in $[0,1)$ then one can take N_0 as the smallest integer $\geq n_0/2 + 1$.

The same method of proof will also allow us to find "subharmonic" solutions. These are periodic solutions of a period q, q being an integer ≥ 2, i.e. a multiple of the period of the forcing term.

<u>Theorem 10.2.</u> Under the assumptions of Theorem 10.1 and for any two positive integers p, q with $p \geq N_0 q$ there exists a periodic solution of (10.1) of period q which has $2p$ zeros in the interval $[0,q)$. Moreover solutions belonging to different ratios p/q are different functions.

<u>Remark 1.</u> If p,q are relatively prime then the solution belonging to p,q has q as the *smallest* period.

<u>Remark 2.</u> This is a special case of a theorem due to P. Hartman,(1) which was preceded by a theorem by Jacobowitz,(2) who had to impose the additional restriction $x\, f(t,x) > 0$. Here we follow essentially Hartman's approach.

We give some simple examples illustrating the above theorems and their limitations.

<u>Example 1.</u>

For the t-independent differential equation

$$\ddot{x} + \alpha\, x + \beta x^3 = 0\,, \qquad \alpha,\beta > 0,$$

actually all solutions are periodic. More generally the same is true for

$$\ddot{x} + f(x) = 0 \tag{10.2}$$

with $f(0) = 0$, f continuously differentiable, $f'(x) > 0$ and

$$F(x) = \int_0^x f(s)\, ds \to +\infty \quad \text{as} \quad x \to \pm\infty\,.$$

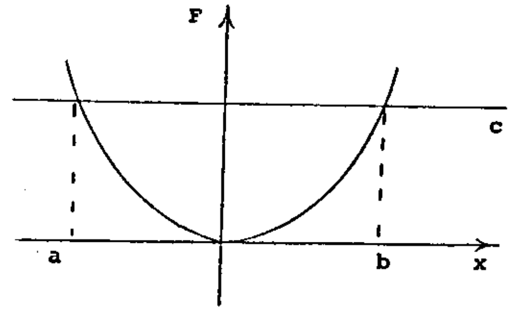
(1) P. Hartman: On boundary value problems for superlinear second order differential equations, Jour. Diff. Eq. <u>26</u>, 37-53, 1977.

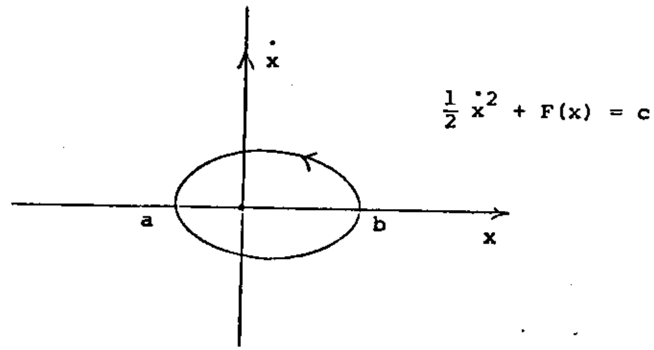
(2) H. Jacobowitz: Periodic solutions of x" + f(t,x) = 0 via the Poincaré Birkhoff fixed point theorem, Jour. Diff. Eq. <u>20</u>, 37-52, 1976.

Indeed, multiplying (10.2) by $\dot{x}$ and integrating we have

$$\frac{1}{2}\dot{x}^2 + F(x) = c\,, \tag{10.3}$$

where c is a constant for each solution (energy conservation). Since $F''(x) > 0$ the above function is convex and (10.3) defines closed convex curves in the x-$\dot{x}$-plane. These are the solution curves.





Their period is given by

$$T(c) = \sqrt{2} \int_a^b \frac{dx}{\sqrt{c-F(x)}} ,$$

where $a < b$ are the two solutions of $F(a) = F(b) = c$. If $x^{-1}f(x) \to +\infty$ as $x \to \pm\infty$, one verifies (Exercise 1) that $T(c) \to 0$ as $c \to \infty$, so that we find c solving the equation

$$T(c) = \frac{1}{N}$$

for sufficiently large integer N. The corresponding solution $x(t)$ describes the closed curve in the time $T(c) = N^{-1}$, and thus has also the period $N\,T(c) = 1$ during which time it runs N times monotonically through the closed curve. The zeros of $x(t)$ correspond to the passage of the $\dot{x}$-axis. During N resolutions $2N$ such passages will take place.

This example is, of course, trivial. A nontrivial example is the periodically forced Duffing equation

$$\ddot{x} + \alpha x + \beta x^3 = \gamma \sin(2\pi t)$$

with $\beta > 0$. Theorem 10.1 implies the existence of orbits of period 1 with an arbitrary large number of zeros. Theorem 10.^ implies the existence of periodic solutions with an arbitrary long (integer) period.

If $f = f(t,x,\dot{x})$ depends on $\dot{x}$ also the above theorems fail in general — the equation has to be dissipationless — as the following example shows.

Example 2. Let

$$f(x,\dot{x}) = h(x) + g(\dot{x})$$

where $h(0) = 0$, $s \cdot g(s) > 0$ for $s \neq 0$, $g(\dot{x})$ representing the "dissipation term". Then the only periodic solution of

$$\ddot{x} + f(x,\dot{x}) = 0$$

is the trivial solution $x \equiv 0$, no matter how small g is.

To prove this we set

$$H(x) = \int_0^x h(s)\, ds .$$

Then, along any solution we have

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + H(x)\right) = \dot{x}\left(\ddot{x} + h(x)\right) = -\dot{x}\, g(\dot{x}) .$$

For a nontrivial periodic solution of period $T > 0$ we find by integration

$$0 = \frac{\dot{x}^2}{2} + H(x)\Big|_0^T = -\int_0^T \dot{x}\, g(\dot{x})\, dt < 0$$

which is a contradiction.

(b) Outline of proof.

For the proof we shall consider the mapping ϕ of the initial values $x(0) = x_0$, $\dot{x}(0) = y_0$ of a solution $x(t)$ of (10.1) into the point (x_1, y_1) where $x_1 = x(1)$; $y_1 = \dot{x}(1)$. This mapping preserves the area-element. This follows, for example, from the fact that (10.1) can be written as a Hamiltonian system

$$\dot{x} = H_y(t,x,y) \ ; \quad \dot{y} = - H_x(t,x,y)$$

with

$$H = \tfrac{1}{2} y^2 + F(t,x) \ ; \quad F(t,x) = \int_0^x f(t,u)\, du$$

(see the corollary to I. Theorem 3.1). Also, ϕ has the origin as a fixed point, since $x \equiv 0$ is a solution. We consider, in polar coordinates, the origin as the inner boundary of an annulus. The outer boundary will be taken as a large circle $x^2 + y^2 = a$; this outer boundary will not be mapped into itself but the argument of $x + iy$ will be advanced arbitrarily much if a is large enough, so that the Poincaré-Birkhoff fixed point theorem becomes applicable.

The main difficulty in this program is that the mapping ϕ is not defined in all of R^2, since we are not assured that the solutions of (10.1) exist for the time interval $0 \leq t \leq 1$. In fact, Coffmann and Ulrich[1] give an example of a positive but only continuous function $g(t) \in C^0[0,1]$ such that $\ddot{x} + g(t)x^3 = 0$ has a solution which does not exist on the whole interval $[0,1]$. To avoid this problem we derive (following Hartman) an "apriori estimate" for those solutions having at most n zeros. This gives us an upper bound $x^2 + \dot{x}^2 \leq B^2$ for the sought periodic solutions with prescribed number of zeros. Then we will modify the function $f(t,x)$ for $|x| > 2B$ so that it grows at most

[1] C. V. Coffmann and D. F. Ulrich, On the continuation of solutions of certain nonlinear differential equations, Monatshefte Math. 71, 385-392 (1967).

linearly, but is unchanged for $|x| \leq B$. Then the so modified equation has solutions for *all* t, i.e. the mapping ϕ for this equation is well defined, and the periodic solution can be constructed in the indicated manner. But since the a-priori bound assures us that these periodic solutions lie in the region $|x| < B$ where the modified and the original one agree they are actually solutions to (10.1). This device of using "a priori" estimates is a frequent and useful device in the theory of ordinary and partial differential equations. The proof of Theorem 10.1 is a rather intricate example of this technique. The difficulty stems from the fact that we impose conditions only on large values of $|x|$ while the solutions pass again and again through $x = 0$.

(c) <u>A priori estimates</u>

We will define a class $\mathcal{F}$ of functions $h(t,x)$ which will be chosen so that

(i) $\mathcal{F}$ contains $f(t,x)$ of (10.1) (Lemma 10.1),

(ii) if $h \in \mathcal{F}$ then also h_0,

$$h_0(t,x) = \begin{cases} h(t,x) & \text{for } |x| \leq c \\ \frac{x}{c}\, h(t,c) & \text{for } x \geq c \\ -\frac{x}{c}\, h(t,-c) & \text{for } x \leq -c \end{cases}$$

belongs to $\mathcal{F}$ for sufficiently large c (Lemma 10.2),

(iii) for every $h \in \mathcal{F}$ we have an a priori estimate for the solutions of

$$\ddot{x} + h(t,x) = 0$$

possessing at most n zeros (Lemma 10.4).

Definition. $F = F_T$ is defined in terms of positive constants T, a, b, M, m satisfying

(10.4) $0 < 2a < b, \quad 0 < M < b$,

and a positive function $g(s) \in C[a,\infty)$ for which $g(s)/s$ is monotone and tends to ∞ as $s \to \infty$, as follows:

F consists of all $h(t,x) \in C([0,T] \times R)$ which are Lipschitz continuous in x and satisfy

(10.5a) $|h(t,x)| \leq M$ for $|x| \leq a$

(10.5b) $x^{-1}h(t,x) \geq 4$ for $|x| \geq a$

(10.5c) $|h(t,x)| \leq g(|x|)$ for $|x| \geq a$

(10.5d) $x^{-1}h(t,x) \geq m^2$ for $|x| \geq b$.

Lemma 10.1. Let f be given as in Theorem 10.1. Then given an integer $n \geq 1$ and a positive $T > 0$, the parameters of F can be chosen such that

(10.6) $f \in F_T$

and

(10.7) $m > 2\pi n T^{-1}$.

Proof: Choose m so large that (10.7) holds, then a so large that (10.5b) holds for h = f. This is possible since by assumption $x^{-1}f(t,x) \to \infty$ for $|x| \to \infty$. Next define

$$g(s) = s\left[1 + \max_{\substack{0 \leq t \leq T \\ a \leq |x| \leq s}} |x|^{-1}|f(t,x)|\right],$$

so that $s^{-1}g(s)$ is monotonic increasing and tends to infinity as $s \to \infty$. Moreover

$$|f(t,x)| \leq \frac{|x|}{s} g(s) \leq g(s) \quad \text{for} \quad a \leq |x| \leq s ,$$

hence (10.5c) holds for h = f. Finally, set

$$M = \max_{\substack{0 \leq t \leq T \\ |x| \leq a}} |f(t,x)|$$

and pick b so large that $x^{-1}f(t,x) \geq m^2$ for $|x| \geq b$.

Lemma 10.2. If f belongs to F_T , so does the function f_0 defined by

$$f_0(t,x) = \begin{cases} f(t,x) & \text{for} \quad |x| \leq c \\ \frac{x}{c} f(t,c) & \text{for} \quad x \geq c \\ -\frac{x}{c} f(t,-c) & \text{for} \quad x \leq -c \end{cases}$$

for any $c > b$. Moreover, the function

(10.8) $$\frac{f_0(t,x)}{1 + |x|}$$

is bounded on $[0,T] \times R$.

Proof: Clearly (10.5b) and (10.5d) are satisfied since these conditions impose only lower bounds. Formula (10.5c) follows from the monotonicity of $s^{-1}g(s)$, while the boundedness of (10.8) is immediate.

Lemma 10.3. Assume $f \in F_T$, such that $mT > 2\pi n$ for an integer $n > 0$. Then there exists a positive number λ_n, depending on n and F_T only, such that every solution x(t) of (10.1) with initial conditions

$$x(0)^2 + \dot{x}(0)^2 \geq 2\lambda_n^2 \quad \text{for} \quad t = 0 ,$$

possesses at least n zeros on its interval of existence in [0,T].

We postpone the lengthy but elementary proof. Lemma 10.3 gives an a-priori estimate for the solutions of (10.1) having at most n zeros. Namely we have

Lemma 10.4. Assume $f \in F_{2T}$, with $mT > 2\pi n$ for an integer $n > 0$. Let x(t) be a solution of $\ddot{x} + f(t,x) = 0$ defined on the interval [0,2T] where it has at most n-1 zeros. Then one has

$$x(t)^2 + \dot{x}(t)^2 < 2\lambda_n^2 \quad \text{for} \quad 0 \leq t \leq 2T. \tag{10.9}$$

Proof: Otherwise one would have

$$x^2 + \dot{x}^2 \geq 2\lambda_n^2$$

for some $t = t_0 \in [0,2T]$. If $0 \leq t_0 \leq T$ we apply Lemma 10.3 to $f(t_0+t,x) \in F_T$ on the interval [0,T] to get there at least n zeros and hence a contradiction. If $T \leq t_0 \leq 2T$ we apply the same argument to $f(-t_0+t,x) \in F_T$, proving Lemma 10.4.

(d) Proof of Theorem 10.1.

We search for periodic solutions of period 1 having 2N zeros in [0,1). Applying Lemma 10.4 with 2T = 1 and n = 2N+1 we get for any such periodic solution the a-priori estimate (10.9). We therefore may replace f by a function f_0 which has only linear growth and which agrees with f for $|x| < \sqrt{2}\,\lambda_n$ by applying Lemma 10.2 with $c = 2\sqrt{\lambda_n}$. We may assume, of course, that c > b. The solutions of the modified equation $\ddot{x} + f_0(t,x) = 0$ exist for all time and we can study the map $\phi: R^2 \to R^2$ defined by

$$\phi: \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} \to \begin{pmatrix} x(1) \\ \dot{x}(1) \end{pmatrix},$$

which is area preserving as was noticed above. We shall describe this map in symplectic polar coordinates

$$x = \sqrt{2R}\,\sin\theta , \qquad \dot{x} = \sqrt{2R}\,\cos\theta ,$$

for which we have

$$dx \wedge d\dot{x} = d\theta \wedge dR ,$$

and $R = \frac{1}{2}(x^2 + \dot{x}^2)$. The differential equations are transformed into

$$\begin{aligned} \dot{\theta} &= P(t,\theta,R) = \cos^2\theta + x^{-1} f_0 \sin^2\theta \\ \dot{R} &= Q(t,\theta,R) = R(1 - x^{-1} f_0) \sin 2\theta , \end{aligned} \tag{10.10}$$

where we have to insert $x = \sqrt{2R}\,\sin\theta$ in $x^{-1} f_0(t,x)$. We consider the map ϕ in the punctured disc

$$0 < x^2 + \dot{x}^2 < 2\lambda_n$$

or

$$0 < R < \lambda_n \quad ,$$

and define it in the above polar coordinates as follows:

(10.11) $(\theta_0, R_0) \to (\theta_1, R_1) = [\theta(1) - 2\pi N, R(1)]$,

where $\theta(t)$, $R(t)$ is the solutuion of (10.10) with initial condition $\theta(0) = \theta_0$, $R(0) = R_0$. This map is area preserving and leaves $R = 0$ invariant since $x = \dot{x} = 0$ is a fixed point. The other boundary, $R = \lambda_n$, is, of course, not preserved. In order to apply the fixed point theorem we have to show that (10.11) is a "twist"-map in the sense that

(10.12)
$$\theta_1 - \theta_0 > 0 \text{ for } R_0 = \lambda_n$$
$$\theta_1 - \theta_0 < 0 \text{ for } R_0 = 0 .$$

This follows from Lemma 10.3: Any solution with $R_0 \geq \lambda_n$ has at least n zeros $t_1, t_2, \ldots, t_r$ in $[0,1)$ so that $\pi^{-1}\theta(t_k)$ is an integer. By (10.10) we have $\dot{\theta}(t_k) = 1$ so that $\theta(t)$ crosses at least n of the lines $\theta = j\pi$ increasingly, hence $\theta(1) - \theta(0) > (n-1)\pi = 2N\pi$, which proves the first inequality of (10.12). At the inner boundary, $R = 0$, we have from (10.10)

(10.13) $$\dot{\theta} = \cos^2\theta + f_x(t,0)\sin^2\theta ,$$

since $f_{0x}(t,0) = f_x(t,0)$. If n_0 is the number of zeros of a nontrivial solution of $\ddot{y} + f_x(t,0)y = 0$ in $[0,1)$, we conclude as above, that for a solution of (10.13)

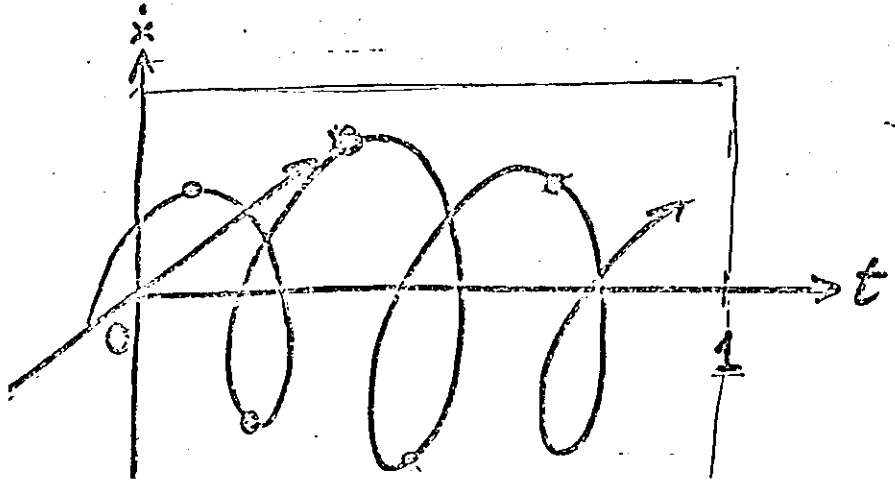
$\theta(1) - \theta(0) \leq (n_0 + 1)\pi$. Therefore, if we choose $2N > n_0 + 1$ the other inequality of (10.12) follows.

The Poincare-Birkhoff fixed point theorem (Theorem 7.3) is applicable and guarantees a fixed point (θ_0, R_0) of ϕ in $0 < R < \lambda_n$. Since $\theta_1 = \theta_0$ we conclude from

$$\theta(1;\theta_0,R_0) = \theta_0 + 2N\pi ,$$

and from the fact that the lines $\theta = j\pi$ are crossed increasingly, that the solution has precisely 2N zeros in its period $0 \leq t < 1$. By the a-priori estimate (Lemma 10.4) the solution lies in $0 < R < \lambda_n$ and is therefore a solution of the original equation $\ddot{x} + f(t,x) = 0$. This proves Theorem 10.1 aside from Lemma 10.3.

Remark. We note that the number of zeros, 2N, is related to the rotation number of the periodic solution. During every increase of θ by 2π one passes twice the plane $x = 0$ (see figure).



The proof of Theorem 10.2 proceeds along the same lines and is left as an exercise. In this case $\frac{p}{q}$ is the ratio of revolutions of the solution about the t-axis and the increase of t, or analytically,

$$\frac{p}{q} = \lim_{t\to\infty} \frac{\theta(t)}{2\pi t} .$$

This makes it evident that different ratios p/q correspond to different solutions.

(e) Proof of Lemma 10.3.

We shall estimate the time a solution spends in the various regions $|x| \leq a$, $|x| \geq a$ and $|x| \geq b$ and estimate $\dot{x}$ in the process. We extend $f(t,x)$ to all real t by setting $f(t,x) = f(T,x)$ for $t > T$ and $f(t,x) = f(0,x)$ for $t < 0$, but in the end these ranges will actually not be needed.

(i) Passage through $|x| \leq a$:

If $\lambda \geq \max(2M, 4a)$ and $-a \leq x(t_0) < a$, $\dot{x}(t_0) \geq \lambda$, then there exists a $t_1 > t_0$ such that

$$x(t_1) = a, \quad \dot{x}(t_1) \geq \frac{1}{2}\lambda \quad \text{and} \quad t_1 - t_0 < \frac{4a}{\lambda} .$$

A similar estimate holds for $x(t_1) = -a$, if $-a < x(t_0) \leq a$ and $\dot{x}(t_0) \leq -\lambda$.

(ii) Passage from $x = a$ to a local maximum:

If $\lambda^2 \geq 2\int_0^{2b} g(s)\,ds$ and $x(t_0) \geq a$, $\dot{x}(t_0) \geq \lambda > 0$, then there exists a $t_1 > t_0$ with

$$\dot{x}(t_1) = 0, \quad x(t_1) > 2b \quad \text{and} \quad t_1 - t_0 < \frac{\pi}{m} .$$

(iii) Passage from a local maximum to $x = a$:

If $x(t_0) \geq \lambda \geq 2b$, $\dot{x}(t_0) = 0$ then there exists a least $t_1 > t_0$ satisfying

$$x(t_1) = a, \quad \dot{x}(t_1) \leq -\lambda \quad \text{and} \quad t_1 - t_0 \leq \frac{\pi}{m} .$$

(iv) Passage from large x-values to $x = a$:

We drop the condition $\dot{x}(t_0) = 0$ in the last case and assume only $x(t_0) \geq \lambda \geq 2b$. Then there exists a $t_1 > t_0$ with

$$x(t_1) = a, \quad \dot{x}(t_1) \leq -\lambda \quad \text{and} \quad t_1 - t_0 \leq \frac{2\pi}{m} .$$

We prove these claims, ad (i): Suppose t is chosen in $0 < \tau \leq 1 \leq \lambda/2M$ such that $-a \leq x(t) \leq a$ for $t_0 \leq t \leq t_0+\tau$. Then we conclude from (10.5a) $\ddot{x} \geq -M$, hence $\dot{x}(t) \geq \dot{x}(t_0) - M(t-t_0) \geq \lambda - M\tau \geq \lambda/2$, and by integration $x(t_0+\tau) \geq -a + (\lambda/2)\tau$. Thus if $\tau > 4a/\lambda$ we obtain $x(t_0+\tau) > a$ and a contradiction to our assumptoin on τ. Hence $\tau \leq 4a/\lambda \leq 1$, and if τ is chosen maximally we have $x(t_0+\tau) = a$ and $\tau \leq 4a/\lambda$, i.e. $t_1 = t_0 + \tau$.

ad (ii): By (10.5b) we have, as long as $x(t) \geq a > 0$, the inequality $\ddot{x} \leq -4x$. Therefore by a Sturm comparison argument a maximum of $x(t)$ must occur within an interval of length $\pi/4$. Indeed, assume the opposite, then $x(t_0) \geq a$, $\dot{x}(t) > 0$ for $t_0 \leq t \leq t_0 + \pi/4 = \tau$. Hence $x(t) \geq a$ and $\ddot{x} \leq -4x$ in this interval. Let $y = \sin 2(t - t_0)$ be a solution of $\ddot{y} + 4y = 0$ over this interval, then

$$\ddot{x}y - \ddot{y}x = \left(-f(t,x) + 4x\right)y \leq 0 ,$$

but integration from t_0 to τ of the left-hand side gives

$$\dot{x}y - x\dot{y}\Big|_{t_0}^{\tau} = \dot{x}(\tau) + 2x(t_0) > 0 ,$$

a contradiction. To get a better estimate for this maximum we use the Liapunov function

$$V(t) = \frac{1}{2}\dot{x}^2 + G(x) , \quad G(x) = \int_a^x g(s)\,ds ,$$

so that by (10.5c)

$$\frac{dV}{dt} = \dot{x}\big(\ddot{x} + g(x)\big) \geq 0$$

as long as $x \geq a$, $\dot{x} \geq 0$. If the first maximum occurs at $t = t_1 > t_0$ we have

$$\frac{1}{2}\lambda^2 \leq V(t_0) \leq V(t_1) = G\big(x(t_1)\big) .$$

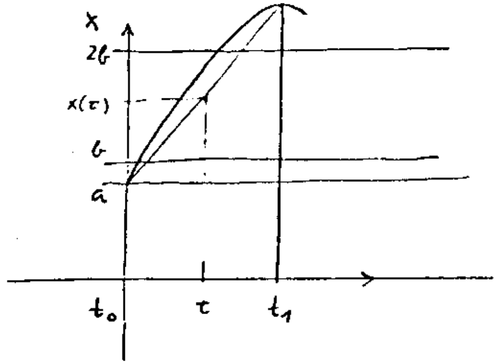
It is convenient to introduce the inverse function $x = \Phi(y)$ of $y = G(x)$. Since $g(s) > 0$ in $x \geq a$ and $g(s) \geq m^2 s$ for large s by (10.5d) we have $G(x) \to \infty$ as $x \to \infty$ and G is monotone. Thus $\Phi(y)$ is a monotone function on $[0,\infty)$ and $\Phi(y) \to \infty$ as $y \to \infty$. With our assumption $\frac{\lambda^2}{2} \geq \int_0^{2b} g(s)\,ds$, the last inequality then reads

$$x(t_1) \geq \Phi(\tfrac{1}{2}\lambda^2) \geq 2b .$$

To get an improved estimate for $t_1 - t_0$ we use the following trick: $x(t)$ is concave since by (10.5b) $\ddot{x} < 0$,

so that for $\tau = \frac{1}{2}(t_1 + t_0)$

$$x(\tau) \geq \frac{1}{2}\big(x(t_1) + x(t_0)\big) > b .$$



Thus $x(t) \geq x(\tau) > b$ on $[\tau,t_1]$ where we have $\ddot{x} \leq -m^2 x$ by (10.5d). Therefore by a Sturm comparison argument we find

$$\frac{1}{2}(t_1 - t_0) = (t_1 - \tau) < \frac{\pi}{2m}$$

proving claim (ii).

ad (iii): Let t_1 be the smallest number $> t_0$ with $x(t_1) = a$ and define the auxiliary function

$$W(t) = \dot{x}^2 + 4(x^2 - a^2) .$$

Then on $[t_0,t_1]$

$$\frac{dW}{dt} = 2\dot{x}(-f + 4x) \geq 0 ,$$

since $\dot{x} \leq 0$ and $f \geq 4x$ by (10.5b). Hence

$$\dot{x}^2(t_1) = W(t_1) \geq W(t_0) = 4(x^2(t_0) - a^2) \geq 3x^2(t_0) > \lambda^2$$

as $a \leq b \leq \frac{1}{2} x(t_0)$. The time interval can be estimated as above.

ad (iv): If $\dot{x}(t_0) \geq 0$ the claim follows from (ii) and (iii). The case $\dot{x}(t_0) < 0$ can be reduced to (iii) by decreasing t_0 until $\dot{x}$ vanishes.

We shall combine the above estimates (i) - (iv) in the following

Lemma 10.5. There exists a monotone increasing function ϕ, $\phi(s) \to \infty$ as $s \to \infty$ and two positive constants λ_0 and c, such that for $\lambda \geq \lambda_0$ the following holds true: if $x(t)$ is a solution of (10.1) satisfying

$$(10.14) \qquad x^2 + \dot{x}^2 > 2\lambda \quad \text{for} \quad t = t_0 ,$$

then there exists a zero of $x(t)$ at time $t_1 > t_0$ with

$$(10.15) \qquad x(t_1) = 0 , \quad \dot{x}^2(t_1) \geq 2\phi(\lambda)^2$$

and

$$(10.16) \qquad t_1 - t_0 < \frac{2\pi}{m} + c\,\frac{a}{\phi(\lambda)} .$$

<u>Proof</u>: Define

$$\phi(s) = \frac{1}{\sqrt{8}} \min\left\{s,\ \Phi\left(\frac{s^2}{8}\right)\right\} ,$$

clearly $\phi(s) \to \infty$ as $s \to \infty$. Pick λ_0 so large that

$$\phi(\lambda_0) \geq 2b ,$$

then if $\lambda \geq \lambda_0$ we have $\lambda \geq 2M, 4a, \sqrt{2G(2b)}, 2b$ since by assumption $M < b$ and $2a < b$. The assumption (10.14) implies $|x(t_0)| \geq \lambda$ or $|\dot{x}(t_0)| \geq \lambda$. In the first case we apply (iv) and reach $|x(t_1)| = a$ in a time $t_1 - t_0 \leq 2\pi/m$ with $|\dot{x}(t_1)| \geq \lambda$. Using then (i) we find a zero for $t_2 > t_1$ where $t_2 - t_1 < \frac{4a}{\lambda}$ and $|\dot{x}(t_2)| \geq \frac{1}{2}\lambda$, i.e. $\dot{x}^2(t_2) \geq \frac{1}{4}\lambda^2 \geq 2\phi(\lambda)^2$ and $t_2 - t_0 \leq \frac{2\pi}{m} + \frac{4a}{\lambda} \leq \frac{2\pi}{m} + \frac{4}{\sqrt{8}}\,\frac{a}{\phi(\lambda)}$ as claimed in (10.16) and (10.15).

If $|\dot{x}(t_0)| \geq \lambda$ we consider the cases $|x(t_0)| \leq a$ and $\geq a$ separately. If $|x(t_0)| \leq a$ we find either (α) a zero $t_1 < t_0 + 4a/\lambda$ with $|\dot{x}(t_1)| \geq \frac{1}{2}\lambda$ or (β) reach $|x(t_1)| = a$ with $|\dot{x}(t_1)| \geq \frac{1}{2}\lambda$ in the time $< 4a/\lambda$. In case (α) we are finished and case (β) reduces to the following and last case. $|x(t_0)| \geq a$, $|\dot{x}(t_0)| \geq \frac{1}{2}\lambda$, where we replaced λ by $\frac{1}{2}\lambda$. Using (ii) and (iii) we return to $|x(t_1)| = a$ for some $t_1 > t_0$ with $t_1 - t_0 < 2\pi/m$ and have $|\dot{x}(t_1)| \geq \Phi\left(\frac{1}{2}\left(\frac{\lambda}{2}\right)^2\right) \geq \sqrt{8}\,\phi(\lambda)$. Then one reaches by (i) a zero at t_2 for $x(t)$ in a time $t_2 - t_1 < \left(1/\sqrt{8}\phi(\lambda)\right)4a$ with $|\dot{x}(t_2)| \geq \frac{1}{2}\,\Phi\left(\frac{1}{8}\lambda^2\right) \geq \sqrt{2}\,\phi(\lambda)$. Adding these contributions we reach a zero in a time

$$< \frac{2\pi}{m} + \frac{4a}{\lambda} + \frac{4a}{\Phi\left(\frac{1}{8}\lambda^2\right)} \leq \frac{2\pi}{m} + c\,\frac{a}{\phi(\lambda)}$$

with $c = \sqrt{8}$, and $\dot{x}(t_2)^2 > 2\phi(\lambda)^2$ at this point, as we have claimed.

We shall now apply the above lemma repeatedly, replacing λ by $\phi(\lambda)$ and so on, and define therefore $\phi_0(s) = s$, $\phi_1(s) = \phi(s)$

as in the lemma, and $\phi_{j+1}(s) = \phi \circ \phi_j(s)$. Clearly $\phi_j(s)$ is monotone increasing and $\phi_j(s) \to \infty$ as $s \to \infty$. For a given $n < mT/2\pi$ we can choose $\lambda = \lambda_n$ so large that

$$\sum_{j=1}^{n} \left(\frac{2\pi}{m} + c\,\frac{a}{\phi_j(\lambda)}\right) = \frac{2\pi n}{m} + ca \sum_{j=1}^{n} \phi_j(\lambda)^{-1} < T .$$

For λ_n sufficiently large we therefore conclude applying Lemma 10.5 n times, that a solution of (10.1) with $f \in F_T$ and with initial conditions $x(0)^2 + \dot{x}(0)^2 \geq 2\lambda_n{}^2$ has at least n zeros in $0 \leq t \leq T$. This finishes the proof of the crucial Lemma 10.3.

Remark 1. We observe that it is not necessary to assume $f(t,x) \in C^1(R^2)$ in Theorem 10.1 and 10.2. Indeed the proof remains the same if $f \in C(R^2)$, f locally Lipschitz-continuous in x and $\frac{\partial f}{\partial x}(t,0)$ continuous. We merely have to check that the local flow ϕ^t of $\ddot{x} + f(t,x) = 0$ is measure preserving, i.e.

$$\int \zeta(\phi^t(x))\, dx = \int \zeta(x)\, dx \tag{10.12}$$

for every smooth function ζ with compact support. As we have seen we can write the equation as a Hamiltonian system with a C^1-Hamiltonian function H. Approximating this Hamiltonian by smoother ones H_n, (10.12) holds true for the corresponding flows ϕ_n^t and hence (10.12) follows in the limit.

Remark 2. As already pointed out one difficulty in our proof is that it is not known whether the solutions of the differential equation (10.1) are bounded. For example, Littlewood posed the problem of deciding whether all solutions of the equation

$$\ddot{x} + g(x) = p(t)$$

are bounded, if $g(x)/x \to +\infty$ as $|x| \to \infty$ and $p(t)$ is periodic (see J. E. Littlewood [1]). In the special case of $g(x) = x^3$ G. R. Morris [2] confirmed this conjecture. One has to expect that for the boundedness more assumptions have to be imposed than we required for Theorem 10.1. One could conjecture that the method which we shall develop in Chapter 5 allows us to establish boundedness of all solutions of (10.1) if

$$f(t,x) = x^{2p+1} + g(t,x) \qquad (p \geq 0, \text{ integer})$$

where g has period 1 in t and $g(t,x)$ with derivatives in x behaves like a polynomial of degree $\leq 2p$ as $x \to \pm\infty$.

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[1] J. E. Littlewood, Unbounded solutions of an equation $\ddot{y} + g(y) = p(t)$, with $p(t)$ periodic and bounded, and $g(y)/y \to \infty$ as $y \to \pm\infty$, J. London Math. Soc. 41, 1966, 497-507.

[2] G. R. Morris, A case of boundedness in Littlewood's problem on oscillatory differential equations, Bull. Australian Math. Soc., 14, 1976, pp. 71-93.

Exercise 1

Let f be continuously differentiable with $f(0) = 0$, $f'(x) > 0$ and $x^{-1}f(x) \to \infty$ as $|x| \to \infty$. Then every solution of $\ddot{x} + f(x) = 0$ lies on a closed curve $\frac{1}{2}\dot{x}^2 + F(x) = c$, $F(x) = \int_0^x f(s)\,ds$, and is periodic with period $T(c)$. Prove that $T(c) \to 0$ as $c \to \infty$.

Hint. Break up the integral for $T(c)$ in Section (a) representing the period into

$$\int_0^b = \int_0^{b/2} + \int_{b/2}^b ,$$

similarly for a, and use

$$F(b) - F(x) \geq f(\tfrac{b}{2})(b-x) \quad \text{on} \quad [\tfrac{b}{2}, b]$$

$$F(b) - F(x) \geq \tfrac{1}{2} F(b) \quad \text{on} \quad [0, \tfrac{b}{2}] ,$$

which grows faster than b^2.

Remark. The statement $T(c) \to 0$ holds actually without the monotonicity of f, as a consequence of Lemma 10.3.

Exercise 2

Prove Theorem 10.2.

11. Closed Geodesics on a Riemannian Manifold

In the last sections of this chapter we shall establish the existence of periodic solutions of Hamiltonian systems by "direct methods of calculus of variations", i.e., by minimizing a functional. The best known example is that of closed geodesics on a compact Riemannian manifold without boundary. If this manifold is not simply connected, e.g. a torus, we can consider the family of closed curves in a homotopy class and obtain a closed geodesic by constructing the curve of shortest length. We need not take the length as the functional to be minimized, but any functional whose minimum gives rise to a geodesic. Technically this approach requires that the functional attains its minimum, a problem which was clearly seen and attacked by D. Hilbert. [1] Actually the thrust of Hilbert's contribution was aimed at the Dirichlet problem and partial differential equations but his ideas have important consequences also for the simpler problems of ordinary differential equations.

[1] D. Hilbert, "Uber das Dirichlet'sche Prinzip", Jahresbericht der Deutschen Mathematiker-Vereinigung, VIII, Erstes Heft 1900, p. 184-188.

We give a proof of the above statement, namely that in every nontrivial homotopy class of closed curves (i.e. of noncontractible curves) on a compact Riemannian manifold there exists at least one closed geodesic. Intuitively we may think of shrinking a rubber band which is slung around a torus to its minimum length position — in which case it will represent a closed geodesic. Our argument is based on an idea of G. D. Birkhoff.[1]

It has to be said that the more difficult and more interesting results about closed geodesics refer to simply connected manifolds (on which our rubber band slips off). However, it is still possible to find nontrivial closed geodesics which are not characterized as minima but as saddle points, or extrema of the length functional. By constructing appropriate families of closed curves these closed geodesics are constructed as minimaxima, i.e. the minima of the maxima on certain families. This leads us into topology and Morse theory. We will not follow this line since it requires a whole book in itself. We merely mention that the Morse theory which gives assertions about the number of critical points of a C^1 function on a manifold in terms of its topological properties arose out of this quest for closed geodesics on a sphere. In this case the manifold in question is an infinitely dimensional space of closed curves, the loop space, and the function on it is

[1] G. D. Birkhoff, "Dynamical systems with two degrees of freedom," Trans. Amer. Math. Soc. 1917 Vol. 18 p. 199-300 (in particular pp. 219-220).

the length functional. Roughly simultaneously with Morse another theory was developed by Lyusternik and Schnirelman who were the first to prove that on every smooth convex 2-dimensional manifold there exist at least 3 different closed geodesics without selfintersection. For these developments see Klingenberg.[1]

In more recent times "direct methods of calculus of variations" have been used to establish periodic orbits of Hamiltonian systems which do not arise from geodesic problems. We will prove two such results after our brief discussion of the geodesic problem. In particular we will show that a Hamiltonian system in R^{2n} with a strictly convex, compact energy surface $M \subset R^{2n}$ without boundary has at least one closed orbit on M.

(a) Statement

We first introduce some notation. The metric tensor g on a Riemannian manifold M defines a scalar product in every trangent space, which we denote by

$$\langle v,w\rangle = g_x(v,w) \ , \qquad v,w \in T_xM \ ,$$

similarly we shall write for the length of a vector $v \in T_xM$

$$|v| = \langle v,v\rangle^{1/2}.$$

[1] W. Klingenberg, "Lectures on Closed Geodesics," Springer-Verlag, 1978.

Locally geodesics can be described as curves c on M of minimal arc length $\int |\dot{c}(t)|\, dt$. Alternatively viewing the geodesic mechanically we can describe it as minimizing the energy

$$E(c) = \frac{1}{2} \int |\dot{c}(t)|^2 dt ,$$

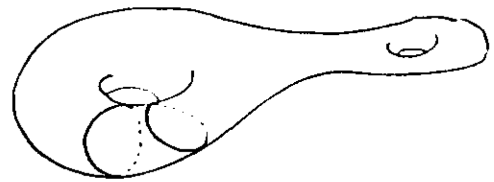
in which case the parameter t plays the role of the time. We prefer the latter approach since it provides a distinguished parameter (proportional to the arc length) on the geodesics, while the length integral is invariant under parameter changes. Our aim is to find a geodesic on M which is a closed curve. We shall prove

<u>Theorem</u> 11.1. On every compact Riemannian manifold M which is not simply connected, i.e. $\pi_1(M) \neq 0$, there exists at least one closed geodesic (in every homotopy class).

Since M is not simply connected there is a closed curve $c: S^1 \to M$, which cannot be deformed into a constant map. We shall denote by F the class of piecewise differentiable closed curves on M which are homotopic to this curve, i.e. which can be deformed into c. The main point is to prove that

$$(11.1) \quad \inf_{c \in F} E(c) = \mu, \quad E(c) = \frac{1}{2} \int_0^1 |\dot{c}(t)|^2 dt$$

is taken on by a curve in F which turns out to be the desired closed geodesic.



(b) Local Geodesics

The proof of the theorem will be based on the following <u>local</u> characterization of geodesics.

<u>Lemma</u> 11.1. (i) Every point on M lies in an open neighborhood U which contains an open $V \subset U$ such that two points $p,q \in V$ can be joined by a unique geodesic $c_{pq}: [0,1] \to U$ with $c_{pq}(0) = p$ and $c_{pq}(1) = q$. Moreover the energy

$$E(p,q) = E(c_{pq}) = \frac{1}{2} \int_0^1 |\dot{c}_{pq}(t)|^2 dt$$

depends differentiably on p and q. (Notice that we parametrize the geodesics proportional to arc length.)

(ii) The geodesic c_{pq} minimizes the energy; in other words, if $c: [0,1] \to M$ is any piecewise differentiable curve with $c(0) = p$ and $c(1) = q$, then

$$(11.2) \qquad E(c_{pq}) \leq E(c) ,$$

and the equality sign holds only for $c = c_{pq}$.

<u>Proof</u>: (i) Working in local coordinates we may assume that $M = R^n$ with the metric being given by

$$\langle v,w\rangle = \sum_{i,k=1}^{n} g_{ik}(x)\, v^i w^k \ , \qquad x \in R^n \ ,$$

where $v,w \in R^n$. We have seen in Section 2 of Chapter I that a geodesic $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ satisfies the Euler equations

$$\ddot{x}_j = - \sum_{s,k} \Gamma^j_{sk}(x(t))\, \dot{x}_s \dot{x}_k \ , \tag{11.3}$$

with Γ^j_{sk} the Christoffel symbols of g. We also recall that for a geodesic x(t), the function

$$\langle \dot{x}(t), \dot{x}(t)\rangle = |\dot{x}(t)|^2$$

is independent of t. This can also be seen directly; namely

$$\frac{d}{dt}\, |\dot{x}(t)|^2 = 2 \sum_{i,k} g_{ik} \ddot{x}_k \dot{x}_i + \sum_{i,j,k} g_{ikx_j} \dot{x}_j \dot{x}_i \dot{x}_k \ .$$

Using the formulae from Exercise 3, Chapter 1, Section 2, we can write the second term in the form

$$\sum_{i,j,k} (g_{ikx_j} + g_{ijx_k} - g_{kjx_i}) \dot{x}_j \dot{x}_i \dot{x}_k = 2 \sum g_{i\ell}\, \Gamma^{\ell}_{jk}\, \dot{x}_i \dot{x}_j \dot{x}_k$$

which is indeed zero due to (11.3). Writing the Euler equations as a system of first order equations we find that a geodesic $\{x(t), \dot{x}(t) = v(t)\}$ is an orbit of the following vector field on TM

$$\begin{aligned} \dot{x}_k &= v_k \\ \dot{v}_k &= - \sum_{r,s=1}^{n} \Gamma^k_{rs}(x)\, v^r v^s \ , \qquad k = 1,2,\ldots,n. \end{aligned} \tag{11.4}$$

We shall denote by ϕ^t the flow of this vector field:

$$\phi^t : (x,v) \mapsto (\psi_t(x,v),\ \chi_t(x,t)).$$

We try to find a solution connecting two given points x,y. We claim that the equation $y = \psi_1(x,v)$ has a unique solution v for y near x. It then follows that the curve

$$c_{xy}(t) = \psi_t(x,v) \ , \qquad 0 \le t \le 1$$

is the desired geodesic joining the point $x = c_{xy}(0)$ with the point $y = c_{xy}(1)$.

To prove the claim we first observe that the system (11.4) has the special property of being invariant under $t^* = \lambda t$, $v^* = \lambda^{-1} v$. To put it differently, the flow satisfies

$$\mu_\lambda \circ \phi^t = \phi^{\lambda t} \circ \mu_\lambda \ , \qquad \mu_\lambda(x,v) = (x, \lambda^{-1} v).$$

For the first component of the flow we conclude in particular

$$\psi_t(x,v) = \psi_1(x,tv) \ ,$$

and differentiating in t we have

$$\frac{d}{dt}\, \psi_1(x,tv)\Big|_{t=0} = \frac{d}{dt}\, \psi_t(x,\ v)\Big|_{t=0} = v \ ,$$

and therefore

$$\frac{\partial}{\partial v}\,\psi_1(x,0) = \mathrm{id}\ .$$

Consequently by the implicit function theorem there is a unique differentiable function $v = v(x,y)$ such that

$$y = \psi_1(x,v) \qquad \text{and} \qquad v(x,0) = 0$$

for y near x, proving our claim.

Moreover, since $\langle \dot{x},\dot{x}\rangle$ is an integral of the geodesic flow ϕ^t, we see that the function

$$E(x,y) = \frac{1}{2}\int_0^1 |\dot{c}_{xy}(t)|^2\,dt = \frac{1}{2}\,|v|^2 = \frac{1}{2}\,|v(x,y)|^2$$

is differentiable in x and y and we have proved the first part of the lemma.

For x fixed, the map

$$v \to \psi_1(x,v) = \exp_x(v)$$

is a diffeomorphism of a neighborhood of 0 onto an open neighborhood U of x. This map is called the exponential map, it satisfies $\exp_x(0) = x$. The curves $t \mapsto \exp_x(tv)$ are the local geodesics issuing from the point x. We point out that this exponential map should not be confused with our notation $\exp tX = \phi^t$ for the flow of a vector field X introduced in Section 1 of Chapter I.

(ii) Although the second statement of the lemma is also well known in calculus of variations we outline a proof and introduce first the so called "extremal integral". We have seen in the first part of the lemma that given two distinct points ξ and x sufficiently close and given $t > t_0$, then there is a unique local geodesic $x(s)$ issuing from ξ at $s = t_0$ and arriving at the point x at $s = t$. It is given by

$$x(s) = \exp_\xi\left(\frac{s-t_0}{t-t_0}\,v\right), \quad \text{if} \quad x = \exp_\xi(v)\ .$$

If we write $x(s) = x(s;t,x,t_0,\xi_0)$ then we have:

$$\text{(11.5)} \qquad \begin{aligned} x(s;t,x,t_0,\xi) &= x \quad \text{if} \quad s = t \\ x(s;t,x,t_0,\xi) &= \xi \quad \text{if} \quad s = t_0\ . \end{aligned}$$

By integrating

$$\text{(11.6)} \qquad F(x,\dot{x}) = \frac{1}{2}\langle g(x)\dot{x},\dot{x}\rangle$$

along the above extremals we define the function S,

$$S(t,x,t_0,\xi) = \int_{t_0}^{t} F(x(s),\ \dot{x}(s))\,ds\ .$$

This is the so called extremal integral. The derivatives of this function can easily be computed as (we suppress the variable t_0 in the notation)

$$\text{(11.7)} \qquad \begin{aligned} S_x(t,x,\xi) &= F_{\dot{x}}\left(x(s;t,x,\xi),\ \frac{\partial x}{\partial s}(s;t,x,\xi)\right)\Big|_{s=t} \\ S_\xi(t,x,\xi) &= -F_{\dot{x}}\left(x(s;t,x,\xi),\ \frac{\partial x}{\partial s}(s;t,x,\xi)\right)\Big|_{s=t_0} \end{aligned}$$

(11.8) $$S_t = -H(x,S_x) ,$$

where H is the corresponding Hamilton-function given by

$$H(x,y) = \frac{1}{2} \langle g^{-1}(x)y,y\rangle .$$

Here y is related to x, $\dot{x}$ by the Legendre transformation:

(11.9)
$$y = F_{\dot{x}}(x,\dot{x}) = g(x)\dot{x} , \quad \dot{x} = H_y(x,y) = g^{-1}(x)y$$
$$F(x,\dot{x}) + H(x,y) = \langle y,\dot{x}\rangle .$$

To prove the above identities (11.7) and (11.8) we denote by a any of the components of x or ξ and find by differentiation of the extremal integral

$$\frac{\partial S}{\partial a} = \int_{t_0}^{t} \frac{\partial}{\partial a} F(x,\dot{x})\, ds$$

and

$$\frac{\partial}{\partial a} F = \langle F_x, x_a\rangle + \langle F_{\dot{x}}, \frac{d}{ds} x_a\rangle$$
$$= \langle F_x - \frac{d}{ds}(F_{\dot{x}}), x_a\rangle + \frac{d}{ds}\langle F_{\dot{x}}, x_a\rangle .$$

Since the extremals satisfy the Euler equations, the first bracket vanishes, and therefore the integration gives

$$\frac{\partial S}{\partial a} = \langle F_{\dot{x}}, \frac{\partial x}{\partial a}\rangle \Big|_{s=t_0}^{s=t} ,$$

from which in view of (11.5) the relations (11.7) follow.

To prove (11.8) we proceed similarly and find

$$S_t = F(x,\dot{x})\Big|_{s=t} + \langle F_{\dot{x}}, \frac{\partial x}{\partial t}\rangle \Big|_{s=t_0}^{s=t} .$$

Using (11.7) and (11.5) and the definition of the Hamilton-function (11.9) we obtain (11.8).

If we fix the initial point ξ and the initial parameter-value t_0 , then S becomes a function of t,x alone, and the extremals issuing from the point ξ form a family of curves satisfying a first order differential equation. Indeed if x(s), $\dot{x}(s)$ is a solution of the Euler equations, then x(s), y(s) defined by (11.9) is a solution of Hamilton's equations, and for s = t, the first equation of (11.7) amounts to

$$y = S_x(t,x) ,$$

and therefore we find by (11.9)

$$\dot{x} = H_y(x, S_x(t,x)) .$$

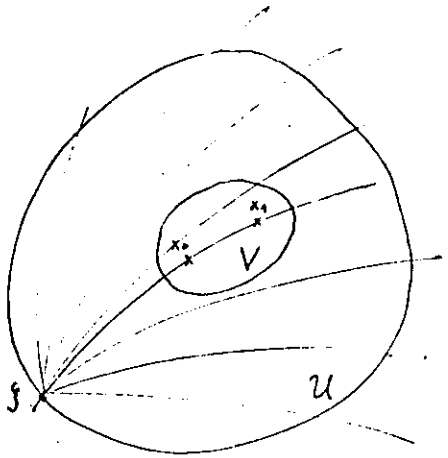
Abbreviating the function

(11.10) $$m(t,x) = H_y(x, S_x(t,x)) ,$$

the extremals issuing from ξ satisfy the differential equation

(11.11) $$\dot{x} = m(t,x) .$$

In order to show that the extremal $c_{x_0x_1}(t) = c_0(t)$, $0 \leq t \leq 1$, connecting two points $x_0 = c_0(0)$ and $x_1 = c_0(1)$ in a sufficiently small neighborhood V minimizes the energy, we choose a larger neighborhood U containing V so that any curve connecting x_0, x_1 but also containing points outside of U has a larger energy than c_0, so that we have to consider curves in U only. On the other hand we choose V and U so small that for a point ξ on ∂U, lying on the extension of the extremal c_0 through x_0, x_1 to the boundary ∂U, the extremals issuing from ξ cover U simply i.e. there is a unique extremal in U connecting ξ and any point in U.



Hence $S = S(t,x)$ is a function on $R \times U$. In particular

$$E(c_0) = S(1,x_1) - S(0,x_0) = \int_0^1 (S_t + \langle S_x, \dot{x}\rangle)\, dt .$$

If $c(t) = z(t)$, $0 \leq t \leq 1$, is any other curve in U connecting x_0 and x_1, then its energy is given by

$$E(c) = \int_0^1 F(z,\dot{z})\, dt ,$$

where $z(0) = x_0$ and $z(1) = x_1$. To compare $E(c)$ with $E(c_0)$ we rewrite $E(c_0)$ as an integral over the curve c, which allows us to compare the integrands. For this purpose we note that the line-integral

$$dS = \int (S_t + \langle S_x, \dot{x}\rangle)\, dt$$

clearly is independent of the path in U and depends on the end points only. Therefore $E(c_0)$ can be written as

$$E(c_0) = \int_0^1 (S_t + \langle S_x, \dot{x}\rangle)\, dt = \int_{c_0} dS = \int_c dS ,$$

i.e. as an integral over the curve c. Using (11.8) and (11.8) we have with (11.10) the identity

$$S_t + \langle S_x, \dot{x}\rangle = - H(x,S_x) + \langle S_x, \dot{x}\rangle = F(x,m) + \langle \dot{x}-m, F_{\dot{x}}(x,m)\rangle ,$$

where $m = m(t,x)$. We therefore find:

$$E(c) - E(c_0) = \int_0^1 \left(F(z,\dot{z}) - F(z,m) - \langle \dot{x}-m, F_{\dot{x}}(x,m)\rangle\right) dt ,$$

where $m = m(t, z(t))$. Since $F(x,\dot{x})$ is quadratic in $\dot{x}$, (see (11.6)) this can be rewritten as

$$E(c) - E(c_0) = \int_0^1 F(z, \dot{z}-m)\, dt = \frac{1}{2}\int_0^1 \langle g(z)(\dot{z}-m), (\dot{z}-m)\rangle\, dt$$

which is ≥ 0 and equal to zero only if $\dot{z} = m$. In the latter case we must have $c = c_0$, since the only solution of this differential equation with $z(0) = x_0$ is the extremal c_0 in view of (11.11). This proves the second part of Lemma 11.1.

Remarks. (1) We mention that the differential equation (11.8) is called the time dependent Hamilton-Jacobi equation. Since in our case the integrand $F = F(x,\dot{x})$ is independent of t it is easy to make the t-dependence of $S(t,x)$ explicit, we find (with $t_0 = 0$)

$$S(t,x) = t^{-1} S(1,x) .$$

Therefore the function $S^*(x) = S(1,x)$ satisfies the partial differential equation

$$S^*(x) = H(x, S^*_x) .$$

(2) We also notice that the relations (11.7), rewritten as $S_x(t,x,\xi) = y$ and $S_\xi(t,x,\xi) = -\eta$ express the fact that the function $S(t,x,\xi)$ is a generating function of the flow $(\xi,\eta) \to \phi^t(\xi,\eta) = (x,y)$ belonging to the Hamiltonian vector field H, with initial condition $\phi^{t_0} = id$.

(c) Proof of the theorem.

Going back to the proof of the theorem we pick a minimizing sequence $c^n \in F$, $n = 1,2,\ldots,$ of our functional (11.1) such that

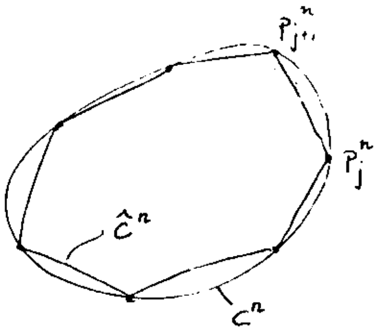
$$\lim_{n\to\infty} E(c^n) = \inf_{c\in F} E(c) = \mu .$$

If we define an equidistant partition of S^1: $0 = t_0 < t_1 < \ldots < t_N = 1$ by setting $t_j = j/N$, then there is an N which is independent of n such that two consecutive points $p_j^n = c^n(t_j)$ and $p_{j+1}^n = c^n(t_{j+1})$ on the curve c^n belong to an open set V on which the above lemma applies. In fact this follows from the compactness of M and from the following estimate of the arc length of c^n between p_j^n and p_{j+1}^n:

$$\int_{t_j}^{t_{j+1}} |\dot{c}^n|\, dt \leq |t_{j+1} - t_j|^{1/2} \sqrt{2E(c^n)}$$

$$\leq \frac{1}{\sqrt{N}} K ,$$

for some constant K which is independent of n. We can use Lemma 11.1 to define another minimizing sequence which we denote by $\hat{c}^n$ and which we define for $t_j \leq t \leq t_{j+1}$ as the unique geodesic joining the two points p_j^n and p_{j+1}^n, the parameters being proportional to arc length.



It then follows that $\hat{c}^n \in F$, $E(\hat{c}^n) \le E(c^n)$ and $\lim_{n\to\infty} E(\hat{c}^n) = \mu$. We have used here the second part of the lemma. Since M is compact, there is by Weierstrass' theorem a subsequence of (N-tuples of) points p^n_j which does converge:

$$p^n_j \to p^*_j \in M\,, \qquad j = 1,2,\ldots,N.$$

By the first part of the lemma $E(p,q)$ is a continuous function of its endpoints, hence

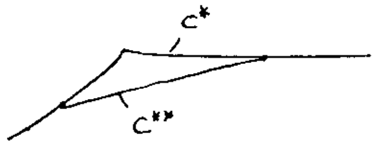
$$E(p^n_j, p^n_{j+1}) \to E(p^*_j, p^*_{j+1})$$

and therefore

$$(11.12) \qquad \mu = E(c^*)\,,$$

where $c^* \in F$ is the closed curve of broken geodesics joining p^*_j with p^*_{j+1}, and we have shown that the minimum is taken on. This curve c^* is our sought closed geodesic. In fact if $p = c^*(t_1)$ and $q = c^*(t_2)$ are any two points on c^*

which are sufficiently close they can be joined by a unique geodesic $\gamma(t)$, $t_1 \le t \le t_2$ which agrees with $c^*(t)$ on this interval. Indeed otherwise we replace c^* by the different curve $c^{**} \in F$ which coincides with c^* outside the interval $[t_1, t_2]$ and with γ on this interval and by (11.2) $E(c^{**}) < E(c^*)$ which contradicts (11.12).



The closed curve c^* is, of course, not a constant since it belongs to F.

It is an interesting fact that, at least if M is a two dimensional surface, the minimal closed geodesics constructed in the theorem have real Floquet multipliers if considered as periodic orbits of the geodesic vector-field (11.4) on TM. If one ignores the possibilities ± 1 this means that they are unstable periodic orbits (Exercise 1).

Exercises

1. Prove that if on a Riemann manifold M with dim(M)=2 one has a closed geodesic which minimizes the energy integral then its Floquet multipliers are real.

If we ignore the degenerate case where all eigenvalues are equal to one this implies that minimizing closed geodesics are unstable (see H. Poincaré, Les Méthodes Nouvelles) de la mécanique célèste, Tome III, Chap. 29, No. 355).

Hint: (1) Near the closed geodesic one can introduce local coordinates u,v so that

$$ds^2 = E(u,v)\, du^2 + dv^2 \; ; \quad E(u,0) = 1, \quad E_v(u,0) = 0$$

and the closed geodesic is given by v = 0.

Normalizing the length of this geodesic to be 1 we have $E(u+1,v) = E(u,v)$ and the energy integral of a curve $c(t) = \big(u(t),v(t)\big)$, is given by

$$I[u,v] = \frac{1}{2}\int_0^1 \big(E(u,v)\dot{u}^2 + \dot{v}^2\big)\, dt \; .$$

(2) For the reference solution u = t, v = 0 one has $I = \frac{1}{2}$, and for any nearby closed curve $c(t) = (u(t),v(t))$ with $u(t+1) = u(t)+1$, $v(t+1) = v(t)$ one has $I \geq \frac{1}{2}$ since the reference solution is minimal. In particular, the second variation at the reference orbit u = t and v = 0:

$$J_2(\phi,\psi) = \left(\frac{d}{d\varepsilon}\right)^2 I[u+\varepsilon\phi, v+\varepsilon\psi]\Big|_{\varepsilon=0} = \int_0^1 (\dot{\phi}^2+\dot{\psi}^2-K(t)\psi^2)\, dt \; ,$$

with $K(u) = -\frac{1}{2}E_{uv}(u,0)$, must be ≥ 0 for all periodic functions ϕ,ψ in $C^1(R)$.

(3) Show that the linearized differential equation for the geodesic flow along our periodic reference orbit agrees with the Euler equation of J_2, i.e. with

$$\ddot{\phi} = 0 \; , \quad \ddot{\psi} + K(t)\psi = 0 \; .$$

Hence the Floquet multipliers are $(1,1,\lambda,\lambda^{-1})$ where λ,λ^{-1} are the Floquet multipliers of the so-called Jacobi equation

(*) $$\ddot{\psi} + K(t)\psi = 0 \; .$$

(4) Set $\phi \equiv 0$ and show: If the Floquet multiplier λ is not real, then there exists a periodic function ψ in C^1 with $J_2(0,\psi) < 0$. For this purpose, show

(i) If λ is not real, then $|\lambda| = 1$, i.e. $\lambda = e^{i\alpha}$, α real, $\pi\alpha \neq$ integer (see Chapter 4).

(ii) There exists a nontrivial solution

$$\chi(t) = \mathrm{Re}\,\big(e^{i\alpha t}\, p(t)\big)$$

where $p(t)$ is complex valued, $\not\equiv 0$ and of period 1.

(iii) χ has infinitely many zeros. Let t_1,t_2 be two consecutive zeros with $\chi(t) > 0$ in $t_1 < t < t_2$ and set

$$\chi_0(t) = \begin{cases} \chi(t) & \text{in } t_1 < t < t_2 \\ 0 & \text{otherwise} \end{cases} \; .$$

Show that the function

$$\psi_0(t) = \sup_{n\in Z} \chi_0(t+n)$$

has period 1 and satisfies

$$J_2(0,\psi_0) \leq 0 ,$$

by using the formula

$$\int_0^1 \left(\dot{\psi}^2 - K(t)\psi^2\right) dt = -\int_0^1 (\ddot{\psi} + K\psi)\psi \, dt + \sum_j \left(\dot{\psi}(t_j+0) - \dot{\psi}(t_j-0)\right)\psi(t_j)$$

where ψ is piecewise differentiable and where t_j are the discontinuities of $\dot{\psi}(t)$.

(Show that $\dot{\psi}_0(t_j+0) - \dot{\psi}_0(t_j-0) < 0$.)

(iv) By rounding off the corners of ψ_0 construct a C^1-function $\psi(t)$ of period 1 with $J_2(0,\psi) < 0$.

2. Give a direct proof of the minimizing property of extremals (see Lemma 11.1) avoiding the extremal function S and using geodesic polar coordinates.

Hint: Denote by $c_{x_0x_1}(t) = \exp_{x_0}(tv)$, $0 \leq t \leq 1$ the unique geodesic joining x_0 with $x_1 = \exp_{x_0}(v)$ so that $E(c_{x_0x_1}) = \frac{1}{2}|v|^2$. If $c(t)$, $0 \leq t \leq 1$, is any piecewise differentiable curve in the domain of the exponential map joining x_0 with x_1, define $w(t)$ by

$$c(t) = \exp_{x_0}(w(t)),$$

with $w(0) = 0$ and $w(1) = v$. Write $w(t)$ in the form $w(t) = r(t)v(t)$, with the scalar $r(t) = |w(t)||v|^{-1}$ and $v(t) = w(t)|v||w(t)|^{-1}$ as long as $w(t) \neq 0$.

If t_0 is the smallest number such that $w(t) \neq 0$ in $t_0 < t \leq 1$ then:

$$E(c) \geq \frac{1}{2} \lim_{\varepsilon \to 0} \int_{t_0+\varepsilon}^1 |\dot{c}(t)|^2 \, dt \geq \frac{1}{1-t_0} \frac{1}{2}|v|^2 .$$

To prove this introduce $f(r,t) = \exp_{x_0}(rv(t))$ such that $c(t) = f(r(t),t)$, hence

$$\dot{c} = f_r\dot{r} + f_t ,$$

and prove that

(1) $|f_r|^2 = |v|^2$

(2) $\langle f_r, f_t\rangle = 0$

(3) $f_t = 0$ if and only if $\dot{v}(t) = 0$,

where $\langle \, , \, \rangle$ is the scalar product given by the Riemannian metric.

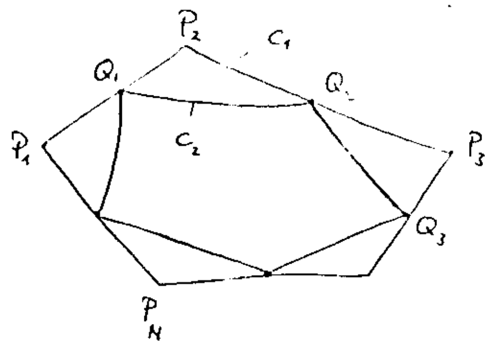
Clearly $E(c) \geq \frac{1}{2}|v|^2$. If $E(c) = \frac{1}{2}|v|^2$, show that $t_0 = 0$ and $v(t) = v(1) = v$ is independent of t, hence $w(t) = r(t)v$ with a piecewise differentiable function $r(t)$ and

$$E(c) = \frac{1}{2}\int_0^1 |\dot{c}(t)|^2 = \frac{1}{2}|v|^2 \int_0^1 \dot{r}^2 \, dt .$$

Show that $r(t) = t$.

There is a direct construction of a minimizing sequence in a given homotopy class which we describe now. We begin with an arbitrary closed curve $c(t)$ $(0 \leq t \leq 1)$ in a nontrivial homotopy class of a compact manifold M. Choose a large integer N and construct a closed curve $c(t)$ $(0 \leq t \leq 1)$ in a nontrivial homotopy class of a compact manifold M. Choose a large integer N and construct a closed curve by joining the points $P_j = c(jN^{-1})$, $P_{j+1} = c((j+1)N^{-1})$ for $j = 1,2,\ldots, N-1$ and P_N, P_1 by minimizing geodesics. By introducing a parameter proportional to arc lengths and so that the period is 1 we obtain the closed curve $c_1(t)$ with $c_1(t_j) = P_j$ $j = 1,2,\ldots, N$, $0 = t_1 \leq t_2 \ldots < t_N < 1$.

Now we construct a new closed curve $c_2(t) = D(c_1(t))$ by connecting the midpoints $Q_j = c_1\left(\frac{t_j + t_{j+1}}{2}\right)$ $j = 1,2,\ldots,N-1$ and $Q_N = c_1\left(\frac{t_N + t_1 - 1}{2}\right)$ by geodesics and paramterizing this curve again proportional to arc lengths so that its period is 1 (see figure)



Then one shows readily that

$$E(c_2) \leq E(c_1) \tag{11.13}$$

Repeating the construction and defining a sequence c_n by $c_{n+1} = D(c_n)$ one gets a sequence of closed curves, which are broken geodesics.

EXERCISE 3. Prove the inequality (11.13) and show that the above sequence c_n has a subsequence c_{n_j} converging to a closed external $c^*(t)$.

Hint: Use that $E(c_1)$ depends continuously on the points P_j and that c^* does not have any "corners".

12. Periodic Orbits on a Convex Energy-Surface

(a) Flow on an Energy Surface.

We consider a Hamiltonian vector field X_H, $H \in C^2(R^{2n})$, on the symplectic manifold R^{2n} with the symplectic structure $\omega = \sum dq_j \wedge dp_j$. The function H is an integral and the vector field X_H is tangential to any level surface (energy surface)

$$M = \{x \mid H(x) = c\}, \quad c \in R,$$

and hence defines a vector field on M. We shall assume that c is not a critical value, i.e. $dH(x) \neq 0$ for every point $x \in M$. Then M is a C^2-submanifold and the vector field X_H does not vanish on M. It is crucial for the following to observe that the vector field X_H on M is up to multiplication with a scalar function $\lambda \neq 0$ on M determined by M and the symplectic form ω, i.e. we can approach a direction as a line to each point on M. Indeed, if F is another defining function for M, i.e.:

$$M = \{x \mid H(x) = c\} = \{x \mid F(x) = c'\}$$

with dH, $dF \neq 0$ on M, then $dF(x) = \lambda(x)\, dH(x)$ at every point $x \in M$, with $\lambda(x) \neq 0$, and therefore

$$X_F = \lambda X_H, \quad \lambda \neq 0 \quad \text{on} \quad M.$$

It then follows that X_H and X_F have the same orbits on M although their parametrization will be different in general.

In fact if ϕ^t is the flow of X_H on M, then the flow ψ^s of X_F on M is given by the formula

$$\psi^s(x) = \phi^t(x), \quad t = t(s,x),$$

where the function t is defined by

$$\frac{ds}{dt} = \lambda(\phi^t(x)), \qquad t(0,x) = 0,$$

for $x \in M$. In particular, X_H has a periodic solution on M if and only if X_F has one. For example, if X_H has the energy surface $M = \{x \in R^{2n} \mid \frac{1}{2}|x|^2 = R > 0\}$, then all the solutions of X_H on M are periodic since the flow of X_F, with the Hamiltonian given by $F(x) = \frac{1}{2}|x|^2$, is $\phi^t(x) = e^{tJ}x$ and hence periodic.

This remark that a direction field on the manifold M is determined by the symplectic structure can also be seen this way: Let $j: M \to R^{2n}$ define the inclusion map, and $j^*\omega$ the restriction of ω onto M. Then this form $j^*\omega$ is skew symmetric and bilinear on the (2n-1)-dimensional tangent spaces TM and is of rank (2n-2). Therefore there exists an $X \neq 0 \in TM$ such that

$$j^*\omega(X,Y) = 0 \quad \text{for all} \quad Y \in TM$$

and this vector is unique up to scalar multiplication, i.e. X spans the null space of $j^*\omega$ and defines a direction or a line $\lambda X \in TM$ at every point of M. This line agrees with the direction field defined above. Indeed assume X_H has M as energy surface, then every tangent vector $Y \in TM$ satisfies

$dH(Y) = \omega(X_H, Y) = 0$ and therefore $j^*\omega(X_H, Y) = 0$.

We have seen that the vector field X_H on M is defined up to a factor by M (and $\omega = dp \wedge dq$). The purpose of this subsection b is to prove the following surprising result with the help of direct methods of calculus of variations.

Theorem 12.1. If M is a compact strictly convex C^2-surface in (R^{2n}, ω) then any vector field X_H with $H \in C^2(R^{2n})$ and having M as energy surface, i.e.

$$M = \{x \in R^{2n} \mid H(x) = c\}, \qquad dH \neq 0 \text{ on } M,$$

has at least one periodic orbit on M.

This result is due to P. Rabinowitz [1] and A. Weinstein.[2] The proof we shall give below is based on an idea due to F. Clark.[3]

Postponing the proof we first mention an application. In the case of Liapunov's theorem we needed a nonresonance condition in order to guarantee a periodic solution on the energy surface $H(x) = \varepsilon$, for ε small. However if the largest frequency is multiple, e.g., if

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1 P. Rabinowitz, "Periodic solutions of Hamiltonian systems" Comm. Pure Appl. Math. 31 (1978) pp. 157-184.

2 A. Weinstein, "Periodic orbits for convex Hamiltonian systems" Ann. of Math. 108 (1978) pp. 507-518.

3 F. H. Clarke, "A classical variational principle for periodic Hamiltonian trajectories"

$$(12.1) \qquad \langle H_{xx}(0)\xi, \xi\rangle = \alpha \sum_{j=1}^{n} \rho_j(\xi_j^2 + \xi_{j+n}^2), \qquad \alpha \neq 0$$

with $\rho_j = \pm 1$, then the result is not applicable. But in case $H_{xx}(0)$ is a definite matrix, which in the above example means that all ρ_j are of the same sign, then we conclude from Theorem 12.1 that every energy surface $H(x) = \varepsilon$ carries a periodic solution provided ε is sufficiently small. If the signs of ρ_j in (12.1) are different this statement is not true as we have seen at the end of Section 2 in Chapter II.

(b) Convex Energy Surfaces.

In order to prove Theorem 10.2 we shall introduce a particular Hamiltonian function H which is homogeneous of degree 2 and which describes M by $M = \{x \in R^{2n} | H(x) = 1\}$. Here M is a compact strictly convex C^2-surface in R^{2n}. In other words if M is represented locally in U by a function f with $df \neq 0$ as

$$M \cap U = \{x \in U \subset R^{2n} \mid f(x) = 0\},$$

then the Hessian f_{xx} on M restricted to the tangent space of M is positive definite, i.e.

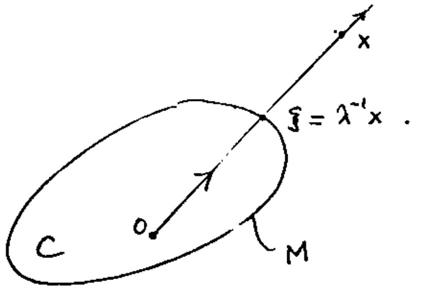
$$(12.2) \qquad \langle f_{xx}(x)\xi, \xi\rangle > 0, \qquad x \in M \cap U$$

for every $\xi \neq 0 \in R^{2n}$ satisfying

$$\langle f_x(x), \xi\rangle = 0 .$$

This is the case if the sectional curvatures of M are positive.

M is the boundary of a compact strictly convex region in R^{2n} which we denote by C. We may assume that C contains the origin in its interior. Then each ray issuing from the origin meets M in exactly one point nontangentially. Thus if $x \neq 0$ is given, then there is a unique point $\xi = \lambda^{-1}x$, $\lambda \neq 0$ on M.



If we define the function F on R^{2n} by

(12.3) $$F(x) = \lambda \quad \text{if} \quad \xi = \lambda^{-1}x \in M$$

for $x \neq 0$ and $F(0) = 0$, then the manifold M is represented by

$$M = \{x \mid F(x) = 1\} .$$

Moreover, $F \in C^2(R^{2n} \setminus \{0\})$ and $dF \neq 0$ on M, since the rays meet M nontangentially. It also is clear that

$$F(\rho x) = \rho F(x) \quad \text{for} \quad \rho \geq 0$$

i.e. F is homogeneous of degree 1. Differentiation with respect to ρ at $\rho = 1$ gives the Euler relation

$$\langle F_x(x), x\rangle = F(x) .$$

Since M is assumed to be strictly convex, we conclude by the homogeneity of F, that the Hessian $F_{xx}(x) \geq 0$ is positive for every $x \neq 0$. We would like to describe M in terms of a strictly convex funtion $H = H(x)$ as $\{x | H(x) = 1\}$, i.e. a function whose Hessian $H_{xx}(x) > 0$ is positive definite at every point $x \neq 0$. Since F is homogeneous of degree 1 it is obviously never strictly convex. Indeed differentiating $\langle F_x, x\rangle = F$ gives $F_{xx}x = 0$. We therefore define the function

$$H(x) = (F(x))^2$$

which is strictly convex precisely if M is strictly convex as one proves without difficulty.

Summarizing we have shown that every strictly convex C^2-submanifold M containing the origin in the interior can be represented as

(12.4) $$M = \{x \in R^{2n} \mid H(x) = 1\}$$

with $H \in C^2(R^{2n} \setminus \{0\})$, $H(0) = 0$, where H has positive definite Hessian at every point $x \neq 0$ and satisfies

(12.5) $$H(\rho x) = \rho^2 H(x) \quad \text{for} \quad \rho \geq 0 ,$$

i.e. is homogeneous of degree 2.

Although we do not need it we briefly discuss the situation for a C^1-manifold M, which is called strictly convex if the region C it bounds is strictly convex, i.e. C contains two points $x, y \in C$ and $\alpha x + (1-\alpha)y$, $0 \leq \alpha \leq 1$ and this point is an interior point of $x \neq y$ and $0 < \alpha < 1$. We shall

(*)

It follows from the definition that F is homogeneous of degree 1, i.e

$$F(\rho x) = \rho F(x) \quad \text{for } \rho \geq 0 .$$

Moreover, $F \in C^2(\mathbb{R}^{2n} \setminus \{0\})$. In fact the function $F(x) = \lambda$ satisfies the equation $f(\frac{x}{\lambda}) = 0$, if M is ~~[illegible]~~ represented by ~~[illegible]~~ $f = 0$, and $\langle f_x, x \rangle \neq 0$ for $x \in M$ since the rays meet M non-tangentially. Therefore the claim follows ~~with the help of the~~ by the implicit function theorem and the homogeneity of F. Differentiation of the above homogeneity relation with respect to ρ at $\rho = 1$ gives the Euler relation

$$\langle F_x, x \rangle = F(x) ,$$

from which we conclude that $dF \neq 0$ on M. Since M is assumed to be strictly convex it follows from $F_x = \alpha(x) f_x$, $x \in M$, that $F_{xx} / TM = \alpha(x) f_{xx} / TM$ hence

(*)

$F_{xx} / TM > 0$, and we conclude by the homogeneity of F, that $F_{xx} \geq 0$ is positive ~~for~~ at every $x \neq 0$. We would like to ....

show that the function F defined by (12.3) is a convex function in the sense that

(12.6) $$F(\alpha x + \beta y) \le \alpha F(x) + \beta F(y)$$

if $\alpha,\beta \ge 0$ and $\alpha+\beta = 1$ for any two points $x,y \in R^{2n}$. We will have equality only if x,y lie on the same ray through 0. Since C is convex, we have for any two points $\xi,\eta \in M$ that the points $\alpha\xi+\beta\eta$ also lie in C if $\alpha,\beta \ge 0$ and $\alpha+\beta = 1$, i.e.

$$F(\alpha\xi + \beta\eta) \le 1 .$$

If $x,y \neq 0$ are any two points we can represent them as $x = \lambda\xi$ and $y = \mu\eta$, $\lambda,\mu > 0$ with $\xi,\eta \in M$. Therefore

$$F(\alpha x + \beta y) = F(\alpha\lambda\xi + \beta\mu\eta) ,$$

and setting

$$a = \frac{\alpha\lambda}{\alpha\lambda+\beta\mu} , \qquad b = \frac{\beta\mu}{\alpha\lambda+\beta\mu}$$

we find by the homogeneity of F

$$F(\alpha x + \beta y) = (\alpha\lambda + \beta\mu)\, F(a\xi + b\eta)$$

$$\le \alpha\lambda + \beta\mu = \alpha\, F(x) + \beta\, F(y)$$

proving (12.6) for $x,y \neq 0$. If $x = 0$ or $y = 0$ it is a consequence of the homogeneity of F. The function $H(x) = (F(x))^2$ is then strictly convex in the sense that in (12.6) the inequality holds if $\alpha > 0$ and $\beta > 0$ and $x \neq y$.

(c) Proof of Theorem 12.1.

In view of our discussion in Sections (a) and (b) it suffices to prove the theorem for a Hamiltonian vector field X_H with a homogeneous strictly convex function H satisfying (12.4) and (12.5). We have to show that there is a periodic solution of

$$\dot{x} = JH_x \quad \text{on} \quad M .$$

Normalizing the period we can ask for a periodic solution of period 2π on M for the equation

(12.7) $$\dot{x} = \lambda\, JH_x \quad \text{for some} \quad \lambda \neq 0 .$$

For instance the variational principle

$$\min \int_0^{2\pi} H(x(t))\, dt \quad \text{under} \quad \frac{1}{2}\int_0^{2\pi} \langle Jx,\dot{x}\rangle\, dt = 1 ,$$

where the functions x(t) are assumed to be 2π periodic, has the above equations as Euler equations. But one shows easily (Exercise 1) that neither the infimum nor the supremum is taken on, even if $H(x) = \frac{1}{2}|x|^2$. The trick now is to use an alternate variational principle having the same differential equations for which however the infimum is taken on. We shall form the Legendre transformation of H but this time with respect to all variables in contrast to the definition in Section 2.6 of Chapter I.

The function G(y) related to H(x) by a Legendre transformation can be defined by

(12.8) $G(y) = \max_{\xi \subset R^{2n}} (\langle\xi,y\rangle - H(\xi)) = \langle x,y\rangle - H(x)$.

There is indeed a unique maximum given by $y = H_x(x)$, since H is strictly convex (the Hessian $H_{xx}(x)$ is positive definite for $x \neq 0$) and satisfies

$$\frac{1}{C}|x|^2 \leq H(x) \leq C|x|^2 , \quad x \in R^{2n} ,$$

for some constant $C > 1$. This estimate is an immediate consequence of the homogeneity of H. Summarizing we have

(12.9) $G(y) \; H(x) = \langle x,y\rangle$,

where

(12.10) $y = H_x(x)$ and $x = G_y(y)$.

Clearly $G(0) = 0$, $G \in C^2(R^{2n} \setminus \{0\})$ and G is homogeneous of degree 2. Since

$$H_{xx}(x) \cdot G_{yy}(y) = I , \quad x \neq 0 ,$$

G is also strictly convex if $y \neq 0$. (For an alternate proof of this fact for C^1-functions we refer to Exercise 4.)

We now consider the following alternate variational principle for 2π-periodic functions z:

(12.11) $\min \int_0^{2\pi} G(\dot{z})\, dz$ under $\frac{1}{2}\int_0^{2\pi} \langle Jz,\dot{z}\rangle = 1$,

which we shall solve by the standard direct variational methods.

To be more precise, we define the following function space F of periodic functions $z(t) = z(t + 2\pi)$ having mean value 0 in order to fix the arbitrary constant in (12.11):

$$F = \left\{ z \in H_1(S^1) \mid \frac{1}{2\pi}\int_0^{2\pi} z(t)\, dt = 0 \right\}$$

where $H_1(S^1)$ is the Hilbert space of absolutely continuous 2π-periodic functions whose derivatives are square integrable, i.e. belong to $L_2(S^1)$. By $A \subset F$ we shall denote the subset

(12.12) $$A = \left\{ z \in F \mid \frac{1}{2}\int_0^{2\pi} \langle Jz,\dot{z}\rangle = 1 \right\} .$$

Following the standard procedure we will show that the functional

$$I(z) = \int_0^{2\pi} G(\dot{z}(t))\, dt$$

(i) bounded from below on A

(ii) taken on its minimum on A, i.e. there exists $z_* \in A$ with

$$\int_0^{2\pi} G(\dot{z}_*)\, dt = \inf_{z \in A} \int_0^{2\pi} G(\dot{z})\, dt = \mu$$

(iii) $z = z_*$ satisfies the Euler equations

$$\nabla G(\dot{z}_*) = \alpha\, Jz_* + \beta$$

with some constants α, β, $\alpha \neq 0$.

(iv) z_* belongs to C^2 and satisfies

$$\dot{z}_* = \nabla H(\alpha J z_* + \beta)$$

and $x = c(\alpha J z_* + \beta)$ is the desired solution.

Ad (i). We need some estimates. Since every z belonging to F has the mean value zero, the Poincaré inequality

(12.13) $$\|z\| \le \|\dot{z}\| , \qquad z \in F$$

holds, where $\|\ \|$ denotes the L_2-norm

$$\|z\|^2 = \int_0^{2\pi} |z(t)|^2 \, dt .$$

This follows simply from the Fourier series. For $z \in A$ we then have by Hölder's inequality

(12.14) $$2 = \int_0^{2\pi} \langle Jz, \dot{z} \rangle \, dt \le \|z\| \, \|\dot{z}\| \le \|\dot{z}\|^2 , \quad z \in A .$$

The function G being strictly convex and homogeneous of degree 2 satisfies the estimate

$$\frac{1}{K} |y|^2 \le G(y) \le K |y|^2 , \qquad y \in R^{2n}$$

for some constant $K \ge 1$. Therefore, by means of (12.14) we find for $z \in A$

(12.15) $$\int_0^{2\pi} G(\dot{z}) \, dt \ge \frac{1}{K} \|\dot{z}\|^2 \ge \frac{2}{K} > 0 ,$$

i.e. the functional is bounded from below by a positive constant, in particular $\mu > 0$.

Ad (ii). We pick a minimizing sequence $z_j \in A$ such that

$$\lim_{n \to \infty} \int_0^{2\pi} G(\dot{z}_j) \, dt = \mu .$$

By (12.15) and (12.14) there is a constant $M > 0$ with

$$2 \le \|\dot{z}_j\|^2 \le M .$$

By (12.14) we obtain $\|z_j\| \ge 2 \|\dot{z}\|^{-1} \ge \frac{2}{\sqrt{M}}$, and so by (12.13)

(12.16) $$\frac{2}{\sqrt{M}} \le \|z_j\| \le \sqrt{M} .$$

In particular z_j is a bounded sequence in the Hilbert space $H_1(S^1)$ and therefore a subsequence, also denoted by z_j, converges weakly in $H_1(S^1)$ to an element $z_* \in H_1(S^1)$:

(12.17) $$z_j \rightharpoonup z_* \quad \text{weakly in } H_1(S^1) .$$

In addition

(12.18) $$\sup_t |z_j(t) - z_*(t)| \to 0 ,$$

i.e. z_j converges uniformly to z_*. For the first statement we used the fact that the closed unit ball of a Hilbert space is weakly compact (see e.g. the textbook "Real Variables" by Royden, p. 173). The second statement follows from the estimates

$$|z_j(t) - z_j(\tau)| \le \left| \int_t^{\tau} \dot{z}_j(s) \, ds \right|$$

$$\le |t-\tau|^{1/2} \|\dot{z}_j\| \le |t-\tau|^{1/2} \sqrt{M}$$

by means of Ascoli's theorem.

We shall now show that $z_* \in A$. By (12.18)

$$0 = \frac{1}{2\pi}\int_0^{2\pi} z_j(t)\,dt \to \frac{1}{2\pi}\int_0^{2\pi} z_*(t)\,dt\ ;$$

Also,

$$2 = \int_0^{2\pi} \langle Jz_j, \dot z_j\rangle = \int_0^{2\pi} \langle J(z_j - z_*), \dot z_j\rangle + \int_0^{2\pi} \langle Jz_*, \dot z_j\rangle\ .$$

The first term on the right-hand side tends to zero by (12.18) since $|\dot z_j| \le \sqrt{M}$, and the second term converges by (12.17); hence

$$\int_0^{2\pi} \langle Jz_*, \dot z_*\rangle = 2$$

and therefore $z_* \in A$. We next prove that this z_* is indeed the minimum. From the convexity of G we deduce (see Exercise 2) that

$$\langle G_y(y_1), y_2 - y_1\rangle \le G(y_2) - G(y_1) \le \langle G_y(y_2), y_2 - y_1\rangle\ ,$$

which applied to $y_2 = \dot z_*$ and $y_1 = \dot z_j$ gives

$$\int_0^{2\pi} G(\dot z_*)\,dt - \int_0^{2\pi} G(\dot z_j)\,dt \le \int_0^{2\pi} \langle G_y(\dot z_*), \dot z_* - \dot z_j\rangle\,dt$$

Observing that G_y being homogeneous of degree 1 satisfies the estimate $|G_y(y)| \le C|y|$, $y \in R^{2n}$ for some positive constant C, we conclude that indeed $G_y(\dot z_*) \in L_2$ and hence the right-hand side tends to zero since $\dot z_* - \dot z_j \rightharpoonup 0$ weakly in L_2. Thus

$$\int_0^{2\pi} G(\dot z_*)\,dt \le \varliminf_{j\to\infty} \int_0^{2\pi} G(\dot z_j)\,dt = \mu\ ,$$

and we have proved that indeed

(12.19) $$\int_0^{2\pi} G(\dot z_*)\,dt = \mu\ ,$$

where

$$\mu = \min_{z\in F} \int_0^{2\pi} G(\dot z)\,dt \quad \text{under} \quad \frac{1}{2}\int_0^{2\pi} \langle Jz, \dot z\rangle\,dt = 1\ .$$

Ad (iii). Since z_* is a minimum,

(12.20) $$\int_0^{2\pi} \langle \nabla G(\dot z_*), \dot \zeta\rangle\,dt = 0$$

for every test function $\zeta \in F$ satisfying

(12.21) $$\int_0^{2\pi} \langle Jz_*, \dot\zeta\rangle\,dt = 0\ .$$

We choose ζ, in particular, so that

$$\dot\zeta = \nabla G(\dot z_*) - \alpha\, Jz_* - \beta.$$

In order that ζ is periodic we pick β so that the mean value of $\dot\zeta$ is zero:

$$\int_0^{2\pi} \dot\zeta\,dt = \int_0^{2\pi} \nabla G(\dot z_*)\,dt - 2\pi\beta = 0\ ,$$

and then determine the constant α such that (12.21) holds true. Using again that z_* has mean value zero, we find that

$$0 = \int_0^{2\pi} \langle Jz_*, \dot\zeta\rangle = \int_0^{2\pi} \langle Jz_*, \nabla G(\dot z_*)\rangle - \alpha \int_0^{2\pi} \langle Jz_*, Jz_*\rangle$$

has a unique solution α, since by (12.16)

$$\int_0^{2\pi} \langle Jz_*, Jz_*\rangle = \int_0^{2\pi} \langle z_*, z_*\rangle = |z_*|^2 > 0 .$$

With this test function ζ, the condition (12.20) becomes

$$\int_0^{2\pi} |\dot\zeta|^2 = \int_0^{2\pi} |\nabla G(\dot z_*) - \alpha Jz_* - \beta|^2 dt$$

$$= \int_0^{2\pi} \langle \nabla G(\dot z_*), \dot\zeta\rangle - \alpha \int_0^{2\pi} \langle Jz_*, \dot\zeta\rangle - \langle \beta, \int_0^{2\pi} \dot\zeta\rangle = 0 ,$$

and we conclude, that z_* satisfies the following Euler equations

$$(12.22) \qquad \nabla G(\dot z_*) = \alpha Jz_* + \beta$$

for constants α and β, as was of course to be expected. The constant α has the following interpretation:

$$\alpha = \mu ,$$

hence $\alpha > 0$. In fact, using the Euler equations and the homogeneity of G we have

$$2\mu = 2 \int_0^{2\pi} G(\dot z_*) = \int_0^{2\pi} \langle \nabla G(\dot z_*), \dot z_*\rangle = \alpha \int_0^{2\pi} \langle Jz_*, \dot z_*\rangle = 2\alpha$$

Ad (iv). It is easy to see that z_* belongs to C^2. In fact remembering that $x = \nabla G(y)$ is inverted by $y = \nabla H(x)$ pointwise we find from (12.22)

$$(12.23) \qquad \dot z_*(t) = \nabla H(\alpha J z_*(t) + \beta) .$$

The right-hand side is continuous and we conclude that $z_* \in C^1$ and inserting it into the right-hand side we get $z_* \in C^2$. Finally, setting

$$x(t) = c(\alpha J z_*(t) + \beta) , \quad c = \sqrt{\frac{2\pi}{\alpha}} > 0 ,$$

we obtain from (12.23) using the homogeneity of ∇H

$$\dot x = c\alpha J \, \nabla H(\alpha Jz_* + \beta) = \alpha J \, \nabla H(x) ,$$

and, since H is of degree 2

$$\int_0^{2\pi} H(x) = \frac{1}{2} \int_0^{2\pi} \langle \nabla H(x), x\rangle = \frac{1}{2} c^2 \alpha \int_0^{2\pi} \langle \dot z_*, Jz_*\rangle = c^2 \alpha$$

which is equal to 2π. Therefore $x(t)$ is a 2π periodic solution of the equation $\dot x = \alpha J \nabla H(x)$ which lies on the energy surface $H = 1$. Consequently $y(t) = x(\alpha^{-1} t)$ is the sought periodic solution of $\dot x = J\nabla H(x)$ on $H = 1$, which has the period $T = 2\pi\alpha$. But this is rather arbitrary since the period of the periodic solution on M depends on our choice of the Hamilton function H.

(d) Remarks. Recalling the transformation properties of Hamiltonian vector fields one concludes from Theorem 10.2 that in fact every regular energy surface which is symplectically diffeomorphic to a strictly convex one carries a periodic solution. Such a situation is however hard to recognize. Since the convexity property is generally lost under a symplectic diffeomorphism it would be interesting to find invariant conditions on an energy surface guaranteeing a periodic solution. For instance it is not known whether an energy surface diffeomorphic to a sphere carries a periodic solution. In this connection one has to keep in mind that there are vector fields on odd dimensional spheres having no periodic orbits (see [1] and [2]) and most likely such vector fields do exist in the more restricted class of measure preserving vector fields.

We point out that the convexity requirement in Theorem 10.2 can be relaxed, it is sufficient to require the surface to be starlike with respect to an interior point. This was proved by an entirely different method, but using functional analytic techniques by P. Rabinowitz cited above.

[1] P. A. Schweitzer, "Counterexamples to the Seifert conjecture and opening closed leaves of foliations," Annals of Math. 100 (1974), 386-400.

[2] T. W. Wilson, "On the minimal sets of nonsingular vector fields," Annals of Math. 84 (1966) 529-536.

Exercises. Ch. II, Sec. 12

1. Assume $H(x) \leq C|x|^2$, $x \in R^{2n}$, for some constant $C > 0$.

(a) Show that

$$\inf_{z \in A} \int_0^{2\pi} H(z(t))\, dt = 0 ,$$

where A is defined by (10.19).

Hint: construct a minimizing sequence of the form z_n

$$z_n(t) = A \frac{1}{\sqrt{n}} \cos nt + B \frac{1}{\sqrt{n}} \sin nt .$$

(b) Prove for any minimizing sequence: $|\dot{z}_n| \to \infty$ (L_2-norm).

2. Let $F \in C^1(R^n)$. Prove that F is convex if and only if

$$F(x) - F(y) \geq \langle \nabla F(y), x-y \rangle$$

and F is strictly convex if and only if the inequality holds if $x \neq y$. (Here a C^1 function is called convex if $F(\alpha x + \beta y) \leq \alpha F(x) + \beta F(y)$, where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$; it is called strictly convex if the inequality holds for $\alpha, \beta > 0$ and $x \neq y$.)

3. Assume $G(y)$ is the Legendre transformation of $H(x)$ such that

$$G(y) + H(x) = \langle x, y \rangle$$

and $y = H_x(x)$, $x = G_y(y)$.

Prove that H is strictly convex if and only if G is strictly convex.

Hint: Use Exercise 2.

4. A map $g: R^n \to R^n$ is called monotone increasing if $\langle g(x) - g(y), x-y\rangle \geq 0$, and it is called strictly monotone increasing if the inequality holds for $x \neq y$. Prove that $F \in C^1(R^n)$ is (strictly) convex if and only if ∇f is (strictly) monotone increasing.

13. Periodic Orbits Having Prescribed Periods.

(a) Statement

In the previous section we established the existence of periodic solutions on a given energy surface. We shall now apply the same method in order to find periodic solutions with prescribed period. We consider a time dependent Hamiltonian vector field with $H(t,x) = H(t+T,x)$, i.e. H is periodic in time of period $T > 0$, and look for T-periodic solutions of the time dependent Hamiltonian equations

$$\dot{x} = JH_x(t,x).$$

The following result is due to F. Clarke and I. Ekeland. [1]

Theorem 13.1. Let $H \in C^2$ be periodic in time of period $T > 0$, and assume H is strictly convex in x having a minimum at the origin, $H(t,0) = 0$ and $H_x(t,0) = 0$, moreover assume that H satisfies

$$\frac{c}{2} |x|^2 - M \leq H(t,x) \leq \frac{b}{2} |x|^2 + M$$

for all t and $x \in R^{2n}$, and

$$H(t,x) \geq \frac{a}{2} |x|^2$$

for all t and $|x| \leq \varepsilon$, for some $\varepsilon > 0$, where a,b,c and M are positive constants. Then, if $a > b$ there is a nonconstant periodic solution having period T provided

[1] F. H. Clarke and I. Ekeland, "Hamiltonian trajectories having prescribed minimal period,"

$$b < \frac{2\pi}{T} < a$$

In particular this solution is different from the trivial solution $x \equiv 0$.

One might wonder whether the condition $a > b$ is compatible with the convexity of H. Here is an example of such a function

$$2H(x) = b|x|^2 + \sqrt{1+\varepsilon|x|^2} \quad \text{for} \quad \varepsilon \geq a - b > 0 .$$

(b) Proof.

With $G(t,y)$ we denote the Legendre transformation of $H(t,x)$ defined by

$$G(t,y) = \max_{\xi \in R^{2n}} \left(\langle\xi,y\rangle - H(t,\xi)\right) .$$

One verifies easily that $G(t,0) = 0$, that G is strictly convex and that it satisfies the estimates

(13.1) $$\frac{1}{2b}|y|^2 - M \leq G(t,y) \leq \frac{1}{2c}|y|^2 + M$$

for $y \in R^{2n}$, and

(13.2) $$G(t,y) \geq \frac{1}{2a}|y|^2$$

if $|y| \leq \delta$ for some $\delta > 0$. We consider the variational principle

$$\min_{z \in F} \Phi(z) ,$$

where Φ is defined by

$$\Phi(z) = \int_0^T \left(G(t,\dot{z}) - \frac{1}{2}\langle Jz,\dot{z}\rangle\right) dt ,$$

and where F is the following space of periodic functions having period T and mean value 0 :

$$F = \left\{z \in H_1(S^1) \bmod T \;\middle|\; \frac{1}{T}\int_0^T z\, dt = 0\right\} .$$

We proceed in principle as before. The functional Φ is bounded from below. In fact, if $z \in F$ we conclude from Poincare's inequality $\|z\| \leq \frac{T}{2\pi}\|\dot{z}\|$, that

$$\int_0^T \langle Jz,\dot{z}\rangle \leq \frac{T}{2\pi}\|\dot{z}\|^2 ,$$

and therefore by (13.1)

$$\Phi(z) \geq \frac{1}{2}\left(\frac{1}{b} - \frac{T}{2\pi}\right)\|\dot{z}\|^2 - TM ,$$

but by assumption $1/b > T/2\pi$. As G is convex one finds from (13.1) the estimate $|\nabla G(y)| \leq C(|y| + 1)$, $y \in R^{2n}$ for some constant C (see Exercise 1). We can therefore proceed as in the proof of Theorem 11.1 to show that a minimizing sequence converges weakly to some $z_* \in F$ satisfying

$$\mu = \min_{z \in F} \Phi(z) = \Phi(z_*) ,$$

and hence

$$\int_0^T \langle\nabla G(t,\dot{z}_*) - Jz_*,\dot{\zeta}\rangle = 0$$

for every test function $\zeta \in F$. Consequently z_* solves the

Euler equations

$$\nabla G(t,\dot{z}_*) - Jz_* = \beta$$

for some constant $\beta \in R^n$, and so $\dot{z}_* = \nabla H(t, Jz_* + \beta)$. Therefore $x = Jz_* + \beta$ is periodic with period T and satisfies the equations $\dot{x} = J\dot{z}_* = J\nabla H(t,x)$. It remains to show that x hence z_* is not a constant solution. But this follows from $a > 2\pi/T$ as follows. We define the test function z by

$$z(t) = A\cos\omega t + JA\sin\omega t\ , \quad \omega = \frac{2\pi}{T}\ , \quad A \in R^{2n},$$

lying in F, and substitute it into Φ. One verifies readily using (13.2) that

$$\Phi(z) \leq \frac{T}{2}\,\omega^2|A|^2\,\left(\frac{1}{a} - \frac{T}{2\pi}\right)$$

if $|A|$ is sufficiently small. The expression on the right-hand side is negative, hence $\mu = \Phi(z_*) \leq \Phi(z) < 0$ and therefore $\dot{z}_*$ cannot be zero. This finishes the proof of the theorem.

(c) Relation to the Poincare-Birkhoff Theorem.

We shall show that for $n = 1$ periodic solutions can be established without the convexity assumption. We consider in R^2 a Hamiltonian system $H(t,x) = H(t+T,x)$ which satisfies

$$JH_x(t,x) = aJx + o(|x|) \quad \text{as} \quad |x| \to 0$$

$$JH_x(t,x) = bJx + o(|x|) \quad \text{as} \quad |x| \to \infty\ ,$$

uniformly in t for two constants a and b. This system has 0 as an equilibrium point and is asymptotically linear. The linear systems given by aJx and bJx represent two harmonic oscillators.

We claim that the system $\dot{x} = JH_x(t,x)$ has a nontrivial periodic solution of period T provided there is a $j \in Z$ such

$$a < j\,\frac{2\pi}{T} < b \tag{13.3}$$

or $b < j\,\frac{2\pi}{T} < a$. This condition is satisfied for every $T > 0$ if a and b have different signs. We shall prove this claim by means of the Poincare-Birkhoff fixed point theorem. Proceeding as in the proof of Theorem 9.1 we introduce symplectic polar coordinates $x_1 + ix_2 = \sqrt{2R}\,e^{i\theta}$ such that the system becomes

$$\dot{\theta} = \hat{H}_R(t,\theta,R)$$

$$\dot{R} = -\hat{H}_\theta(t,\theta,R)$$

with $\hat{H}(t,\theta,R) = H(t,x)$. We define a measure preserving homomorphism of an annulus by

$$(\theta_0,R_0) \to (\theta_1,R_1) = \big(\theta(T) - 2\pi j,\ R(T)\big)\ ,$$

where $\theta(t)$ and $R(t)$ are the solutions of the equations having the initial conditions $\theta(0) = \theta_0$ and $R(0) = R_0$. The inner boundary $R_0 = 0$ remains invariant and on R_0

$$\theta_1 - \theta_0 = Ta - 2\pi j = 2\pi\left(a \frac{T}{2\pi} - j\right) < 0$$

by (13.3). Similarly, if R_0 is sufficiently large we find:

$$\theta_1 - \theta_0 = Tb - 2\pi j + O\left(\frac{1}{R_0}\right) > 0 .$$

Therefore the twist condition required in Theorem 7.3 is met and we conclude two fixed points for the map which by construction gives rise to nonconstant periodic solutions having period T.

Incidentally the last statement can be generalized to higher dimensions, that is to systems $H(t,x) = H(t+T,x)$ satisfying

$$JH_x(t,x) = JAx + o(|x|) \quad \text{as} \quad |x| \to 0$$
$$JH_x(t,x) = JBx + o(|x|) \quad \text{as} \quad |x| \to \infty$$

uniformly in t for two symmetric and time independent matrices A and B. There is an integer $\text{ind}(A,B,T) \in Z$ depending only on T and the purely imaginary eigenvalues of JA and JB with the property that if $\text{ind}(A,B,T) \neq 0$ the Hamiltonian system possesses a nontrivial periodic solution of period T. This integer does not vanish for instance if $A < 0 < B$ or $B < 0 < A$. In dimension two the condition $\text{ind}(A,B,T) \neq 0$ is precisely the condition (13.3) if JA and JB have the imaginary eigenvalues ia and ib respectively. The proof of this generalization uses as topological tool the generalized Morse theory developed by C. Conley, we refer to the paper [1].

[1] H. Amann and E. Zehnder, "Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations," Ann. Sc. norm. sup. di Pisa, 1980.

We discussed the above results because of their intrinsic interest and the use of calculus of variations to establish the existence of periodic orbits but actual applications of these results are not yet available. As for the comparison with the Poincare-Birkhoff fixed point theorems it is to be said that a genuine generalization of this theorem to symplectic mappings of higher dimensions has not been found. Of course, if additional assumptions are imposed in the interior such results do exist as we have seen in Section 8 but have the drawback of not being applicable to higher iterates. It is one of the main strengths of the Poincare-Birkhoff fixed point theorem to yield at once infinitely many periodic orbits. This cannot be said about the results discussed in this section.

Chapter II, Section 13 EXERCISES

1. Assume $G \in C^1(R^n)$ is convex and satisfies $G(x) \leq C|x|^2 + M$, $x \in R^n$. Prove that ∇G is linearly bounded i.e. $|\nabla G(x)| \leq C'(|x| + 1)$.

   Hint: $G(x + \varepsilon y) - G(x) \geq \langle \nabla G(x), \varepsilon y \rangle$, $\varepsilon > 0$. Prove

$$|\nabla G(x)| \leq \frac{2}{\varepsilon}\left\{C(|x| + \varepsilon)^2 + M\right\}.$$

2. Let $x(t)$ be the T-periodic solution found in Theorem 10.3 for the case that H is independent of t and let $H(x(t)) = h$.

   (a) Prove

$$T \cdot h = -\mu + \frac{1}{2}\int_0^T \langle Jz_*, \dot{z}_* \rangle \, dt ,$$

   where $x = Jz_* + \beta$ and $\mu = \Phi(\dot{z}_*)$ are as in the proof of Theorem 10.3

   (b) Deduce the following estimate relating the energy h and the period T of this periodic solution:

$$h \leq \frac{M}{\frac{1}{b} - \frac{T}{2\pi}} \cdot \frac{T}{2\pi}$$

   Hint: Integrate $G(y) + H(x) = \langle x, y \rangle$ for $x(t)$ and $y(t) = \dot{z}_*(t)$ and use the estimates in the proof of Theorem 10.3

3. Assume H is as in Theorem 10.3 but independent of t. Then the theorem guarantees a nonconstant periodic solution z_* for every given period T in the interval $b < 2\pi/T < a$. Prove that this period is minimal.

   Hint: Argue by contradiction and assume that z_* has period $n^{-1}T$ for some integer $n \geq 2$. Show that the test function $z(t) = n\, z_*(\frac{t}{n})$ satisfies

$$\Phi(z) = n\,\Phi(z_*) - (n-1)\int_0^T G(\dot{z}_*)\, dt .$$

i.e $G(\dot{z}_*)$

CHAPTER III. INTEGRABLE HAMILTONIAN SYSTEMS

1. A Theorem of Arnold and Moser

(a) Definition and Examples

In this chapter we consider a particularly simple class of Hamiltonian systems, the so-called integrable ones. They are characterized by the property of having sufficiently many integrals such that the task of solving (or "integrating) the differential equations becomes essentially trivial. For a general system of differential equations in R^m this would require $m-1$ integrals but for Hamiltonian systems in R^{2n} this requires only n integrals.

Before giving the formal definition we recall that $\omega = \sum_{j=1}^{n} dy_j \wedge dx_j$ defines a symplectic structure in R^{2n} with Poisson brackets

$$\{F,G\} = \sum_{j=1}^{n}\left(\frac{\partial F}{\partial x_j}\frac{\partial G}{\partial y_j} - \frac{\partial F}{\partial y_j}\frac{\partial G}{\partial x_j}\right) .$$

For any function $H \in C^2(D)$, where D is a domain in R^{2n} we define

$$X_H = \sum_{j=1}^{n}\left(\frac{\partial H}{\partial y_j}\frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j}\frac{\partial}{\partial y_j}\right)$$

so that

$$X_H F = -\{H,F\} .$$

A function $F \in C^1(D)$ is called an integral of the vector field X_H if $X_H F = 0$, but $dF \neq 0$ in D. Thus F is constant along the orbits of X_H.

Definition. A Hamiltonian vector field X_H is called integrable in D if it possesses n integrals $F_j \in C^2(D)$ with the following properties:

$$(1.1)\quad \begin{cases} \text{(i)} & dF_1, dF_2, \ldots, dF_n \text{ are linearly independent in } D \\ \text{(ii)} & \{H,F_j\} = 0 \text{ in } D \\ \text{(iii)} & \{F_j,F_k\} = 0 \text{ in } D . \end{cases}$$

It is not difficult to generalize this concept to symplectic manifolds M with a symplectic 2-form ω. In this case we defined (see Chapter 1, Section 7) X_H by

$$\omega(X_H,\cdot) = -dH$$

for any $H \in C^2(M)$ and

$$\{F,G\} = \omega(X_F,X_G) .$$

Then we call X_H integrable on (M,ω) if the conditions (1.1) hold with D replaced by M. In most applications it suffices to work in a domain D in R^{2n}.

In the above definition it is essential that we require the existence of the integrals F_j "in the large", since locally, i.e. in the neighborhood of a point $p \in M$ which is not a stationary point any Hamiltonian system is integrable. Indeed by Theorem 4.2 of Chapter I we can introduce local coordinates such that $H = y_1$ so that $F_j = y_j$ define the integrals.

As a rule we will require that the domain D or the manifold M are invariant under the flow ϕ^t of X_H, i.e.

$$\phi^t(D) = D , \quad t \in R .$$

A simple example of an integrable system is given by

Example 1: If

$$H = H(y_1,\dots,y_n) \in C^2(D_2) , \quad D_2 \subset R^n$$

is independent of the x_j then $F_j = y_j$ define n integrals satisfying (1.1) in $D = R^n \times D_2$.

Example 2: The motion of n harmonic oscillators is given by

$$H = \frac{1}{2}\sum_{j=1}^{n} (p_j^2 + \omega_j^2 q_j^2) , \quad \omega_j > 0$$

in R^{2n}. This system is integrable with

$$F_j(p,q) = p_j^2 + \omega_j^2 q_j^2 \qquad (j = 1,2,\dots,n),$$

but we have to restrict ourselves to the set

$$D = \{p,q \mid F_1 \cdot F_2 \cdots F_n \neq 0\}$$

where the dF_j are linearly independent.

Actually Example 2 is a special case of Example 1 since the symplectic transformation

$$(1.2) \quad \begin{cases} q_j = \sqrt{\dfrac{2y_j}{\omega_j}} \cos x_j \\ p_j = \sqrt{2\omega_j y_j} \sin x_j \end{cases}$$

takes the Hamiltonian of Example 2 into

$$\sum_{j=1}^{n} \omega_j y_j$$

and F_j into $2\omega_j y_j$, and this is a special case of Example 1. But in this case the x_j are to be identified modulo 2π. Since the x_j-variables do not occur in Example 1 we can equally well consider

Example 3: Set

$$H = H(y_1,\dots,y_n) \in C^2(D_2) , \quad D_2 \in R^n$$

and consider the flow X_H on the manifold

$$M = T^n \times D_2$$

where $T^n = R^n/Z^n$ is the torus obtained by identifying $x,x' \in R^n$ if $x - x'$ has integer coordinates, i.e. $x - x' \in Z^n$, the lattice of vectors with integer coefficients. This system is again integrable. It has the property that the manifolds

$$\{x,y \mid y_j = c_j\}$$

are tori and hence compact. It is a typical example and we shall show that many integrable systems can be brought into this form, if the variables are introduced in an appropriate way. In physics or mechanics these variables are often called action-angle-variables: The x_j are the angle variables and the y_j are the action variables. The last term is related to the action

$$\int_\gamma \sum_{j=1}^{n} p_j \, dq_j$$

for a closed curve γ. For example if we take γ to be the ellipse

$$p_1^2 + \omega_1^2 q_1^2 = a_1 \ ; \quad p_j = p_j^0 \ , \quad q_j = q_j^0 \quad \text{for} \quad j \geq 2 \ ,$$

then the above action is equal to

$$\frac{\pi a_1}{\omega_1} = \int_0^{2\pi} y_1 \, dx_1 = 2\pi \, y_1 \quad ,$$

if x,y are defined by (1.2). Hence the y_j are obtained as "action integrals" by different curves on the manifolds $F_j = c_j$. Incidentally, integrable Hamiltonian systems played a role in the old quantum theory of Bohr-Sommerfeld where the values of the action integrals were quantized. One major shortcoming of that theory is that it is only applicable to integrable systems which constitutes a very restrictive, even exceptional class of Hamiltonian systems.

(b) Statement of the theorem

Since the F_j are constant along the orbits of X_H it is natural to consider the "level sets"

$$N_c = \{m \in M \mid F_j(m) = c_j, \ j = 1,2,\ldots,n\},$$

which are n-dimensional submanifolds of M. It follows from (1.1) (i) and (iii) that $X_{F_1}, X_{F_2}, \ldots X_{F_n}$ span the tangent space of N_c since they are linearly independent and

$$X_{F_j} F_k = \{F_j, F_k\} = 0 \ .$$

Similarly, by (1.1) (ii) we have

$$X_H F_k = \{H, F_k\} = 0 \ ,$$

which shows that X_H is tangential to N_c and defines a vector field on N_c, the restriction of X_H to N_c. Moreover, the manifolds N_c are Lagrange manifolds which means that

$$\omega(X_{F_j}, X_{F_k}) = \{F_j, F_k\} = 0 \ .$$

We note that the choice of the individual integrals is rather unimportant and they can be replaced by functions of the F_j, say $\mu_k(F)$, where $F = (F_1, F_2, \ldots, F_n)$. Since

$$\{\mu_k(F), \mu_\ell(F)\} = \sum_{i,j=1}^{n} \frac{\partial(\mu_k, \mu_\ell)}{\partial(F_i, F_j)} \{F_i, F_j\}$$

these $\mu_k(F)$ are also in involution, and if $\mu = (\mu_1, \mu_2, \ldots \mu_n)$ has a nonvanishing Jacobian then the $d\mu_k(F)$ are also linearly independent. Of course the level sets N_c are up to the parameter choice independent of μ. We will reserve the freedom to replace the vector function $F = (F_1, F_2, \ldots, F_n)$ by $\mu \circ F$.

The following theorem will show that if one of the manifolds N_c which we take to be N_0 is compact and connected, then it is an n-dimensional torus, and in a neighborhood of this torus the vector field X_H can be written in the form of Example 3 if appropriate coordinates are introduced, i.e. we can introduce action-angle variables. We describe the statement on a symplectic manifold (M,ω) of dimension 2n.

Theorem 1.1.* Let $F = (F_1, F_2, \ldots, F_n)$ be a C^2-vector-function on a symplectic manifold (M,ω), where dim $M = 2n$,

* This theorem is due to Arnold for the special case $M = R^{2n}$, however, he required an additional assumption which was removed by R. Jost who proved it in the general case.

and assume that F satisfies (1.1) (i) and (iii). Moreover, assume that the n-dimensional submanifold $N := F^{-1}(0) \subset M$ is compact and connected. Then

(a) $N = F^{-1}(0)$ is an embedded n-dimensional torus T^n,

(b) in addition there is an open neighborhood $U(N) \subset M$ which can be described by coordinates x_j, y_j in the following manner:

If $x = (x_1, x_2, \dots, x_n)$ are variables on the torus $T^n = R^n/Z^n$ and $y = (y_1, \dots, y_n) \in D_1$ where D_1 and D_2 are some domains of R^n containing $y = 0$ then there exist diffeomorphisms

$$\psi: T^n \times D_1 \to U(N) \quad ,$$
$$\mu: D_2 \to D_1 \,, \quad \mu(0) = 0 \quad ,$$

such that

(1.3) $$\mu \circ F \circ \psi = y \quad ,$$

(1.4) $$\psi^* \omega = \sum_{j=1}^{n} dy_j \wedge dx_j \; .$$

In particular, ψ maps the torus $T^n \times \{0\}$ onto $N_0 = F^{-1}(0)$ and the torus $T^n \times \{y\}$ onto $N_c \cap U$ where $y = \mu(c)$.

Corollary to Theorem 1. Any integrable Hamiltonian system given by H with integrals F_j is by the symplectic diffeomorphism ψ transformed into the following system on $T^n \times D_1$:

$$H \circ \psi = h(y_1, \dots, y_n) \; ,$$

where x_j, y_j are canonical coordinates and $x \in T^n$ angular variables mod 1. In particular on U(N) the Hamiltonian is a function of the integrals $F_1, \dots, F_n$.

Proof of Corollary: $F_j \circ \psi$ and therefore by (1.3) the coordinate functions y_j are integrals, therefore

$$0 = \{H \circ \psi, y_j\} = \{h, y_j\} = \frac{\partial}{\partial x_j} h \; ,$$

and hence h does not depend on x.

Thus near a compact connected level surface $F^{-1}(0)$ the flow is extremely simple:

$$\dot{x}_j = \frac{\partial h}{\partial y_j} \; , \quad \dot{y}_j = 0 \; .$$

It is easily "integrated" with the solutions

(1.5) $$x_j(t) = x_j(0) + t \frac{\partial h}{\partial y_j} \; , \quad y_j(t) = y_j(0) \; ;$$

hence the name integrable system. In more geometric terms every torus $T^n \times \{y\}$ is invariant under the flow ϕ^t of h, and the restriction of the flow onto such a torus is linear:

(1.6) $$\phi^t | T^n \times \{y\}: (x, y) \mapsto (x + t\omega, y) \quad ,$$

where $\omega = \omega(y) = \frac{\partial}{\partial y} h(y) \in R^n$ are the so-called frequencies. If the integrable Hamiltonian system is nondegenerate in the sense that

(1.7) $$\mathrm{Det}\left(\frac{\partial^2 h}{\partial y^2}(y)\right) \neq 0 \; , \quad y \in D \; ,$$

then the frequencies $\omega(y) = \frac{\partial h}{\partial y}(y)$ vary from torus to torus.

We can distinguish tori on which the $\omega_1, \omega_2 \cdots \omega_n$ are rationally independent and those where there exists a nontrivial relation $j_1\omega_1 + \cdots + j_n\omega_n = 0$ with integer coefficients j_k. In the first case each orbit is dense on the torus, as is well known by a theorem of Kronecker. In the second case the closure of each orbit lies on a lower dimensional torus. Such a relation $j_1\omega_1 + \cdots + j_n\omega_n = 0$ is called a resonance condition. For some tori one will have several resonance conditions and in the case one has n-1 linear independent relations one has $\omega_k = cg_k$ with integers g_k and $c > 0$. Then all orbits are periodic of period c^{-1}. In view of Lemma 1.3 of Chapter II all Floquet multipliers of these periodic solutions are 1 and under small perturbations of H such tori of periodic orbits will generally be destroyed. This fact can be used to show that near an integrable system there are Hamiltonian systems which are not integrable. In other words integrable systems are quite exceptional.

We remark, that if M is a domain in R^{2n}, the above solutions in U(N) are in the original p,q coordinates represented by trigonometrical series in t. Indeed

$$(p,q) = \psi(x,y) = \sum_{j\in Z^n} a_j(y)\, e^{2\pi i<j,x>} ,$$

and therefore we find by means of (1.6)

$$\psi \circ \phi^t \circ \psi^{-1}(p,q) = \psi \circ \phi^t(x,y)$$

$$= \sum_{j\in Z^n} a_j(y)\, e^{2\pi i<j,x+t\omega>}$$

$$= \sum_{j\in Z^n} b_j(x,y)\, e^{2\pi i<j,\omega>\cdot t} ,$$

$$b_j(x,y) = a_j(y)\, e^{2\pi i<j,x>}$$

where $\omega = \omega(y) = \frac{\partial}{\partial y} h(y)$.

Although the solutions of integrable systems have simple behavior it is by no means easy to recognize whether a given Hamiltonian system is integrable or not. This requires finding the integrals and then the action and angle variables whose existence is guaranteed by the above theorem once the integrals are found. In principle, this latter step requires just the integration of functions, or a "quadrature" as it is expressed in the older literature. In the later sections we shall describe such unexpected integrable systems and methods of finding action-angle variables.

We mention that the action angle-variables x,y of Theorem 1.1 are fixed to a large extent. In case x',y' is another set of such variables with the same properties as in that theorem then there exist a scalar function w(y) , a unimodular matrix M and constants $c \in R^n$ such that

$$(1.8) \qquad x' = M(x + \frac{\partial w}{\partial y}) , \quad y' = (M^T)^{-1}y + c$$

provided (1.7) holds.

Indeed under the assumption (1.7) there are many tori with dense orbits from which one concludes that $y' = f(y)$ holds. Since the mapping $(x,y) \to (x',y')$ is canonical it follows from previous considerations that

$$f_y^T(y)x' = x + \frac{\partial w}{\partial y}$$

is the most general extension of $y' = f(y)$ to a canonical map. Finally since this mapping must preserve the lattice Z^n the matrix f_y must have integer coefficients as well as its inverse, hence it is unimodular proving (1.8).

Thus on each fixed torus the angle-variables are fixed up to a phase-shift w_y and up to a change of basis of the fundamental group on T^n, given by M. But straight lines on T^n are mapped into straight lines under (1.8).

(c) Proof of Theorem 1.1.

The proof proceeds in three steps: First we show that N_0 is a torus. Second, we introduce coordinates x,y locally near some point p on N_0 (see Lemma 1.2) which is essentially contained in our previous considerations in Chapter I. Third, we extend these coordinates to a neighborhood of the torus. The normalization of the periods requires an additional change of variables (Lemma 1.3) which concludes the proof.

To prove statement (a) of the theorem we recall that the vector fields X_{F_j} span the tangent space of $N = N_0$. Let $\phi_j^{t_j}$ denote the flow associated with X_{F_j}. Since $[X_{F_j}, X_{F_k}]$ is a Hamiltonian vector field with the Hamiltonian $\{F_j, F_k\} = 0$,

the vector fields commute, hence the flows $\phi_j^{t_j}$ also commute, and we set

$$\phi^t = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \dots \circ \phi_n^{t_n}, \quad t = (t_1,\dots,t_n),$$

wherever it is defined. For $p \in N$ the flow $\phi^t(p)$ leaving N invariant is defined for all $t \in R^n$ as N is by assumption compact. This follows from standard existence theorems of ordinary differential equations. Hence we can define an action of R^n on N by

(1.9) $$t \to \phi^t(p) .$$

for $p \in N$. Clearly we have

$$\phi^t \circ \phi^s(p) = \phi^{t+s}(p), \quad p \in N$$

for $t,s \in R^n$. Now we fix the point $p \in N$ and observe that (1.9) maps R^n into N. It is an immersion, which means that $d\phi^t$ is an isomorphism; this follows from the fact that the vectors

$$d\phi^t\left(\frac{\partial}{\partial t_j}\right) = \frac{\partial}{\partial t_j} \phi^t(p) = X_{F_j}(\phi^t(p))$$

are linearly independent. The mapping (1.9) is onto N since its image is both closed and open in N. Therefore we can use $t = (t_1, t_2, \dots, t_n)$ as local coordinates on N.

We denote the isotropy group of p under (1.9) by

$$\Gamma = \{t \in R^n \mid \phi^t(p) = p\} .$$

Γ is a discrete subgroup of R^n, i.e. a lattice, and therefore it is generated by vectors $v_1, v_2, \ldots, v_d \in R^n$ which are linearly independent over R. Every element $\gamma \in \Gamma$ is of the form

$$\gamma = \sum_{k=1}^{d} g_k v_k$$

with integers g_k. Since the map (1.9) takes $t+\gamma$ into

$$\phi^{t+\gamma}(p) = \phi^t(p)$$

it gives rise to a diffeomorphism

$$\phi_0 : R^n/\Gamma \to N .$$

Since N is compact, hence also R^n/Γ compact we have $d = n$ i.e. the lattice is n-dimensional and so N is a torus.

For the second step we need the construction of local coordinates near the point $p \in N$ which we keep fixed.

Lemma 1.3. If F_j are n functions on a symplectic manifold (M,ω) satisfying (1.1) (i) and (iii), then every point p has a neighborhood U and a diffeomorphism ψ_0 of a neighborhood V of the origin in R^{2n} onto U such that $\psi_0(0,0) = p$,

$$(1.10) \quad \psi_0^*\omega = \sum_{k=1}^{n} dy_k \wedge dx_k \quad \text{and} \quad F_j \circ \psi_0 = y_j ,$$

where (x,y) are the coordinates in $V \subset R^{2n}$.

Proof: By Darboux's theorem (see Chapter I, Section 8) there exists a diffeomorphism ϕ of $V' \subset R^{2n}$ onto U such that $\phi^*\omega = \sum_{j=1}^{n} dp_j \wedge dq_j$ where p_j, q_j are the variables in V'. If we set $f_j = F_j \circ \phi$ then we have by assumption

$$\{f_j, f_k\} = 0 \quad \text{and} \quad df_j \quad \text{linearly independent.}$$

By the consideration of I §4 (b) one can construct functions $g_j(p,q)$ such that

$$\{g_j, f_k\} = \delta_{jk} , \quad \{g_j, g_k\} = 0 ,$$

so that $x_j = g_j(p,q)$, $y_j = f_j(p,q)$ defines a canonical map $\chi : (p,q) \to (x,y)$ of $V' \to V$ such that $f_j \circ \chi^{-1} = y_j$. Thus $\psi_0 = \phi \circ \chi^{-1}$ is the desired mapping.

Remark. Any other coordinates x', y' with the properties of (1.10) are related to x,y by

$$x' = x + \frac{\partial Q}{\partial y} , \quad y' = y$$

where $Q = Q(y)$ is a scalar function. This is an evident consequence of the canonical character of the map $(x,y) \to (x',y')$ and the relation $y' = y$.

Third step: With the fixed point $p \in N$ we introduce the coordinates x,y by the map

$$\psi_0 : V \to U$$

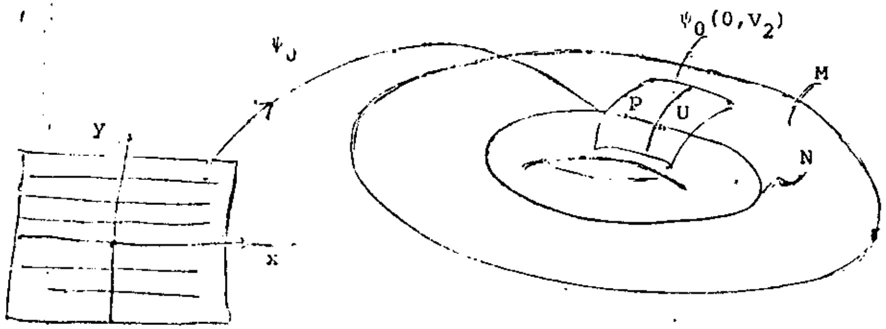
of Lemma 1.3 where U is a neighborhood of p and $V = V_1 \times V_2 \subset R^{2n}$ is an open set containing the origin. In these coordinates

$F_j \circ \psi_0 = y_j$ and therefore the flows $\psi_0^{-1} \circ \phi_j^{t_j} \circ \psi_0$ are given by $(x,y) \to (x + t_j \cdot e_j, y)$ and $\phi^t = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \dots \circ \phi_n^{t_n}$ by

$$\psi_0^{-1} \circ \phi^t \circ \psi_0(x,y) = (x+t,y) ,$$

if $|t|$ is small enough, i.e.

(1.11) $$\phi^t \circ \psi_0(x,y) = \psi_0(x+t,y) .$$



This relation allows us to extend the mapping ψ_0 to a mapping θ defined by

(1.12) $$\theta: (x,y) \to \phi^x \circ \psi_0(0,y) .$$

Geometrically (1.12) says that we apply the flow $\phi^t = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \dots \circ \phi_n^{t_n}$ to a transversal section $S = \psi_0(0,V_2)$ of the vector fields $X_{F_1},\dots,X_{F_n}$, where S is a n-dimensional Lagrange-manifold. Setting $x = 0$ in (1.11) we find $\theta(x,y) = \psi_0(x,y)$ for (x,y) near $(0,0)$.

Since $\phi^t(0,0)$ is defined for all $t \in R^n$ it follows from the standard theorems in differential equations that for any compact ball K about 0 in R^n there exists a neighborhood $D_2(K)$ of 0 in R^n such that $\phi^t(0,y)$ is defined for $t \in K$, $y \in D_2(K)$. Thus the map θ can be defined as a map taking $K \times D_2(K)$ into M. We will choose K so large that it contains the periods $\pm v_k$ in its interior.

Lemma 1.4. The mapping $\theta: K \times D_2(K) \to M$ defined by (1.12) satisfies

$$\theta^* \omega = \sum_{j=1}^{n} dy_j \wedge dx_j .$$

Proof: For (x,y) near $(0,0)$ we have $\theta(x,y) = \psi_0(x,y)$ and the statement follows from (1.10).

To prove the statement for all $x \in K$ we note that $\theta(x,y)$ satisfies

$$\theta(x,y) = \phi^s \circ \theta(x-s,y)$$

if $x, x-s \in K$, $y \in D_2(K)$. Hence if x is near the arbitrary point $s \in X$ we have

$$\theta^*\omega = (\phi^s \circ \theta_{-s})^*\omega = \theta_{-s}^*(\phi^s)^*\omega = \theta_{-s}^*\omega$$

where θ_{-s} designates the mapping $(x,y) \to \theta(x-s,y) = \psi_0(x-s,y)$. Hence we have

$$\theta^*\omega = \theta_{-s}^*\omega = \sum_{j=1}^{n} dy_j \wedge d(x_j - s_j) = \sum_{j=1}^{n} dy_j \wedge dx_j$$

proving our claim.

Next we determine the points (x,y) which are mapped into the same points in M under θ. For y near 0 we determine $w_k(y) \in R^n$ such that

$$\text{(1.13)} \qquad \theta(w_k(y),y) = \theta(0,y) \ , \quad w_k(0) = v_k \ ,$$

where v_k are the basis vectors of Γ for $y = 0$.

To solve the equation (1.13) we consider the mapping

$$\rho\colon (\xi,\eta) \to (x,y) = \psi_0^{-1} \circ \theta(v_k+\xi,\eta)$$

which is well defined for (ξ,η) near $(0,0)$ since $(0,0)$ is a fixed point of ρ. Moreover, since $F_j \circ \psi_0(x,y) = y_j$ is an integral we have

$$y_j = F_j \circ \psi_0(x,y) = F_j \circ \theta(v_k+\xi,\eta) = F_j \circ \phi^{v_k} \circ \psi_0(\xi,\eta)$$
$$= F_j \circ \psi_0(\xi,\eta) = \eta_j \ .$$

From Lemma 1.4 we conclude that

$$\theta^*\omega = \sum_{j=1}^{n} dy_j \wedge dx_j \ , \quad \theta_{v_k}^*\omega = \sum_{j=1}^{n} d\eta_j \wedge d\xi_j \ ,$$

where as above θ_v is defined by $\theta_v(\xi,\eta) = \theta(v+\xi,\eta)$. Hence for $\rho = \theta^{-1} \circ \theta_{v_k}$ we have

$$\rho^*\left(\sum_{j=1}^{n} dy_j \wedge dx_j\right) = \sum_{j=1}^{n} d\eta_j \wedge d\xi_j \ .$$

Hence ξ,η are a set of variables which like x,y satisfy the conditions of Lemma 1.3. By the remark following that lemma ρ is of the form

$$(\xi,\eta) \to \left[x = \xi + \frac{\partial Q_k}{\partial \eta}(y),\ y = \eta\right], \quad \frac{\partial Q_k}{\partial \eta}(0) = 0 \ .$$

We return to the equation (1.13). Setting $w_k(y) = v_k + \xi$ we have to solve the equations

$$\theta(\xi+v_k,y) = \theta(0,y)$$

if y is near 0. We have just shown that $\theta(\xi+v_k,y) = \psi_0 \circ \rho(\xi,y) = \psi_0\left(\xi + \frac{\partial Q_k}{\partial y}(y),y\right)$. On the other hand $\theta(0,y) = \psi_0(0,y)$ by definition. Therefore the above equations are equivalent to

$$\psi_0\left(\xi + \frac{\partial Q_k}{\partial y}(y),y\right) = \psi_0(0,y)$$

which have the unique solutions

$$\xi + \frac{\partial Q_k}{\partial y}(y) = 0 \ .$$

Hence $w_k = v_k - \frac{\partial Q_k}{\partial y}$, or

$$\text{(1.14)} \quad w_k = \frac{\partial W_k}{\partial y} \quad \text{with} \quad W_k(y) = \langle v_k,y\rangle - Q_k(y) \ .$$

This establishes the existence of $w_k = w_k(y)$ as well as the smooth dependence on y. The vectors $w_k(y)$, $k = 1,2,\dots n$ are linearly independent being close to $w_k(0) = v_k$. The relation

$$(1.15)\quad \theta(x+w_k(y),y) = \phi^{x+w_k}\circ\theta(0,y) = \phi^{x}\circ\theta(0,y) = \theta(x,y)$$

allows now the extension θ to $R^n\times D_2 \to M$. This extended map θ also satisfies the statement of Lemma 1.4 and preserves the lattice $\Gamma(y)$ spanned by the linear combination of the $w_k(y)$ with integer coefficients. Therefore it gives rise to a diffeomorphism

$$(1.16)\qquad \theta_0:\ R^n/\Gamma(y) \to F^{-1}(y)\cap U(N)\ .$$

One has to convince oneself that this mapping is injective if y is in some possibly smaller neighborhood of 0. If this were not the case there would exist a sequence of pairs of points (x_ν,y_ν), (x'_ν,y_ν) with

$$\theta(x_\nu,y_\nu) = \theta(x'_\nu,y_\nu)\ ,\qquad y_\nu \to 0\ ,$$

for which $x_\nu-x'_\nu \notin \Gamma(y_\nu)$. Taking a subsequence we can assume these sequences to be convergent and $x_\nu \to x^*$ and $x'_\nu \to x'^*$. From the last relation we find $\theta(x^*,0) = \theta(x'^*,0)$, i.e. $x^*- x'^*\in \Gamma = \Gamma(0)$, hence $x_\nu-x'_\nu$ is close to a point in $\Gamma(y_\nu)$.

Since $\Gamma(y_\nu)$ is discrete with the distance between points of $\Gamma(y_\nu)$ greater than a constant $\delta > 0$, uniformly in y_ν, it follows that $x_\nu-x'_\nu$ belongs to $\Gamma(y_\nu)$ for sufficiently large ν. This contradiction proves that (1.16) is injective.

Finally we normalize the periods w_k by the canonical map $\sigma:\ (\tilde{x},\tilde{y}) \to (x,y)$ given implicitly by

$$\tilde{y}_j = W_j(y)$$

$$x_j = \sum_{k=1}^{n} \tilde{x}_k \frac{\partial W_k(y)}{\partial y_j}\ ,$$

where W_k is given by (1.14). This mapping is canonical since it has the generating function

$$S(\tilde{x},y) = \sum_{k=1}^{n} \tilde{x}_k W_k(y)\ ,$$

with nonsingular $S_{\tilde{x}_k y_j}$ as the $w_k(y)$ are linearly independent. We denote by μ the mapping $y \to \tilde{y} = W(y)$. Then we see that σ maps the lattice points $(e_j,\tilde{y})$ into $(w_j,y) \in \Gamma(y)$ where $y = \mu^{-1}(\tilde{y})$. Therefore the mapping

$$\psi = \theta\circ\sigma:\ R^n \times D_1 \to U(N) \subset M$$

satisfies by (1.15) and by Lemma 1.4

$$\psi(\tilde{x}+e_j,\tilde{y}) = \psi(\tilde{x},\tilde{y})\ ,$$

$$\psi^*\omega = \sum_{j=1}^{n} d\tilde{y}_j \wedge d\tilde{x}_j\ ,$$

and induces the desired coordinate map

$$R^n/Z^n \times D_2 \to U(N) \subset M\ .$$

Moreover, $\mu \circ F \circ \psi = \tilde{y}$ and relabeling $\tilde{x}, \tilde{y}$ by x, y we have the statement of the theorem.

(d) Generalizations

If in Theorem 1 we drop the assumption that the submanifold $N = F^{-1}(0)$ is compact, and merely assume N to be connected, then N is not a torus anymore, but our proof shows that N is an embedded cylinder $T^{n-k} \times R^k$, for some $0 < k \leq n$. We have to require however that the flows of the vector fields X_{F_j}, $1 \leq j \leq n$, do exist for all times on N. For N compact this was automatically the case. The lattice Γ is then n-k dimensional and we have only n-k periods. If, in addition, we assume that the flows of X_{F_j} exist for all times on the symplectic manifold M, then Theorem 1 holds true with T^n replaced by $T^{n-k} \times R^k$. In this case we have n-k angle variables. We shall meet such a situation with k = n later on.

It can happen that a system has more than n independent integrals, where dim M = 2n; in fact the Kepler system is an example, as we have seen. We have to keep in mind that these integrals cannot be all in involution, since the dimension of an isotropic subspace of a symplectic vector space is at most n. In order to generalize Theorem 1 to such a situation we consider on a symplectic manifold (M,ω), dim M = 2n, n+k functions $(0 \leq k < n)$

$$F_1, \ldots, F_{n-k};\ F_{n-k+1}, \ldots, F_{n+k}$$

with the properties

(1.17) (i) $dF_1, dF_2, \ldots, dF_{n+k}$ are linearly independent
(ii) $\{F_i, F_j\} = 0$, $1 \leq i \leq n-k$ and $1 \leq j \leq n+k$,

i.e. the first n-k functions are in involution and commute with the remaining functions F_j, $n-k+1 \leq j \leq n+k$. This generalizes the case k = 0 considered so far.

If F denotes the map $F = (F_1, \ldots, F_{n+k}): M \to R^{n+k}$, it follows from (1.17) that the level sets

$$F^{-1}(\xi) \subset M \quad \text{for} \quad \xi \in R^{n+k}$$

are (n-k)-dimensional submanifolds of M which are isotropic.

Theorem 2 Let $F = (F_1, \ldots, F_{n+k})$ be a C^2-vector function on the symplectic manifold (M,ω) with dim M = 2n. Assume the functions F_j satisfy the properties (i) and (ii) of (1.17). Assume moreover that the (n-k)-dimensional submanifold $F^{-1}(0) = N \subset M$ is connected and compact. Then

(a) $F^{-1}(0)$ is an embedded (n-k)-dimensional torus T^{n-k};

(b) there is an open neighborhood $U(N) \subset M$ of N which can be described by "angle and action" variables in the following manner. There is a diffeomorphism

$$\psi: T^{n-k} \times D_1 \to U(N), \quad D_1 \subset R^{n+k},$$

$T^{n-k} \times D_1$ having the coordinates $x_1, \ldots, x_{n-k}$ (mod 1), $y_1, \ldots, y_{n-k}, p_1, \ldots, p_k, q_1, \ldots, q_k$, and there are two diffeomorphism $\mu_1: D_2 \to D_1$ with $\mu_1(0) = 0$ and $\mu_2: D_3 \to D_4$ with $\mu_2(0) = 0$, where D_1, D_2 are open neighborhoods of 0 in R^{n+k} and D_3 , D_4 are open neighborhoods of 0 in R^{n-k}, such that

(i) $\mu_1 \circ (F_1,\dots,F_{n+k}) \circ \psi = (y,p,q)$

$\mu_2 \circ (F_1,\dots,F_{n-k}) \circ \psi = y$

(ii) $\psi^*\omega = \omega_0$, with

$$\omega_0 = \sum_{j=1}^{n-k} dy_j \wedge dx_j + \sum_{j=1}^{k} dp_j \wedge dq_j .$$

The diffeomorphism ψ maps $T^{n-k}\times\{0\}$ onto N and $T^{n-k}\times\{\xi\}$ onto $F^{-1}(\eta) \cap U(N)$ where $\eta = \mu_1^{-1}(\xi)$.

As a consequence any Hamiltonian system X_H having the integrals F_j , $1 \le j \le n+k$ satisfying (1.17) is by the symplectic map ψ transformed into the system

$$H \circ \psi = h(y_1,\dots,y_{n-k}) \text{ on } T^{n-k} \times D_1 .$$

Hence on U(N) the Hamiltonian is a function of the integrals $F_1,\dots,F_{n-k}$ alone. On $T^{n-k}\times D_1$ the Hamiltonian equations become

$$\dot{x} = \frac{\partial}{\partial y} h(y) , \quad \dot{y} = 0 \quad \text{and} \quad \dot{p} = \dot{q} = 0 .$$

We omit the proof of Theorem 2 since it is similar to that of Theorem 1 (see Exercise 4).

Exercises: Ch. 3, Sec. 1.

1. Assume the functions F_j , $1 \le j \le n+k$, satisfy (1.17). Let $N = F^{-1}(0)$ be compact and connected. Prove that N is an embedded (n-k)-dimensional torus.
Hint: Proceed as in the proof of Theorem 1.

2. Let $f_1,\dots,f_s$, $s < n$ be functions on a symplectic manifold (M,ω) with dim M = 2n such that
(1) $df_1,\dots,df_s$ linear independent,
(2) $\{f_i,f_j\} = 0$, $1 \le i,j \le s$.
Prove that every point $p \in M$ has an open neighborhood $U \subset M$ and functions $f_{s+1},\dots,f_n$ defined on U such that the functions $f_1,f_2,\dots,f_n$ satisfy (1) and (2) on U with s = n .
Hint: By the Frobenius theorem there are local coordinates $x_1,\dots,x_{2n}$ such that $X_{f_j} = \frac{\partial}{\partial x_j}$, $1 \le j \le s$.

3. Assume $F_1,\dots,F_{n+k}$ satisfy (1.17). Prove that every point $p \in N = F^{-1}(0) \subset M$ has an open neighborhood $U \subset M$ and a diffeomorphism $\psi: V \to U$, where V is an open neighborhood of 0 in R^{2n} with coordinates $x_1,\dots x_{n-k}$, $y_1,\dots,y_{n-k}$, $q_1,\dots,q_k$, $p_1,\dots,p_k$, such that $\psi(0) = p$ and
(1) $\psi^*\omega = \sum_{j=1}^{n-k} dy_j \wedge dx_j + \sum_{j=1}^{k} dp_j \wedge dq_j$,
(2) $F_j \circ \psi = y_j$, $1 \le j \le n-k$,
(3) $F_j \circ \psi = f_j(y,p,q)$, $n-k+1 \le j \le n+k$, with $f_j(0) = 0$.

Show that $(y,p,q) = g(G_1,\ldots,G_{n+k}),\ G_j = F_j \circ \psi$.

Hint: Use Exercise 2 and Darboux's theorem as in the proof of Theorem 1.

4. Prove Theorem 2.

Hint: Proceed as in the proof of Theorem 1 using Exercises 1 and 3.

2. The Delaunay Variables

(a) Commuting integrals for the Kepler problem

We will introduce action-angle variables for the Kepler problem in R^n. Its Hamiltonian is given by

$$H = \frac{1}{2}\,|p|^2 - |q|^{-1} \qquad (2.1)$$

On account of its rotation symmetry it has the integrals

$$\Gamma_{ij} = q_i p_j - q_j p_i , \quad 1 \le i < j \le n ,$$

which, however, are not in involution. In fact one computes

$$\left\{\Gamma_{ij} , \Gamma_{\alpha\beta}\right\} = \Gamma_{i\alpha}\,\delta_{j\beta} - \Gamma_{j\alpha}\,\delta_{i\beta} - \Gamma_{i\beta}\,\delta_{j\alpha} + \Gamma_{j\beta}\,\delta_{i\alpha} \qquad (2.2)$$

(See I, §6, Exercise 2).

The question arises whether one can construct integrals in involution. Such integrals are indeed provided by the following

Lemma 2.1 The n-1 functions

$$G_k^2 = \sum_{1 \le i < j \le k} \Gamma_{ij}^2 = \sum_{1 \le i < j \le k} (q_i p_j - q_j p_i)^2 , \qquad (2.3)$$

$k = 2, 3, \ldots, n,$

are in involution.

Proof: From (2.2) we obtain

$$\left\{G_k^2 , \Gamma_{\alpha\beta}\right\} = \sum_{1\leq i<j\leq k} 2\,\Gamma_{ij} \left\{\Gamma_{ij} , \Gamma_{\alpha\beta}\right\}$$

$$= \sum_{i,j=1}^{k} \Gamma_{ij} (\Gamma_{i\alpha}\,\delta_{j\beta} - \Gamma_{j\alpha}\,\delta_{i\beta} - \Gamma_{i\beta}\,\delta_{j\alpha} + \Gamma_{j\beta}\,\delta_{i\alpha})$$

$$= 2 \sum_{i,j=1}^{k} \Gamma_{ij} (\Gamma_{i\alpha}\,\delta_{j\beta} - \Gamma_{i\beta}\,\delta_{j\alpha})$$

since $\Gamma_{ij} = -\Gamma_{ji}$. Therefore, for $\alpha,\beta \leq k$ we find

$$\left\{G_k^2 , \Gamma_{\alpha\beta}\right\} = 2 \sum_{i=1}^{k} (\Gamma_{i\beta}\;\Gamma_{i\alpha} - \Gamma_{i\alpha}\;\Gamma_{i\beta}) = 0$$

and hence for $\ell < k$

$$\{G_k^2 , G_\ell^2\} = 2 \sum_{1\leq\alpha<\beta\leq\ell} \{G_k^2 , \Gamma_{\alpha\beta}\}\,\Gamma_{\alpha\beta} = 0$$

which proves the lemma.

We record :

$$\left\{G_k^2 , \Gamma_{\alpha\beta}\right\} = \begin{cases} -2\sum_{i=1}^{k} \Gamma_{i\alpha}\,\Gamma_{i\beta} & \text{for } 1 \leq \alpha \leq k < \beta \\ +2\sum_{i=1}^{k} \Gamma_{i\alpha}\,\Gamma_{i\beta} & \text{for } 1 \leq \beta \leq k < \alpha \\ 0 \text{ otherwise.} & \end{cases}$$

Thus we have the n integrals in involution:

$$G_2^2 , G_3^2 , \ldots , G_n^2 , H .$$

Their gradients are linerly independent in an open set as we will show below.

(b) <u>The flows for</u> X_{G_j} ; <u>normalization of the periods</u>

We determine the commuting flows generated by the Hamiltonians (2.3) . Obviously it is enough to study the flow of

(2.4) $$F = G_n^2 = \sum_{1\leq i<j\leq n} (q_i p_j - q_j p_i)^2 = |q|^2\,|p|^2 - \langle q,p\rangle^2 ,$$

since those for G_k^2 are analogue in R^{2k}. The above Hamiltonian was discussed already in Chapter I §5 and we saw, that for $G_n^2 > 0$ all solutions are periodic of period

(2.5) $$\frac{2\pi}{2\sqrt{F}} = \frac{2\pi}{2G_n}$$

In general, if $F = F(q,p)$ is a Hamiltonian for which the solutions on the energy surface $\{q,p \mid F(q,p) = E\}$ are periodic and have the period $T = T(E)$ then the system with the Hamiltonian $\phi(F)$ has solutions of period $\frac{d\phi}{dE}^{-1}$ $T(E)$ on the energy surface $\phi(F) = \phi(E)$ if $\phi'(E) \neq 0$.

This follows simply from the fact that the system

$$\frac{dz}{dt} = J \frac{\partial F}{\partial z}$$

transforms under the change of the independent variable

$$t \to s = \phi'(F)^{-1} t$$

into the system

$$\frac{dz}{ds} = \phi'(F) J \frac{\partial F}{\partial z} = J \frac{\partial}{\partial z} (\phi(F))$$

with the Hamiltonian $\phi(F)$.

If we apply this remark to the Hamiltonian (2.4) with $\phi(E) = \sqrt{E}$ then according to (2.5) the period becomes

$$2\sqrt{F} \cdot \frac{2\pi}{2\sqrt{F}} = 2\pi \quad .$$

Thus the system with the Hamiltonian G_n has solutions of period 2π , provided $G_n > 0$.

We apply the same remark to the Hamiltonian (2.1) where we restrict ourselves to negative values of $H(q,p) = E < 0$. Then all solutions have the periods

$$2\pi(-2H)^{-3/2} \quad .$$

To normalize the period to 2π we take

$$\phi(H) = (-2H)^{-1/2}$$

and set

$$(2.6) \quad G_{n+1} = (-2H)^{-1/2} = (-|p|^2 + 2|q|^{-1})^{-1/2} \quad .$$

For definitiveness we take in (2.3) and (2.6) the square root defining the G_k nonnegative for $k \geq 3$, but set

$$G_2 = q_1 p_2 - q_2 p_1$$

which can take both signs.

(c) <u>The flows $\phi_k^{t_k}$ of X_{G_k}</u> .

We turn to the geometric interpretation of the flow

$$\phi_n^t = \exp(tX_{G_n}) .$$

For this purpose we interpret the points $(q,p) \in R^{2n}$ as pairs of vectors q, p in R^n . Every such pair of independent vectors spans a (two-dimensional) plane E in R^n:

$$E = \text{span} \{ q, p \} \subset R^n .$$

It turns out that the pair $q(t)$, $p(t)$ defined by $(q(t), p(t)) = \phi_n^t(q,p)$ stays in E for all t, i.e. E is fixed in R^n . Moreover, the pair $q(t)$, $p(t)$ is obtained from $q(0)$, $p(0)$ by a rotation in this fixed plane by an angle t.

To prove this fact, we apply a canonical transformation $(q,p) \to (Rq, Rp)$ where R is a rotation and bring E into the position

$$RE = \text{span}\{e_1 , e_2 \}.$$

Clearly the vector field X_{G_n} is tangent to the subspace $RE \times RE$ in R^{2n} and on this subspace

(2.7) $$G_n = \pm (q_1p_2 - q_2p_1) ,$$

which, up to the sign, defines the differential equations

$$\dot{q}_1 = -q_2 \qquad \dot{p}_1 = -p_2$$
$$\dot{q}_2 = q_1 \qquad \dot{p}_2 = p_1 .$$

From this one reads off the claim.

Similarly, the flows

(2.8) $$\phi_k^{t_k} = \exp(t_k X_{G_k}) \qquad k = 2,3,\dots, n$$

correspond to rotations about the angle t_k in the plane (or line or point)

$$E_k = \pi_k E$$

obtained from E by the projection $\pi_k : R^n \to R^k$,

$$\pi_k q = \sum_{j=1}^{k} q_j e_j \qquad \text{if } q = \sum_{j=1}^{n} q_j e_j .$$

Finally the flow (2.8) for $k = n+1$ moves points in E along the elliptical orbit at a speed proportional to the actual time but so normalized that one revolution cor-

responds to an increase of t_{n+1} by 2π . In astronomy this parameter is called the mean anomaly.

(d) Linear independence of the dG_j .

Lemma 2.2 The functions $G_2, G_3, \dots G_{n+1}$ defined by (2.3) and (2.6) satisfy the n inequalities

$$0 \le G_2^2 \le G_3^2 \le \dots \le G_{n+1}^2 ,$$

The dG_j^2 $(j = 2, \dots, n+1)$ are linearly independent at precisely those points where

(2.9) $$0 < G_2^2 < G_3^2 < \dots < G_{n+1}^2 \quad \text{holds.}$$

The first n-1 inequalities are evident, with our convention of taking non negative roots. The last one is equivalent to the inequality

(2.10) $$-2HG_n^2 \le 1 .$$

This is obvious if $G_n = 0$, and if $G_n > 0$ we have from (2.4)

$$|p|\,|q| \ge G_n > 0$$

or

$$|q|^{-1} \le \frac{|p|}{G_n}$$

with equality if and only if $\langle p,q \rangle = 0$. Hence

$$-2H = -|p|^2 + 2|q|^{-1} \le -|p|^2 + 2G_n^{-1}|p|$$
$$= -(|p| - G_n^{-1})^2 + G_n^{-2}$$
$$\le G_n^{-2}$$

which proves (2.10). We have equality only if

(2.11) $G_n > 0$, $\langle p,q \rangle = 0$, $|p| = G_n^{-1}$.

This situation corresponds to circular orbits of the Kepler problem.

If

$$G_k^2 - G_{k-1}^2 = \sum_{j=1}^{k-1} (q_j p_k - q_k p_j)^2 = 0, \quad k \le n ,$$

or if G_2 vanishes then obviously $dG_k^2 = dG_{k-1}^2$ or dG_2^2 vanishes making the dG_j^2 linearly dependent. Similarly $G_{n+1}^2 - G_n^2 = 0$ implies $dG_{n+1}^2 - dG_n^2 = 0$, as one verifies from (2.11).

The proof of the converse, namely that the $dG_j^2 = 2\, G_j\, dG_j$, hence the dG_j are linearly independent in the case (2.9) will follow from the consideration below.

Lemma 2.3 For any $k = 3, 4, \ldots, n$ the conditions

$$0 < G_{k-1} < G_k$$

are equivalent to

$$\dim E_{k-1} = 2 , \quad E_k \not\subset \pi_{k-1} R^n = R^{k-1} \quad ;$$

in other words $E_k = \pi_k E$ is transversal to $R^{k-1} = \pi_{k-1} R^n$ in $\pi_k R^n$.

The proof can easily be derived from the explicit expressions for G_{k-1}^2 and $G_k^2 - G_{k-1}^2$. Consequently the first n-1 conditions in (2.9) imply that $E_k = \pi_k E$ are planes for $k = 2,3, \ldots, n$ and that E_k intersects R^{k-1} transversally in a line. The last condition in (2.9) amounts to a violation of (2.11) , i.e. it means the Kepler orbit determined by q,p is not a circle.

(e) Construction of the Section S and the Delaunay Variables.

Now we apply the same construction as in Section 1 for the action-angle variables using as transversal section S the Lagrange manifold

(2.12)

$$S = \{q,p \mid p_1 = 0,\ q_j = 0, p_j > 0 \text{ for } j \ge 2\ ;\ |q_1|\ |p|^2 > 1\ \}.$$

To motivate this choice, and at the same time give a geometrical interpretation of S we proceed as follows: For a point (q,p) in the domain

$$D = \{\ q,p \mid \ 0 < G_2^2 < G_3^2 < \ldots < G_{n+1}^2\ \}$$

we choose t_{n+1} so that

$$\phi_{n+1}^{-t_{n+1}} (q,p) = (q^*, p^*)$$

lies at the perihelium of the ellipse, which is characterized by

(2.13) $\langle q^*, p^* \rangle = 0$, $|p^*| > G_n^{-1}$

(See (2.11)). The point (q^*, p^*) lies in the plane E_n which by $\phi_n^{-t_n}$ is rotated into the line of intersection with R^{n-1}, say given by $q_n = 0$, $p_n > 0$. Proceeding in this way we achieve that

$$\phi_2^{-t_2} \circ \dots \circ \phi_{n+1}^{-t_{n+1}} (q,p) = (\tilde{q}, \tilde{p})$$

satisfies $\tilde{q}_j = 0$, $\tilde{p}_j > 0$ for $j \geq 2$ Hence by (2.13) (which is rotation invariant) we get $q_1 \tilde{p}_1 = 0$, hence $\tilde{p}_1 = 0$ and

$$|\tilde{p}|^2 G_n^2 = |\tilde{p}|^2 |\tilde{p}|^2 \tilde{q}_1^2 > 1 \quad ;$$

hence $(\tilde{q}, \tilde{p}) \in S$.

Thus S describes the states where (q,p) lies at the perihelium of the ellipse, the orbit plane is E_2 and q lies on the q_1 - axis.

Lemma 2.4: If $y \in R^n$ is restricted to the set

$$\Omega = \{ y \in R^n , 0 < |y_1| < y_2 \dots < y_n \}$$

then the equations.

$$G_{k+1}(q,p) = y_k \qquad (k = 1, 2, \dots, n)$$

have a unique solution $(q,p) = \lambda(y)$ in S. Thus mapping $y \to \lambda(y)$ defines a diffeomorphism $\lambda : \Omega \to S \cap D$ with D being the domain of the theorem in section 1.

Proof: For $(q,p) \in S$ the equations in question become

$$(2.14) \quad \begin{cases} q_1 p_2 = y_1 \\ |q_1| \, |\pi_{k+1} p| = y_k & \text{for } k = 2, 3, \dots, n-1 . \\ - |p|^2 + 2|q_1|^{-1} = y_n^{-2} \end{cases}$$

The middle equation for $k = n-1$ gives the relation $|q_1| \, |p| = y_{n-1}$ with which we eliminate $|q_1|$ from the last equation

$$- |p|^2 + 2|p| \, y_{n-1}^{-1} = y_n^{-2}$$

or

$$(|p| - y_{n-1}^{-1})^2 = y_{n-1}^{-2} - y_n^{-2} > 0 .$$

This defines two positive values for $|p|$, but only one with $|p| \, y_{n-1} > 1$ which is equivalent to the requirement $|p|^2 |q_1| > 1$ in S. This defines $|p|$ hence $|q_1|$ and therefore p_{k+1} as positive root of

$$p_{k+1}^2 = |q_1|^{-2} (y_k^2 - y_{k-1}^2) \quad \text{for } k = 2,3,\dots, n-1 .$$

Finally we choose the sign of q_1 so as to satisfy the first equation of (2.14). The rest of the proof is clear.

Now we can verify that the dG_j are linearly independent at points in D. It suffices to check this at points $S \cap D$. We could use q_1 , p_j , $j \geq 2$ as variables, but it is more convenient to use $\alpha_j = q_1 p_{j+1}$, $j = 1, \ldots, n-1$ and $\alpha_n = q_1$, since $q_1 \neq 0$ in D. In these variables the Jacobian

$$\left(\frac{\partial G_k}{\partial \alpha_j} \right)$$

is a triangular matrix which evidently does not vanish since the $\alpha_j = q_1 p_{j+1} \neq 0$.

Now we define the Delaunay variables x_j , y_j $(j = 1,2, \ldots, n)$ like in §1 by the mapping

$$\psi : (x,y) \to \phi_2^{x_1} \circ \phi_3^{x_2} \circ \ldots \circ \phi_{n+1}^{x_n} (\lambda(y))$$

which by construction maps the following domain

$$\psi : T^n \times \Omega \to D \quad \text{where } T^n = R^n / (2\pi Z)^n$$

as a canonical diffeomorphism. Moreover,

$$G_{k+1} \circ \psi(x,y) = y_k \qquad 1 \leq k \leq n-1$$

$$H \circ \psi(x,y) = -2y_n^{-2}$$

(f) Interpretation and Poincaré Variables.

These variables are of practical interest only in the case n=3 . Usually they are introduced by solving the Hamilton-Jacobi equations for the Kepler problem using separation

of variables in polar coordinates. This well known procedure is possibly shorter but the interpretation of the variables is not at all transparent.

In case $n = 3$ we see that

$$G_2 = q_1 p_2 - q_2 p_1$$

is the angular momentum about the q_3 - axis and

$$G_3 = |q \wedge p|$$

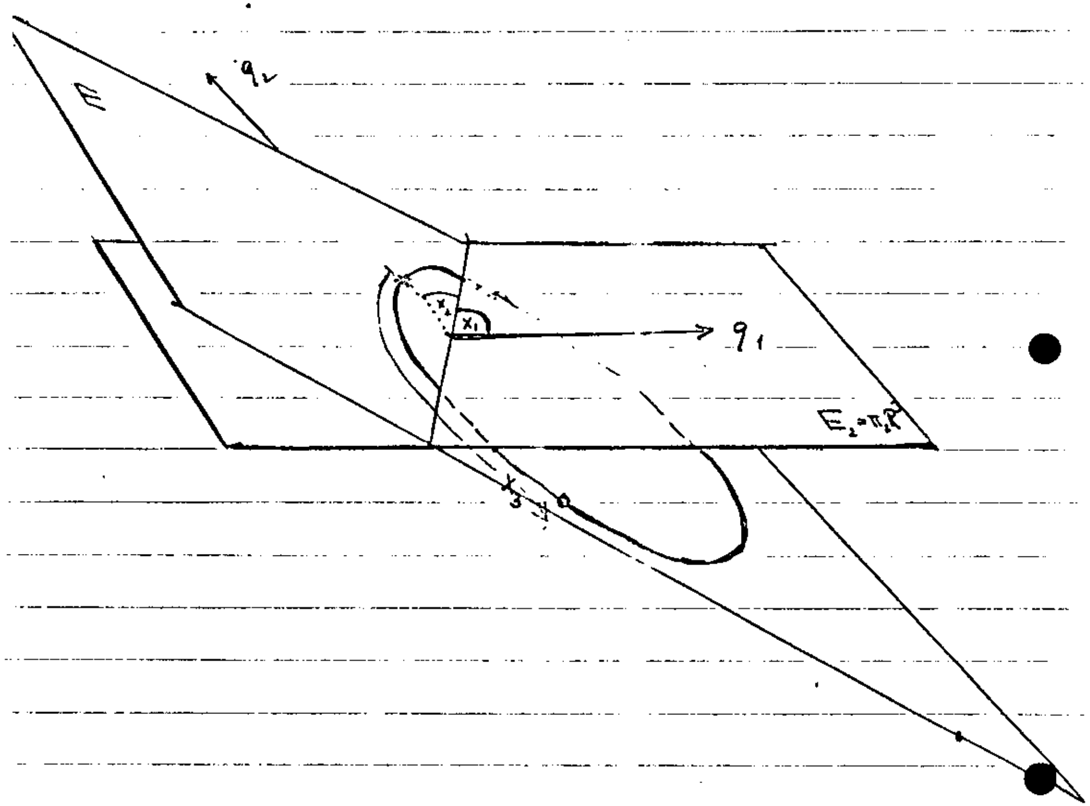
is the length of the total angular momentum, so that

$$G_2 = G_3 \cos j$$

where j is the inclination of the orbit plane E against $R^2 = \pi_2 R^3$. From the formulae of Chapter I we see that

$$G_4 = \sqrt{a} \quad , \quad G_3 = G_4 \sqrt{1 - \varepsilon^2} = \sqrt{a(1-\varepsilon^2)}$$

where a is the semi-major axis of the ellipse, and ε its eccentricity. Thus the conditions (2.9) exclude collision orbits, circular orbits and those with inclination zero. For those orbits one needs different coordinate patches. (see below).



(g) The Floquet multipliers for the periodic solutions of the first kind.

In Chapter II, §4 we studied the three body problem in rotating coordinates. The equations (4.7) (Chap. II) become for $\mu = 0$

$$\ddot{w} + 2i\alpha\dot{w} - \alpha^2 w^2 = - \frac{w}{|w|^3}$$

which have among their solutions the circular orbits

$$w = r\, e^{i\beta t} \quad , \text{ with } (\beta + \alpha)^2 \; r^3 = 1 \; .$$

We want to use the Delaunay variables x_1 , x_2, y_1 , y_2 with $n = 2$ to compute the Floquet multipliers of these orbits.

For this purpose we use that the Kepler problem is given by the Hamiltonian

$$- \frac{1}{2y_2^2}$$

and the rotation by the Hamiltonian $G = y_2$. Hence the Kepler problem in the rotating coordinates is given by

$$H = - \frac{1}{2y_2^2} + \alpha\, y_1 \quad ,$$

and the above circular orbits by

$$y_1 = y_2 = \sqrt{r} \quad ; \quad x_1 = y_2^{-3}\, t + x_1(0) \; ; \; x_2 = \alpha t + x_2(0) \; .$$

The angular variables x_1 , x_2 , x_3 are the angles of the node from the q_1-axis , the angle of the perihelion from the node and the mean anomaly (see figure). The traditional notation in astronomy for these variables is $(L, G, \Theta, \ell, g, h)$ for $(y_3, y_2, y_1, x_3, x_2, x_1)$ in the indicated order. Incidentally, the domain D has two components distinguished by the sign of G_2.

The near circular orbits of small inclination are described by "Poincaré variables" which are constructed as follows: The canonical map $(x,y) \to (\xi, \eta)$ with

$$\xi_k = \sum_{j=1}^{k} x_j \qquad \eta_k = y_{k+1} - y_k , \quad k \leq n-1$$

$$\xi_n = \sum_{j=1}^{n} x_j \qquad \eta_n = y_n$$

takes the domain D into that where $\eta_k > 0$. Now we introduce symplectic polar coordinates by

$$u_k = \sqrt{2\eta_k} \cos \xi_k \quad (k = 1, 2, \ldots, n-1)$$

$$v_k = \sqrt{2\eta_k} \sin \xi_k$$

so that the orbits of interest have $u_k^2 + v_k^2$ small for $k \leq n-1$.

Near the circular orbits the Delaunay variables are to be replaced by the Poincaré variables u_1, v_1, u_2, v_2 where

$$\begin{cases} u_1 = \sqrt{2(y_2 - y_1)} \cos x_1 \\ v_1 = \sqrt{2(y_2 - y_1)} \sin x_1 \\ u_2 = x_1 + x_2 \\ v_2 = y_2 \end{cases}$$

so that

$$y_2 - y_1 = \frac{1}{2} (u_1^2 + v_1^2)$$

and

$$H = - \frac{1}{2v_2^2} + \alpha \left(v_2 - \frac{1}{2} (u_1^2 + v_1^2) \right) .$$

The circular orbit corresponds to

(2.15) $u_1 = v_1 = 0 ; \quad v_2 = \sqrt{r} \quad ; \quad u_2 = (\alpha + v_2^{-3}) t + u_2(0)$

with period $T = 2\pi (\alpha + r^{-3/2})^{-1} = 2\pi/\beta$.

Now it is easy to determine the Floquet multipliers. The full differential equations are

$$\begin{cases} \dot{u}_1 = -\alpha v_1 \\ \dot{v}_1 = \alpha u_1 \\ \dot{u}_2 = + v_2^{-3} \\ \dot{v}_2 = 0 \end{cases}$$

After we linearize these equations on (2.15) we obtain the Floquet multipliers 1, 1, $e^{\pm i\alpha T}$, which is evident from the first two equations. Since $T = 2\pi/\beta$ this agrees with the statement below (4.12) in Chap. II. This provides a simpler computation than the one suggested in excercise 2 of Chap. II, §4.

§3 Integrals via Asymptotics; the Störmer Problem

(a) Integrals of the Störmer Problem

If the solutions of a Hamiltonian system all escape to infinity it is usually very easy to establish the existence of integrals in involution. We illustrate this observation with the example of a charged particle in a dipole field, which is described by the differential equations

(3.1) $$\frac{d^2}{dt^2} q = \frac{dq}{dt} \wedge B(q) , \quad q \in R^3 ,$$

where

(3.2) $$B(q) = \nabla (q_3 r^{-3}) = - \frac{\partial}{\partial q_3} \left(\frac{q}{r^3}\right) = \frac{1}{r^5} (-3q\, q_3 + e_3 r^2)$$

with $r = |q|$.

This system has the Hamiltonian

(3.3) $$H = \frac{1}{2}\left(\left(p_1 - \frac{q_2}{r^3}\right)^2 + \left(p_2 + \frac{q_1}{r^3}\right)^2 + p_3^2 \right)$$

as one integral (section 2 in Chapter I) and as a second integral the angular momentum:

$$G = q_1 p_2 - q_2 p_1 = q_1 \dot{q}_2 - q_2 \dot{q}_1 - \frac{q_1^2 + q_2^2}{r^3} ,$$

where the first term on the right hand side is the angular momentum ordinarily encountered, while the second term is the contribution due to the dipole. Is it possible to find a third integral, or more precisely, are there 3 integrals in involution which would make this system an integrable one?

We will show that in the domain given by

(3.4) $$H > 0 \quad , G > 0$$

there exist indeed 3 integrals in involution. On the other hand in some regions where $G < 0$ one has evidence that no such three integrals exist. This state of affairs which seems paradox at first has a very simple explanation. We will show that in the domain (3.4) all solutions escape and have the asymptotic behavior

$$q(t) \sim at + b + 0(t^{-1}) \quad \text{as } t \to \infty \ .$$

The vectors $a, b \in R^3$ can be viewed as functions of the initial values $q(0)$, $\dot{q}(0)$, and, moreover, a_1 , a_2 , a_3 will turn out to be 3 integrals in involution.

On the other hand there are other open regions in the phase space in which particles are trapped and in which these integrals are not defined.

One can view the situation also in another way: Locally, in a sufficiently small domain Ω of a 2n-dimensional space one can always find n integrals in involution, say $G_j(z)$. Using then the fact that

$$G_j(\phi^t(z)) = G_j(z) \ ,$$

one can extend their domain of definition to the region of accessibility $\cup_t \phi^t(\Omega)$. But this may give rise to multiple valuedness of the extended functions if the orbits through Ω return to Ω. If, however, they escape to ∞ without recurrence then such an extension is indeed possible. Thus the

difficulties of finding integrals in the large are closely tied to the recurrence of the orbits. The point of this section is to show that this difficulty disappears in case the orbits escape. This example is presented for its instructional value; otherwise it has little importance. The differential equations (3.1), however, played an important role in the study of the motion of charged particles in the magnetic field of the earth. Early numerical studies were made by Störmer and therefore it is often referred to as the Störmer problem.

(b) Escape of the Solutions

Next we show that any solution with $G > 0$, $H > 0$ satisfies

(3.5) $$q(t) = at + b + 0(t^{-1}) \quad \text{for } t \to \infty \ .$$

To prove this remark we compute

(3.6) $$\frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 = \langle q,\ddot{q}\rangle + |\dot{q}|^2 = \langle q,\dot{q}\wedge B\rangle + |\dot{q}|^2 \ .$$

By (3.2) we have

$$\langle q,\dot{q}\wedge B\rangle = -\langle \dot{q}, q\wedge B\rangle = \frac{-1}{r^3}\langle \dot{q},q\wedge e_3\rangle = \frac{q_1\dot{q}_2 - q_2\dot{q}_1}{r^3}$$

$$= \frac{1}{r^3}\left(G + \frac{q_1^2+q_2^2}{r^3}\right) > 0$$

since

$$\dot{q}_1 = p_1 - \frac{q_2}{r^3} \ , \ \dot{q}_2 = p_2 + \frac{q_1}{r^3} \ , \ \dot{q}_3 = p_3$$

hence

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 > |\dot{q}|^2 \quad 2H$$

and $H - \frac{1}{2}|\dot{q}|^2$. Therefore the velocity is a constant and without loss of generality we may take $|\dot{q}| = 1$ or $H = \frac{1}{2}$. Then

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 \geq 1$$

and by integration

$$|q|^2 \geq t^2 - c_1|t| - c_2 \quad \text{for all } t.$$

With this information we go into the differential equation (3.1). Using $B(q) = 0(|q|^{-3}) = 0(t^{-3})$ we obtain $\ddot{q} = 0(t^{-3})$ which yields (3.5) by integration. We also obtain

$$(3.7) \quad \dot{q}(t) = a + 0(t^{-2}) \;;\; p(t) = a + 0(t^{-2}).$$

We remark that the solution does not pass through $q = 0$, since

$$|q| \geq \frac{G}{\sqrt{2H}} > 0$$

in the region $H > 0$ and $G > 0$. This estimate is a simple consequence from the formula (2.10) below and the following discussion under (c).

We reformulate this result: We combine (q,p) to a vector $z \in R^6$ and denote the flow by $z \to \phi^t(z)$. Similarly, let

$$\phi_0^t(z) = \begin{pmatrix} q + tp \\ p \end{pmatrix}$$

denote the "free flow" corresponding to $B = 0$. Then (3.5) and (3.7) can be expressed by saying that the limit

$$(3.8) \quad \phi_0^{-t} \circ \phi^t(z) \to \psi(z) \qquad \text{for } t \to +\infty$$

exists and maps $z = (q,p)$ into the vector (b,a).

We will use - but not prove here - that in (3.8) also the derivatives of $\phi_0^{-t} \circ \phi^t$ converge to the corresponding derivatives of ψ uniformly for z in a compact domain. Since ϕ_0^{-t} and ϕ^t both are canonical maps, it therefore follows that also ψ is a canonical map. This map ψ assigns the "scattering data at $t = +\infty$" $(b,a) = \psi(q,p)$ to the initial values (q,p) at $t = 0$ of a solution. This way the components a_j , b_j of a,b become differentiable functions of q,p , moreover

$$\{a_j , a_k\} = 0 \quad , \{a_j , b_k\} = -\delta_{jk} , \{b_j , b_k\} = 0 ,$$

since ψ is canonical. Finally we show that

$$(3.9) \qquad \psi \circ \phi^s = \phi_0^s \circ \psi \quad ,$$

i.e. ψ maps the given flows ϕ^t into the free flow ϕ_0^t . To prove this we replace t by $t + s$ in (3.8) so that

$$\begin{aligned}\psi(z) &= \lim_{t \to +\infty} \phi_0^{-(t+s)} \circ \phi^{t+s}(z) \\ &= \lim_{t \to +\infty} \phi_0^{-s} \circ (\phi_0^{-t} \circ \phi^t)(\phi^s(z)) \\ &= \phi_0^{-s} \circ \psi \circ \phi^s(z)\end{aligned}$$

proving our claim.

Since the free flow is described by the Hamiltonian

$$H_0 = \frac{1}{2} |\xi|^2$$

we conclude from (3.9) that

$$H_0 \circ \psi = H$$

on the domain $H > 0$ and $G > 0$. Furthermore, since a_j are integrals in involution of X_{H_0} we have in

$$F_j(q,p) = a_j \circ \psi\,(q,p) \quad , \quad j = 1, 2, 3 ,$$

three integrals in involution of the given system. It is also clear that dF_j are linearly independent, and our claim is proven, namely that (3.1) is integrable in $G > 0$, $H > 0$.

(c) Allowed region for the Störmer problem

We give a rough description of the cases when $G < 0$. For this purpose it is useful to introduce cylinder-coordinates

$$q_1 = \rho \cos\theta \quad , \quad q_2 = \rho \sin\theta \quad , \quad q_3 = z$$

and extend it to a canonical transformation with the generating function

$$W = \rho(p_1 \cos\theta + p_2 \sin\theta) + p_3 z \quad .$$

The equation

$$p_\theta = \frac{\partial W}{\partial \theta} \quad , \quad p_\rho = \frac{\partial W}{\partial \rho} \quad , \quad p_z = \frac{\partial W}{\partial z}$$

can be solved:

$$\begin{cases} p_1 = -\dfrac{p_\theta}{\rho} \sin\theta + p_\rho \cos\theta \\ p_2 = \dfrac{p_\theta}{\rho} \cos\theta + p_\rho \sin\theta \\ p_3 = p_z \end{cases} ,$$

and one finds

$$\begin{cases} H = \frac{1}{2}\left(p_\rho^2 + p_z^2 + \left(\dfrac{p_\theta}{\rho} + \dfrac{\rho}{r^3}\right)^2 \right) \\ G = p_\theta \end{cases} .$$

Hence θ does not occur explicitly in H, i.e. θ is an ignorable variable, which simply expresses that $p_\theta = G$ is an integral. If we assign G a constant value

$$G = 2\gamma$$

the Hamiltonian system can be written as

$$\ddot\rho = -\frac{1}{2}\frac{\partial V}{\partial \rho} , \quad \ddot z = -\frac{1}{2}\frac{\partial V}{\partial z} \quad , \quad V = \left(\frac{2\gamma}{\rho} + \frac{\rho}{r^3}\right)^2$$

with Hamiltonian

$$H = \frac{1}{2}\left(\dot\rho^2 + \dot z^2 + V(\rho, z)\right) \quad . \tag{3.10}$$

As above we restrict ourselves to the domain

$$D(\gamma) = \{q,p \mid G = 2\gamma , H = \tfrac{1}{2}\} .$$

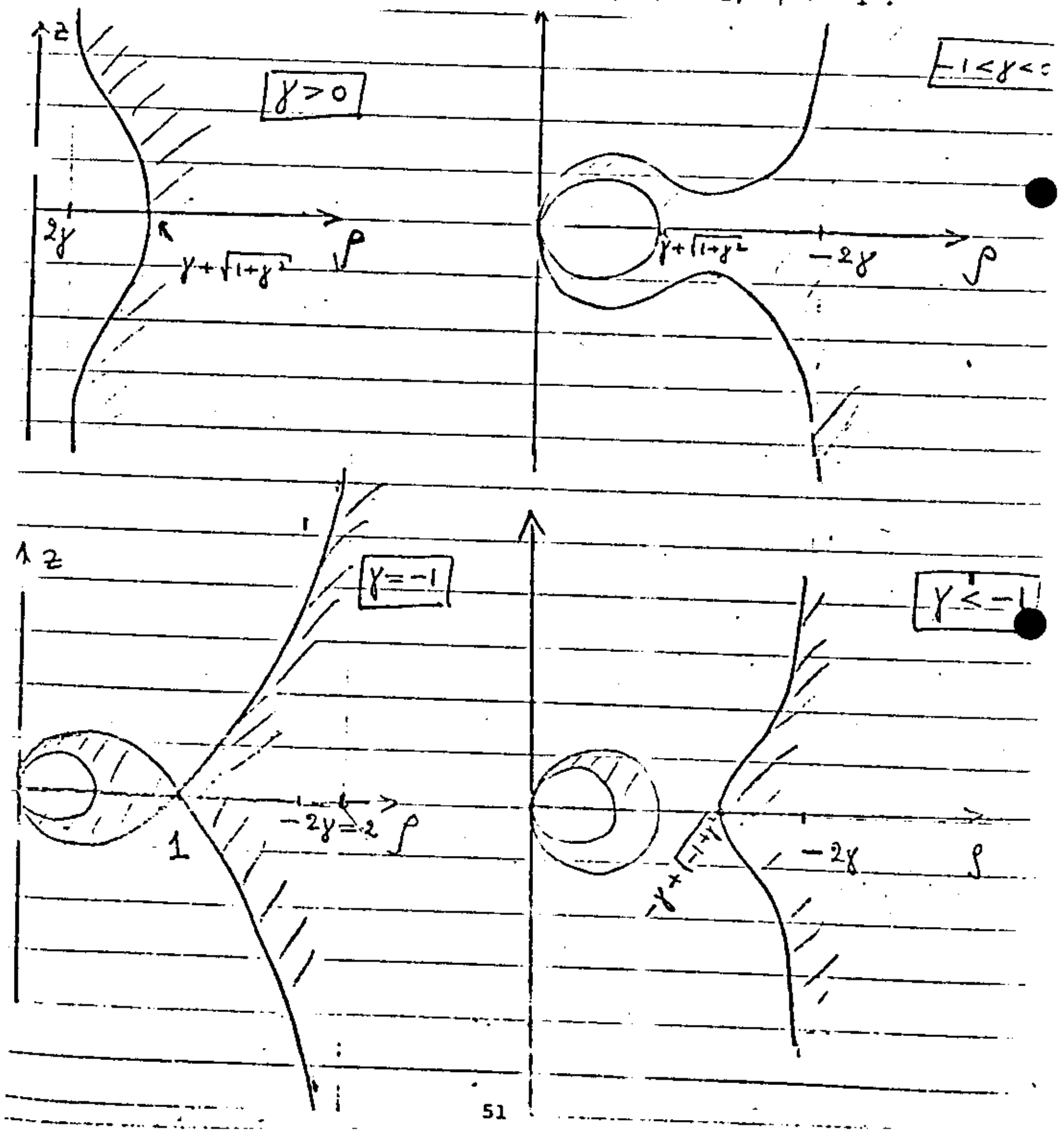
We project this domain into the configuration space and obtain by (3.10)

$$\Omega(\gamma) = \{\rho, z \mid \rho \geq 0 , V \leq 1\} ;$$

on the boundary one has $\dot\rho^2 + \dot z^2 = 0$. For this reason the boundary curves of $\Omega(\gamma)$ are called zero-velocity

$$\frac{2\gamma}{\rho} + \frac{\rho}{r^3} = \pm 1 .$$

Clearly for $\gamma \geq 0$ only the plus sign can occur. But for $\gamma < 0$ both signs occur and $\Omega(\gamma)$ has two components if $\gamma < -1$. Below the allowable regions $\Omega(\gamma)$ are sketched in four cases $\gamma > 0$, $-1 < \gamma < 0$, $\gamma = -1$, $\gamma < -1$.



Incidentally, for $\gamma = -1$ the point of intersection $(\rho, z) = (1,0)$ corresponds to a circular periodic orbit for the original problem (3.1).

(d) The case $\gamma < -1$

The phase space is restricted by prescribing the values of the integrals H, G. In particular, for $\gamma < -1$ the domain $D(\gamma)$ has two components:

$$D(\gamma) = D_1(\gamma) \cup D_2(\gamma) ,$$

where $D_1(\gamma)$ is the bounded region and $D_2(\gamma)$ the unbounded region. If we abandon the normalization of $H = \frac{1}{2}$ the condition $\gamma < -1$ has to be replaced by $\gamma < -\sqrt{2H}$ or $G < -\sqrt{H/2}$. We conclude that the domain

$$\{ q, p \mid G < -\sqrt{H/2} \} = I \cup II$$

has two components I, II: I bounded, II unbounded.

In conclusion we show that our system is integrable in the unbounded component II while one has evidence for nonexistence of such integrals in I - due to the presence of homoclinic orbits.

In the domain II all solutions escape again and the previous argument is applicable. Indeed, going back to the normalization $H = \frac{1}{2}$ we have in II

(3.11) $$p = \sqrt{q_1^2 + q_2^2} \geq -\gamma + \sqrt{-1 + \gamma^2} \geq 1 + \delta , \quad \delta > 0$$

and

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 = \langle q, \dot{q} \wedge B \rangle + |\dot{q}|^2$$

$$= \frac{1}{r^3}\left(2\gamma + \frac{\rho^2}{r^3}\right) + 1$$

$$= \frac{\rho}{r^3}\left(\frac{2\gamma}{\rho} + \frac{\rho}{r^3}\right) + 1$$

$$\geq -\frac{\rho}{r^3} + 1 \geq -\frac{1}{\rho^2} + 1 \qquad ,$$

since

$$\frac{2\gamma}{\rho} + \frac{\rho}{r^3} \geq -1$$

in II. Hence, by (3.11)

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 \geq 1 - \frac{1}{(1+\delta)^2} > 0 ,$$

which shows again that $|q| \to \infty$ and $|q|^{-1} = O(t^{-1})$. The rest of the argument is the same.

Exercise 1:

Show: Given two vectors $a, b \in R^3$ with $a \neq 0$ there exists a unique solution $q(t)$ of (3.1) with

$$q(t) = at + b + O(t^{-1}) \text{ as } t \to +\infty .$$

Hint: Convert (3.1) into the integral equation

$$q(t) = at + b - \int_t^\infty (t - \tau)\, f(\dot{q}(\tau), q(\tau))\, d\tau ,$$

$$f(\dot{q}, q) = \dot{q} \wedge B(q)$$

and apply the contraction argument with the norm

$$\sup_{t \geq T} t\,(|\hat{q}(t)| + |\dot{\hat{q}}(t)|)$$

for some large T where $q(t) = at + b + \hat{q}(t)$, $\dot{q}(t) = a + \dot{\hat{q}}(t)$.

Exercise 2:

Show that for the solution $q(t,a,b)$ constructed above

$$\dot{q}(t,a,b) - a \quad , \quad D(\dot{q}(t,a,b) - a) \quad , \quad D(q(t,a,b) - at - b)$$

tend to zero as $t \to \infty$ uniformly on compact domains not containing $a = 0$; here D stands for any first derivation with respect to a_j, b_j.

Hint: You can go into the complex and proceed as in Exercise 1 and use Cauchy estimates to control the derivatives.

Exercise 3:

Use Exercise 2 to establish that ψ defined by (3.8) is a diffeomorphism of the region $(q,p) \in R$ defined by $H > 0$ and $G > 0$ onto the region $(b,\omega) \in R^6$ with $a \neq 0$.

Exercise 4:

Show that the differential equation for the angular motion is governed by

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial p_\theta} = -\left(\frac{2\gamma}{\rho} + \frac{\rho}{r^3}\right)\frac{1}{\rho} .$$

Exercise 5:

Let H_1 and H_2 be the Hamiltonians of two vector fields with flows ϕ_1^t and ϕ_2^t respectively. Prove that

$$\psi_t(z) = \phi_1^{-t} \circ \phi_2^{t}(z)$$

is the flow of the timedependent Hamiltonian $H(t,z) = (H_2 - H_1) \circ \phi_1^t(z)$.

4. The Toda Lattice

(a) Particle Systems on the Line

We turn to the discussion of a particular integrable Hamiltonian system where the integrals have algebraic character, One has to keep in mind that such systems are very exceptional.

The examples of this section describe the motion of mass points of mass $m_j > 0$ moving on the line, which we take to be the x-axis. If q_j denotes the position of a mass point, the differential equation is given by

$$m_j \ddot{q}_j = -U_{q_j}(q) , \quad j = 1,2,\ldots,n,$$

where

$$U = \sum_{1 \le i < j \le n} V_{ij}(q_i - q_j) .$$

Here V_j denotes the potential function for the pair q_i, q_j which would be $c|q_i - q_j|^{-1}$ for Newton's law. Without specifying the V_{ij} or m_j we consider this as an n-particle system on the line.

Sometimes one is interested in systems where not all but only the "neighbors" q_{j-1}, q_{j+1} of q_j exert a force on q_j. In this case the potential would be of the form

$$U = \sum_{j=1}^{n-1} W_j(q_j - q_{j+1})$$

and the differential equations

$$m_j \ddot{q}_j = W'_{j-1}(q_{j-1} - q_j) - W'_j(q_j - q_{j+1})$$

for j = 2,3,...n-1. For j = 1,n one has to replace the undefined term by 0.

As a rule one assumes that $V_{ij}(-x) = V_{ij}(x)$ which expresses that the force exerted by the i-th particle on the j-th is, except for the sign, the same as the force exerted by the j^{th} particle on the i^{th}. The symmetry condition $W_j(-x) = W_j(x)$ has a similar interpretation. However, some of the examples below violate this basic requirement.

In statistical mechanics one considers such particle systems as models where, however, the number of particles is infinite. In that case one speaks of lattices. In this way Toda * studied the following example which we describe* for finitely many particles.

Toda Lattice

If we take $m_j = 1$, $j = 1\ 2,\ldots,n$, and $W_j(x) = e^{-x}$ we get a Hamiltonian system with

(4.1) $$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{q_j-q_{j+1}} .$$

The differential equations take the form

(4.2) $$\ddot{q}_j = e^{q_{j-1}-q_j} - e^{q_j-q_{j+1}} .$$

In order that these equations are valid also for j = 1 and j = n we have to drop the undefined terms, which we express formally by the "boundary condition"

* M. Toda, Wave propagation in anharmonic lattices, Jour. Phys. Soc. Japan 23, 1967, pp. 501-506.

(4.3) $$q_0 = -\infty,\quad q_{n+1} = \infty \quad \text{or} \quad e^{q_1} = 0\ ,\quad e^{-q_{n+1}} = 0$$

We will also consider the periodic case of the Toda lattice where we consider the points q_j on the circle R/PZ , i.e. we identify points x, $\tilde{x} \in R$ if they differ by a multiple of a positive constant P. This means effectively that the Hamiltonian becomes

(4.4) $$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n} e^{q_j-q_{j+1}}$$

where, however, (4.3) is replaced by

(4.5) $$q_{n+1} = q_1 + P\ ,\qquad p_{n+1} = p_n$$

It is convenient to replace P by 0 by means of the following formal consideration — losing some of the physical interpretation. We note that (4.4) admits an equilibrium solution, mainly if $p_j = 0$ and if all $q_j - q_{j+1}$ are equal and by (4.5)

$$q_{j+1} - q_j = \frac{P}{n} .$$

Setting $\lambda = P/2n$ we have

$$q_j = j2\lambda + c\ ,\qquad p_j = 0\ ,$$

with some constant c. Now we subject the system to the transformation $(x,y,s) \to (q,p,t) = \psi(x,y,s)$ given by

$$q_j = x_j + 2\lambda j$$
$$p_j = e^{-\lambda} y_j$$
$$t = e^{\lambda} s$$

This transformation is canonical in the generalized sense and the new Hamiltonian

$$e^{2\lambda}H \circ \psi = \frac{1}{2}\sum_{j=1}^{n} y_j^2 + \sum_{j=1}^{n} e^{x_j - x_{j+1}}$$

is of the same form but the boundary condition becomes

$$x_{n+1} - x_n = q_{n+1} - q_n - 2\lambda n = P - 2\lambda n = 0 \ ,$$

i.e. P is replaced by 0. Therefore we will from this point on consider the periodic Toda lattice with the boundary condition

$$(4.6) \qquad q_{n+1} = q_1 \ , \qquad p_{n+1} = p_1 \ .$$

The Calogero system is an n-particle system, where all particles interact and where $m_j = 1$, $V_{ij}(x) = ax^{-2} + bx^2$ so that the Hamiltonian is given by

$$(4.7) \qquad H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\le i<j\le n} \left(a(q_i - q_j)^{-2} + b(q_i - q_j)^2\right)$$

In contrast to the Toda lattice this potential is symmetric: $V_{ij}(-x) = V_{ij}(x)$.

Also this system has a periodic analogue, at least for $b = 0$, but its derivation is somewhat different. Since all particles interact with each other one has to take into account all fictitious particles at $q_j + kP$ with integer P. If we take $P = \pi$ and use the well known formula

$$\sum_{k=-\infty}^{+\infty} (x + k\pi)^{-2} = \sin^{-2} x$$

then we are led to the periodic problem

$$(4.8) \qquad H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\le i<j\le n} \frac{a}{\sin^2(q_i - q_j)}$$

where q_j are considered modulo π. Notice that this potential is singular not only for $q_i - q_j = 0$ but for $q_i - q_j = k\pi$, i.e. when q_i meets a fictitious point $q_j = k\pi$.

The above two—or four examples—are integrable Hamiltonian systems, a fact which is by no means obvious. It has to be said that these examples do not have a natural physical origin, but they can be used as approximations to physical systems. In particular, it can be shown that the Toda lattice is the discrete analogue of a partial differential equation, the Korteweg-deVries equation which does have physical significance. Here we will study the above systems as model problems. Since they are integrable the nature of the solution should be particularly simple, as we know from Section 1. This is indeed the case and the main task is to find the normal coordinates, or the action-angle variables in which the integration of the equation becomes simple. To find these variables, in particular, to find the integrals of these systems is not at all easy, however, and it involves surprising new aspects. In this section we will fully discuss the Toda lattice (4.1) and give a complete description of the solutions in the nonperiodic case. The periodic case can also be treated but requires hyperelliptic functions for its solution which are less familiar.

(b) Toda Lattice, Results.

The results to be proven are given in the following

Theorem 4.1. (i) The systems (4.1) and (4.4) (periodic case) possess n integrals $F_j = F_j(q,p)$, $j = 1,2,\ldots,n$ which are polynomials in p_k, $e^{q_k - q_{k+1}}$

(ii) These integrals $F_j(q,p)$ are linearly independent and in involution:

$$\{F_j, F_k\} = 0$$

(iii) For the solution of the system (4.1) $e^{q_j - q_{j+1}}$, p_j are given as rational functions of $e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots, e^{-\lambda_n t}$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct real numbers depending on the initial values. (The statement (iii) does not apply to the periodic Toda system (4.4) but the statements (i), (ii) do.)

(c) Iso-spectral Deformations.

Our first task (see (i)) is to find integrals of this system. We will identify these integrals as eigenvalues of a matrix function L. This relates our problem to spectral theory. The problem of finding the integrals is replaced by the problem of finding these matrices.

We simplify the differential equations by introducing the new variables

$$(4.9)\qquad a_j = \frac{1}{2} e^{(q_j - q_{j+1})/2}, \quad b_j = -\frac{1}{2} p_j \qquad j = 1,2,\ldots,n-1$$

so that

$$\frac{\dot a_j}{a_j} = \frac{1}{2}(\dot q_j - \dot q_{j+1}) = \frac{1}{2}(p_j - p_{j+1}) = b_{j+1} - b_j$$

$$\dot b_j = -\frac{1}{2}\dot p_j = -\frac{1}{2}\ddot q_j = -\frac{1}{2}\left(e^{q_{j-1}-q_j} - e^{q_j - q_{j+1}}\right) = 2(a_j^2 - a_{j-1}^2).$$

Hence the differential equations become

$$(4.10)\qquad \begin{aligned} \dot a_j &= a_j(b_{j+1} - b_j), \qquad & j = 1,2,\ldots,n-1 \\ \dot b_j &= 2(a_j^2 - a_{j-1}^2), \qquad & j = 1,2,\ldots,n \end{aligned}$$

where the right-hand sides are simple quadratic polynomials.

We observe that (4.9) provides a transformation of the p,q variables into the a,b variables, but we have to note that a,b constitute only 2n-1 variables, while the phase space has dimension 2n. However, in (4.9) as well as in (4.1) only the differences of the q_j occur and therefore they are invariant under the translation

$$(4.11)\qquad q_j \to q_j + s, \quad p_j \to p_j$$

which is the flow generated by the vector field X_A,

$$A = \sum_{j=1}^{n} p_j .$$

We shall therefore identify the orbits (4.11) of the R^1-action with points which we call configurations, and denote the space of configurations by

$$M_{2n-1} = R^{2n}/R^1 \cong R^{2n-1}.$$

Then the formulae (4.9) define an invertible mapping $\psi: (q,p) \to (a,b)$ of

$$\psi: M_{2n-1} \to D = \{a_1,\dots,a_{n-1},b_1,\dots,b_n \mid a_j > 0\}.$$

It was observed by H. Flaschka that these equations can be written in matrix form as follows: If $L = L(a,b)$, $B = B(a)$ are defined as the Jacobi matrices — or tridiagonal matrices —

$$(4.12)\quad L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & & \vdots \\ 0 & a_2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & a_{n-1} \\ 0 & 0 & \dots & a_{n-1} & b_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 \\ -a_1 & 0 & a_2 & & \vdots \\ 0 & a_2 & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & a_{n-1} \\ 0 & 0 & \dots & a_{n-1} & 0 \end{pmatrix}$$

then the equation (4.10) can be written in the form

$$(4.13)\qquad \frac{dL}{dt} = BL - LB = [B,L] .$$

This is easily verified. We want to point out that these equations are compatible, i.e. that the right-hand side $[B,L]$ is also a symmetric matrix which is tridiagonal. For this reason it is not so easy to find pairs of matrices B,L that depend on $2n$ or $2n-1$ parameters for which the equations (4.13) are compatible. In any event (4.12) is such a pair.

We observe that the eigenvalues $\lambda_k = \lambda_k(a,b)$, $k = 1,2,\dots,n$ of the Jacobi matrix $L = L(a,b)$ defined by (4.12) are real and, in addition, distinct. Indeed the symmetry

of L implies reality and $a_j > 0$ implies that the components of any eigenvector can recursively be determined from the first component, i.e. the eigenvector is uniquely determined up to a factor showing that the eigenvalue is simple. From this one concludes that the $\lambda_j = \lambda_j(a,b)$ are differentiable functions of a,b. We are going to show that these functions λ_j are integrals of the motion. If $a(t)$, $b(t)$ is a solution of the equations (4.10), or equi- equivalently if $L(t) = L(a(t),b(t))$ is a solution of (4.13), we shall deduce from the form of (4.13) that $L(t)$ and $L(0)$ are similar to each other from which it then follows that the spectrum of $L(t)$ is independent of t. For this reason one says that (4.13) describes an "iso-spectral deformation".

To prove this statement, namely that the spectrum of $L(t)$ is fixed, we adjoin (4.12) to the equation

$$(4.14)\qquad \frac{dU}{dt} = BU.$$

Since B is skew-symmetric it follows that $U(t)$ is orthogonal, if $U(0)$ has this property. Indeed

$$\frac{d}{dt}(U^T U) = U^T(B^T + B)U = 0 .$$

From (4.13), (4.14) we obtain

$$\frac{d}{dt}(U^{-1}L\,U) = -U^{-1}\dot U\,U^{-1}L\,U + U^{-1}\dot L\,U + U^{-1}L\,\dot U$$
$$= U^{-1}(-BL + \dot L + LB)U = 0$$

and therefore $U^{-1}L\,U$ is a constant matrix. If we take the initial condition $U(0) = I$ then

$$U(t)^{-1}L(t)\,U(t) = L(0),$$

i.e. $L(t)$ and $L(0)$ are similar and the eigenvalues $\lambda_j(t) = \lambda_j\big(a(t),b(t)\big)$ of $L(t)$ are indeed independent of t, as we wanted to show.

The eigenvalues $\lambda_k(a,b)$, or their symmetric functions $\pm\sigma_j$ defined by

$$\det(z - L) = z^n + \sigma_1 z^{n-1} + \cdots + \sigma_n$$

are integrals of the motion. The same holds for any other functions of the eigenvalues, like $\operatorname{tr} L^k = \sum_{j=1}^{n} \lambda_j^k$. Expressing these in terms of q,p we obtain

$$F_k(q,p) = \operatorname{tr} L^k, \qquad k = 1,2,\ldots,n,$$

as integrals. We compute

$$F_1 = \sum b_j = -\frac{1}{2}\sum_{j=1}^{n} p_j$$

$$F_2 = \sum_j b_j^2 + 2\sum_{j=1}^{n-1} a_j^2 = \frac{1}{4}\sum_{j=1}^{n} p_j^2 + \frac{1}{2}\sum_{j=1}^{n-1} e^{q_j - q_{j-1}} = \frac{1}{2}H$$

so that $-2F_1$, $2F_2$ are the momentum and energy integrals. However, $F_3, F_4, \ldots$ do not have a physical interpretation. It is clear that F_k are polynomials in a_j, b_j but, more precisely, they are polynomials in a_j^2, b_j (Exercise 4.1). This proves statement (i) of Theorem 4.1.

Incidentally we see that the $F_k(q,p) = \operatorname{Tr} L^k$ have linearly independent gradients. For this purpose it suffices to show that the determinant

$$\det\left(\frac{\partial F_k}{\partial \lambda_j}\right), \qquad (j,k = 1,2,\ldots,n)$$

does not vanish. This is a Vandermonde determinant since $F_k = \sum_{j=1}^{n} \lambda^k$. It does not vanish since the λ_j are distinct.

In the periodic case one proceeds in just the same way after replacing L and B by the cyclic matrices

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 & a_n \\ a_1 & b_2 & \ddots & & & 0 \\ 0 & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & & & \ddots & & a_{n-1} \\ a_n & 0 & \cdots & 0 & a_{n-1} & b_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & & -a_n \\ -a_1 & 0 & & & & 0 \\ 0 & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & & & & 0 & a_{n-1} \\ a_n & 0 & \cdots & 0 & a_{n-1} & 0 \end{pmatrix}$$

Such a pair of linear operators L, B satisfying a differential equation (4.13) is called a Lax pair. Lax derived such an equation for the Korteweg-deVries equation. This approach has been fruitful in a number of other cases as we shall see in the following sections.

(d) The F_k are in involution.

To prove (ii) we show that the eigenvalues λ_j are in involution, which implies that the $F_k = \sum_{j=1}^{n} \lambda_j^k$ are also in involution. The following argument also applies to the periodic case but we carry it out in the nonperiodic case.

Before computing $\{\lambda_j, \lambda_k\}$ we express the Poisson bracket

(4.15) $$\{F,G\} = \langle F_q, G_p\rangle - \langle G_q, F_p\rangle$$

in terms of the variables a_j, b_k defined by (4.9). For any function $\phi = \phi(a,b)$ we define a function $F = \phi \circ \psi$ of q,p and if $G = G \circ \psi$ is a second function the transformed Poisson bracket $\{\ ,\ \}_*$ is defined by

$$\{\phi,\psi\}_* \circ \psi = \{F,G\}$$

It is easily calculated: From (4.9) we find

$$\frac{\partial}{\partial q_j} = \frac{1}{2}\left(a_j \frac{\partial}{\partial a_j} - a_{j-1}\frac{\partial}{\partial a_{j-1}}\right) ; \quad \frac{\partial}{\partial p_j} = -\frac{1}{2}\frac{\partial}{\partial b_j}$$

where we have to replace the undefined terms by 0, or we set $a_0 = 0$, $a_n = 0$ in accordance with (4.3). Then the transformed Poisson bracket is computed as

(4.16) $$\{\phi,\psi\}_* = -\frac{1}{4}\sum_{j=1}^{n}\Big((a_j\phi_{a_j} - a_{j-1}\phi_{a_{j-1}})\psi_{b_j} - (a_j\psi_{a_j} - a_{j-1}\psi_{a_{j-1}})\phi_{b_j}\Big) .$$

It suffices to show that

$$\{\lambda_j, \lambda_k\}_* = 0$$

which we will now do.

For this puprose we have to compute the derivations of $\lambda_j = \lambda_j(a,b)$ with respect to a_k, b_k. To avoid subscripts we write λ in place of λ_j and μ for the eigenvalue λ_k. Let ϕ,ψ be the corresponding eigenvectors of L.

(4.17) $$(L-\lambda)\phi = 0 , \quad (L-\mu)\psi = 0$$

which are normalized by $\langle\phi,\phi\rangle = 1$, $\langle\psi,\psi\rangle = 1$. We We then show that

(4.18) $$\frac{\partial\lambda}{\partial a_j} = 2\phi(j)\phi(j+1) ; \quad \frac{\partial\lambda}{\partial b_j} = \phi^2(j)$$

where we denote the components of ϕ by $\phi(j)$ to avoid confusion with the subscript labeling the eigenvalues.

We prove (4.18). Indicating differentiation with respect to a_j or b_j by a prime we have from (4.17)

$$(L' - \lambda') + (L-\lambda)\phi' = 0$$

or

$$\langle(L'-\lambda')\phi,\phi\rangle + \langle(L-\lambda)\phi',\phi\rangle = 0 .$$

Since L is symmetric the second term vanishes and, on account of the normalization of ϕ,

$$\lambda' = \langle L'\phi,\phi\rangle.$$

From

$$\langle L\phi,\phi\rangle = \sum_{j=1}^{n-1} 2a_j\phi(j)\phi(j+1) + \sum_{j=1}^{n} b_j\phi^2(j)$$

the relations (4.18) follow.

Inserting these relations into (4.16) we find

Comparing this expression with (4.19) we obtain

$$-2\{\lambda,\mu\}_* = (\lambda-\mu)^{-1} \sum_{j=1}^{n} (W_j^2 - W_{j-1}^2) = (\lambda-\mu)^{-1}(W_n^2 - W_0^2)$$

This expression vanishes in the periodic case since $W_n = W_0$ and in the nonperiodic case since then $W_n = 0$, $W_0 = 0$. This proves that (ii) of Theorem 4.1 holds here. Actually, in the periodic case the eigenvalues λ_j can be multiple and we have established (ii) only for those q,p for which the eigenvalues of L are distinct. But this is an open set and since the F_j are real analytic this suffices.

(e) <u>The Explicit Representation of the Solutions</u>

We turn to the proof of the statement (iii) in Theorem 4.1. We introduce the function

(4.20) $$f(z) = \langle (z - L)^{-1} e_n, e_n \rangle ,$$

where e_n is the unit vector (0,0,0,...,0,1). Since L is symmetric it follows that f(z) is an analytic function for Im z ≠ 0 and $\overline{f(z)} = f(\bar{z})$; moreover,

$$\text{Im } f(z) > 0 \quad \text{for} \quad \text{Im } z > 0 .$$

~~Moreover~~ It is a rational function with simple poles at the eigenvalues λ_j of L and so admits the partial fraction expansion

(4.21) $$f(z) = \sum_{j=1}^{n} \frac{x_j^2}{z - \lambda_j} , \quad x_j > 0$$

with positive residues x_j^2. Moreover comparision of the coefficients of z^{-1} as $|z| \to \infty$ gives

$$\sum_{k=1}^{n} x_k^2 = 1 .$$

From the identity

$$\sum_{k=1}^{n} \frac{x_k^2}{z - \lambda_k} = \langle (z - L)^{-1} e_n, e_n \rangle$$

we read off that

$$x_k = \pm \langle \phi_k, e_n \rangle = \pm \phi_k(n) ,$$

where ϕ_k is the eigenvector of λ_k. Thus for a given matrix L we can compute the rational function f(z) which has n simple real poles at λ_j with positive residual x_j^2, since Im f(z) > 0 for Im z > 0. Ordering these poles according to size we can define a map χ,

$$\chi\colon D_1 \to \Lambda\colon (a^2, b) \to (\lambda, x)$$

which associates to every point (a_j^2, b_j), where $(a,b) \in D$, the point (λ, x) in the set

$$\Lambda = \{(\lambda,x) \mid \lambda_1 < \lambda_2 < \dots < \lambda_n , \sum_{k=1}^{n} x_k^2 = 1, x_k > 0\},.$$

We claim that this map $\chi\colon D_1 \to \Lambda$ is one to one and onto. Moreover χ^{-1} is a rational mapping. We shall view it later on as a coordinate transformation and then describe the differential equations (4.13) in the new variables.

The fact that the mapping χ has an inverse $\chi^{-1}: \Lambda \to D_1$ corresponds to the inverse method of spectral theory, which in the elementary form described here goes back to Stieltjes.[1] It is based on the fact that f(z) can be represented as a finite continued fraction expansion

$$(4.22)\qquad f(z) = \cfrac{1}{\lambda - b_n - \cfrac{a_{n-1}^2}{\lambda - b_{n-1} - \cfrac{\ddots}{\; - \cfrac{a_1^2}{\lambda - b_1}}}} ,$$

where the entries a_j, b_j agree precisely with those of $L = L(a,b)$.

In order to prove (4.22) we denote by L_j the matrix obtained from L by cancelling the last n-j rows and columns and accordingly set $L_n = L$. Then the characteristic polynomial

$$\Delta_j(z) = \det(z - L_j)$$

satisfies the recursion formula

$$(4.23)\qquad \Delta_j = (z - b_j)\Delta_{j-1} - a_{j-1}^2\Delta_{j-2}$$

for $j = 3,4,\ldots,n$. It also holds for $j = 1,2$ if we set

$$\Delta_{-1} = 0 , \qquad \Delta_0 = 1 .$$

This can be seen by expanding the determinant with respect to the last row. Hence the ratio $\rho_j = \Delta_j/\Delta_{j-1}$ satisfies

[1] F. Gantmacher, M, Krein, "Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme," Akad. Verlag, Berlin (1960), See: Anhang II.

$$\rho_j = z - b_j - \frac{a_{j-1}^2}{\rho_{j-1}} ,$$

which leads to a finite continued fraction for the rational function

$$\frac{\Delta_{n-1}(z)}{\Delta_n(z)} = \cfrac{1}{z - b_n - \cfrac{a_{n-1}^2}{z - b_{n-1} - \cfrac{\ddots}{\; - \cfrac{a_1^2}{z - b_1}}}}$$

and it remains to verify that

$$(4.24)\qquad f(z) = \frac{\Delta_{n-1}(z)}{\Delta_n(z)} = \langle (z - L)^{-1}e_n, e_n\rangle .$$

To show this one can use the recursion relation (4.23) to see that the vector v with components

$$v_j = \frac{\Delta_{j-1}}{\Delta_n} a_j \cdots a_{n-1} \quad \text{for } j=1,2,\ldots,n-1$$
$$v_n = \frac{\Delta_{n-1}}{\Delta_n}$$

solves the equation $(z - L)v = e_n$ i.e. $v = (z - L)^{-1}e_n$, and so

$$v_n = \langle v, e_n\rangle = \langle (z - L)^{-1}e_n, e_n\rangle = \frac{\Delta_{n-1}(z)}{\Delta_n(z)} ,$$

and we have proved (4.24) and therefore also (4.22).

We come to the "inverse problem" which requires that we determine χ^{-1}. With any point $(\lambda, x) \in \Lambda$ we associate the function f(z) defined by (4.21). Then Im f(z) > 0 if

Im $z > 0$ and $z\,f(z) \to 1$ as $|z| \to \infty$. Thus

$$\frac{1}{f(z)} = z + A - g(z) \; ,$$

where A is a real constant, and where $g(z)$ is a rational function which satisfies

$$\operatorname{Im} g(z) = \operatorname{Im} z + \frac{\operatorname{Im} f(z)}{|f(z)|^2} > 0 \quad \text{for} \quad \operatorname{Im} z > 0 \; .$$

Thus $g(z)$ has only simple poles on the real axis and their number is n-1. One computes easily that

$$-A = \sum_{j=1}^{n} \lambda_j x_j^2 \; , \qquad z\,g(z) \to \sum_{j=1}^{n} \lambda_j^2 x_j^2 - \left(\sum_{j=1}^{n} \lambda_j x_j^2\right)^2 \; (\text{as } |z| \to \infty)$$

which is > 0, since $(\lambda, x) \in \Lambda$. Thus $g(z) = B\,f_{n-1}(z)$ with $B > 0$ and $z\,f_{n-1}(z) \to 1$ as $|z| \to \infty$. Hence

$$f(z) = \frac{1}{z + A - B f_{n-1}(z)} \; ,$$

and by induction we get a unique continued fraction of the form (4.22) with $A = -b_n$, $B = a_{n-1}^2 > 0$ etc. Thus the rational function 4.21 determines a_j^2, b_j and conversely, is determined by the a_j^2, b_j.

This shows that the map χ maps D_1 one to one onto Λ and moreover that χ^{-1} is a rational mapping.

We will express the differential equations in terms of the variables λ_j, x_j. Since the λ_j are integrals we have $\dot{\lambda}_j = 0$ and it remains to determine the differential equations for the x_j, which turn out to be

$$(4.25) \qquad \dot{x}_j = -\left(\lambda_j - \sum_{k=1}^{n} \lambda_k x_k^2\right) x_j \; , \qquad j = 1,2,\dots,n.$$

This system is easily integated if one observes that it is obtained from the linear system

$$\dot{y}_j = -\lambda_j y_j$$

by setting

$$x_j = \frac{y_j}{|y|}$$

Hence the solutions of (4.25) are given by

$$x_j^2(t) = \frac{x_j^2(0)\,e^{-2\lambda_j t}}{\sum_{k=1}^{n} x_k^2(0)\,e^{-2\lambda_k t}}$$

This proves the assertion (iii) of Theorem 4.1 since a_j^2, b_j are rational functions of x_k^2, λ_k and

$$2a_j^2 = e^{q_j - q_{j+1}} \; , \qquad -2b_j = p_j \; .$$

It remains to derive the differential equation (4.25).

From our considerations in (c), in particular, from the relation

$$U(t)^{-1} L(t)\, U(t) = L(0)$$

it is clear that the eigenvectors $\phi_k = \phi_k(t)$ of $L(t)$ satisfy

$$\phi_k(t) = U(t)\,\phi_k(0) \; ,$$

hence in view of (4.14)

$$\frac{d\phi_k}{dt} = B\phi_k \; .$$

Using the form (4.12) of B we see that

$$\frac{d\phi_k(n)}{dt} = - a_{n-1} \phi_k(n-1) .$$

If we combine this with the eigenvalue equation

$$a_{n-1} \phi_k(n-1) + b_n \phi_k(n) = \lambda_k \phi_k(n)$$

we obtain from $x_k = \pm \phi_k(n)$

$$\frac{dx_k}{dt} = \frac{d\phi_k(n)}{dt} = (-\lambda_k + b_n) \phi_k(n) = (-\lambda_k + b_n) x_k .$$

Finally, since $\sum_{k=1}^{n} x_k^2 = 1$ we have

$$0 = \sum x_k \dot{x}_k = - \sum_{k=1}^{n} \lambda_k x_k^2 + b_n .$$

Inserting this value of b_n into the differential equations we obtain (4.22) which completes the proof of Theorem 4.1.

(f) Scattering

In the nonperiodic case the integrable character of the Toda lattice is in fact trivial since one can find the necessary integrals as asymptotic velocities for $t \to +\infty$, similarly as we have seen in Section 3 for the Störmer problem. We have proved however more, namely that the integrals are algebraic functions of p_j and $e^{q_j - q_{j+1}}$. This algebraic character has a surprising consequence for the scattering behavior, which we describe now.

We shall show that for all solutions one has the asymptotic behavior

$$(4.26) \quad \begin{aligned} q_j(t) &= \alpha_j^+ t + \beta_j^+ + O(e^{-\delta t}) \\ q_j(-t) &= -\alpha_j^- t + \beta_j^- + O(e^{-\delta t}) \end{aligned}$$

for $t \to +\infty$ with some $\delta > 0$. We will call α_j^+, α_j^- the asymptotic velocities for $t \to +\infty$ or $t \to -\infty$. Then it turns out that

$$(4.27) \qquad \alpha_j^+ = -2 \lambda_{n-j+1} , \quad \alpha_j^- = -2\lambda_j ,$$

where λ_j are the eigenvalues of L ordered by

$$(4.28) \qquad \lambda_1 < \lambda_2 < \ldots < \lambda_n .$$

In other words, the asymptotic velocities α_j^+ agree up to the factor −2 and up to ordering with these eigenvalues!

Moreover, the relation

$$\alpha_{k-j+1}^+ = \alpha_j^-$$

has the physical interpretation that the asymptotic velocities for $t \to -\infty$ and for $t \to +\infty$ are exactly the same except for reordering. In other words the first particle gives its velocity at $t = -\infty$ to the n^{th} particle at $t = +\infty$ and the j^{th} particle gives its velocity to the $(n-j+1)^{st}$ particle, just like one has for elastic reflection of particles.

This is a highly exceptional situation for a particle system and this is a consequence of the integrable character. Indeed, from the conservation of energy H and momentum we conclude

$$\sum_{j=1}^{n} (\alpha_j^+)^2 = \sum_{j=1}^{n} (\alpha_j^-)^2 , \quad \sum_{j=1}^{n} \alpha_j^+ = \sum_{j=1}^{n} \alpha_j^- .$$

but the conservation of the "unphysical" integrals F_k for $k \geq 3$ gives

$$\sum_{j=1}^{n} (\alpha_j^+)^k = \sum_{j=1}^{n} (\alpha_j^-)^k$$

This implies that the unordered sets $\{\alpha_j^+\}_{j=1,\ldots,n}$ and $\{\alpha_j^-\}_{j=1,\ldots,n}$ agree.

We mention without proof, that the symplectic scattering map taking (α_j^-, β_j^-) into (α_k^+, β_k^+) can be determined explicitly for the Toda lattice and is given by

$$\alpha_{n-j+1}^+ = \alpha_j^- , \quad \beta_{n-j+1}^+ = \beta_j^- - \frac{\partial}{\partial \alpha_j^-} W(\alpha_1^-, \ldots, \alpha_n^-)$$

where

$$W = \sum_{i<j} (\alpha_i^- - \alpha_j^-) \left(\log (\alpha_i^- - \alpha_j^-)^2 - 1\right) .$$

One computes

$$\frac{\partial}{\partial \alpha_k} W = \sum_{j>k} \log (\alpha_j^- - \alpha_k^-)^2 - \sum_{j<k} \log (\alpha_j^- - \alpha_k^-)^2$$

and one interprets this as follows: The "phase shift" $\beta_{n-k+1}^+ - \beta_k^-$ is the sum of the phase shifts $\pm \log (\alpha_j^- - \alpha_k^-)^2$ between the j^{th} and the k^{th} particle. The scattering map is the same as if the particles interacted just pairwise! We will not prove ~~their~~ statement about the phase shift,*

* J. Moser, Finitely many mass points on the line ...," Lecture NOtes in Physics 38, Springer-Verlag, 1975, pp. 467-497.

but restrict ourselves to the proof of (4.27).

From Theorem 4.1 (iii) it is clear that the asymptotic behavior of $a_j(t)$, $b_j(t)$ is given by

$$c_1 e^{\eta_1 t} + c_2 e^{\eta_2 t} + \ldots , \qquad c_1 \neq 0$$

where the exponents η_1, η_2 are linear combinations of the $-2\lambda_k$ with integer coefficients — a fact, which we will use repeatedly. Since

$$H = \frac{1}{2} \sum_{j=1}^{n} b_j^2 + \sum_{j=1}^{n-1} a_j^2$$

is preserved we see that $b_j(t)$, $a_j(t)$ are bounded, and from the differential equations (4.10) we see that also

$$2 \int_{t_1}^{t_2} a_j^2 \, dt = \sum_{k=1}^{j} \left(b_k(t_2) - b_k(t_1)\right)$$

is bounded, i.e.

$$\int_{-\infty}^{+\infty} a_j^2(t) \, dt < \infty .$$

Therefore we have

$$a_j(t) \to 0 \quad \text{for} \quad t \to \pm \infty$$

and because of the above asymptotic behavior the decay is exponential and in particular (4.26) follows.

With this information we see that the matrix $L = L(t)$ of (4.12) approaches a diagonal matrix $L(+\infty)$, $L(-\infty)$ for $t \to \pm \infty$, whose diagonal elements must be the t-independent distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Hence each $b_j(t)$ approaches one of these eigenvalues and it remains to determine which of them is approached. From the first equation of (4.10) we conclude that

$$b_{j+1}(+\infty) - b_j(+\infty) \leq 0$$

since otherwise $a_j(t)$ would not be bounded. Moreover, the $b_j(+\infty)$ must be distinct since the set of the $b_j(+\infty)$ adjoins with the spectrum of L. This shows that

$$b_j(+\infty) = \lambda_{n-j+1}$$

if the eigenvalues are ordered according to (4.28). By the same argument we have

$$b_j(-\infty) = \lambda_j .$$

Translating this information via (4.9) for the p_j, q_j we obtain readily (4.27).

Finally we remark that the periodic Toda lattice is of a different character. We may restrict ourselves to motions where the center of gravity is at rest, i.e.

$$\sum_{j=1}^{n} q_j = 0 .$$

Then the energy surface

$$E = \left\{q,p \,\middle|\, H = \frac{1}{2}\sum_{j=1}^{n}\left(p_j^2 + e^{q_j - q_{j+1}}\right) = h,\ \sum_{j=1}^{n} q_j = 0,\ q_{n+1} = q_1\right\}$$

is compact and the motion bounded, in fact quasiperiodic,

as described in Section 1. Indeed from the boundedness of $e^{q_j - q_{j+1}} \leq 2h$ and from the boundary conditions (4.6) we have

$$1 = \prod_{j=1}^{n} e^{q_j - q_{j+1}} = e^{q_1 - q_{n+1}} \leq (2h)^{n-1} e^{q_k - q_{k+1}},$$

and we see that all differences $q_k - q_{k+1}$, hence the q_k, are bounded. Since the p_k are also bounded E is indeed compact.

In the periodic case the solutions cannot be rationally expressed in terms of exponential functions but are hyperelliptic integrals, Similarly as in the problems of the next section. We will not discuss this case here; see P. van Moerbeke. [1]

* P. van Moerbeke: The spectrum of Jacobi matrices, Inventiones Math. 37, 1976, pp. 45-81.

Exercises

1. Show that $\operatorname{tr} L^k$ are polynomials in a_j^2, b_j.

2. Show that for even n

$$\dot{x}_j = (H_{x_{j+1}} - H_{x_{j-1}}), \quad j = 1,2,\ldots,n,$$

where H_{x_0} are to be replaced by 0, is a Hamiltonian system with respect to the Poisson bracket

$$\{F,G\} = \sum_{j=1}^{n-1} (F_{x_j} G_{x_{j+1}} - G_{x_j} F_{x_{j+1}})$$

or the symplectic two-form

$$\omega = \sum_{\substack{j,k=1 \\ j \geq k}}^{n/2} dx_{2j} \wedge dx_{2k-1}$$

3. With the symplectic structure of Exercise 2 the system

$$\dot{x}_j = e^{x_{j+1}} - e^{x_{j-1}}$$

with the Hamiltonian $H = \sum_{j=1}^{n} e^{x_j}$ is integrable. Its integrals are given as eigenvalues of the Jacobi matrix L in (4.12) where $b_j = 0$, $a_j = e^{x_j/2}$.

4. Let F be defined on the unit sphere $\sum_{j=1}^{n} x_j^2 = 1$ by

$$F = \frac{1}{2} \sum_{k=1}^{n} \lambda_k x_k^2 .$$

Show that the differential equation

$$\dot{x} = - \nabla F$$

where ∇ denotes the gradient in the tangent space of the unit sphere (defined by $dF(v) = \langle \nabla F, v \rangle$ for any v in the tangent space of the unit sphere) is given by

$$\dot{x}_j = (- \lambda_j + F) x_j$$

(which is the system (4.22)).

5. Discuss the flow defined by the system of Exercise 4 for distinct λ_j. Show that every solution approaches one of the equilibrium points $\pm e_k$.

5. Separation of Variables

The classical approach to establish a Hamiltonian system as an integrable one is to solve the Hamilton-Jacobi equation by separation of variables. This approach is due to Jacobi who showed that way that the geodesic flow on an ellipsoid

$$\sum_{j=1}^{n} \frac{x_j^2}{\alpha_j} = 1 , \qquad 0 < \alpha_1 < \ldots < \alpha_n ,$$

with different major axes is integrable. The main difficulty of this method is that it requires finding — or guessing — coordinates in which separation of variables is possible. For the Kepler problem — which is rotation invariant — one can use polar coordinates; this is the standard way to get to the Delaunay variables which we described in Section 2. We will illustrate this approach with a different mechanical system requiring elliptical coordinates on the sphere. The solutions are no longer expressible as rational functions of exponentials but are given in terms of hyperelliptic functions, which are generalizations of elliptic functions.

(a) Two Integrable Mechanical Problems.

(i) Neumann's problem. Consider the motion of a mass point on the sphere $|x| = 1$ in R^n moving under the influence of the linear force $-Ax$, where $A = A^T$ is a symmetric n by n matrix with distinct eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$. The differential equations have the form

$$\ddot{x} = - Ax + \lambda x , \tag{5.1}$$

where λx is the normal force which is required to keep the particle on the sphere. In analytic terms, the identity $|x(t)|^2 = 1$ implies

$$\langle x,\dot{x}\rangle = 0 , \quad \langle x,\ddot{x}\rangle + |\dot{x}|^2 = 0 .$$

Inserting the differential equation into the second equation we find

$$- \langle x,Ax\rangle + \lambda + |\dot{x}|^2 = 0$$

or

$$\lambda = \langle Ax,x\rangle - |\dot{x}|^2 . \tag{5.2}$$

Inserting (5.2) into (5.1) we have the desired differential equation which is nonlinear. It can be viewed as a Hamiltonian system on the tangent bundle $|x|^2 = 1$, $\langle x,\dot{x}\rangle = 0$ of the sphere which turns out to be integrable. This was shown by C. Neumann in 1858.[1]

(ii) The geodesic flow on an ellipsoid

$$Q(x) = \langle A^{-1}x,x\rangle = 1$$

where $A = A^T$ is a positive symmetric matrix with distinct eigenvalues, can also be viewed mechanically, namely as the motion of a particle on the above ellipsoid without any external force. Therefore the differential equations have the form

$$\ddot{x} = - \lambda A^{-1}x , \tag{5.3}$$

[1] C. Neumann, "De problemate quiam mechanico, quod ad primam integralium, ultraellipticorum classem revocatur; Journ. Reine Angw. Math. 56, 1859, 46-63.

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where the right-hand side is the normal force required to keep the particle on the ellipsoid. In this case one computes

(5.4) $$\lambda = Q(\dot{x})|A^{-1}x|^{-2}$$,

and (5.3), (5.4) define the system in question which has to be restricted to the tangent bundle of the ellipsoid:

(5.5) $$\langle A^{-1}x,x\rangle = 1, \quad \langle A^{-1}x,\dot{x}\rangle = 0 \quad .$$

In this and the next section we will show that these two systems are integrable; in fact they possess integrals which can be expressed as quartic polynomials in x, $\dot{x}$ as we will see.

(b) Elliptical coordinates on the sphere $|x| = 1$.

To describe the differential equations we introduce coordinates on the sphere. We do not use polar coordinates but some elliptical coordinates which are essential for the integration of the Hamilton-Jacobi equations. To define them we introduce the quadratic form

(5.6) $$Q_z(x) = \langle(z - A)^{-1}x,x\rangle$$

for z not an eigenvalue of A. We may assume that A is diagonal,

$$A = \text{diag}\ (\alpha_1,\alpha_2,\dots,\alpha_n) \ , \qquad \alpha_1 < \alpha_2 < \dots < \alpha_n$$

and write $Q_z(x)$, which is a rational function of z, in the form

(5.7) $$Q_z(x) = \frac{m(z)}{a(z)}$$

where

$$a(z) = \prod_{j=1}^{n} (z - \alpha_j) \ , \qquad m(z) = \prod_{k=1}^{n-1} (z - \mu_k) \ ,$$

so that $\mu_1,\mu_2,\dots,\mu_{n-1}$ are the zeros of $Q_z(x)$. Comparing the asymptotic behavior for large z we see that $Q_z(x) = |x|^2 z^{-1} + O(|z|^{-2})$, and (5.7) implies $|x|^2 = 1$. Now we consider the n-1 zeros $\mu_1,\mu_2,\dots,\mu_{n-1}$ as coordinates on the unit sphere. Since

$$Q_z(x) = \sum_{j=1}^{n} \frac{x_j^2}{z-\alpha_j}$$

we can express the x_j^2 as the residues of m/a at $z = \mu_j$, i.e.

(5.8) $$x_j^2 = \frac{m(\alpha_j)}{a'(\alpha_j)} = \frac{\prod_{k=1}^{n-1} (\alpha_j-\mu_k)}{\prod_{i\neq j} (\alpha_j-\alpha_i)} \quad .$$

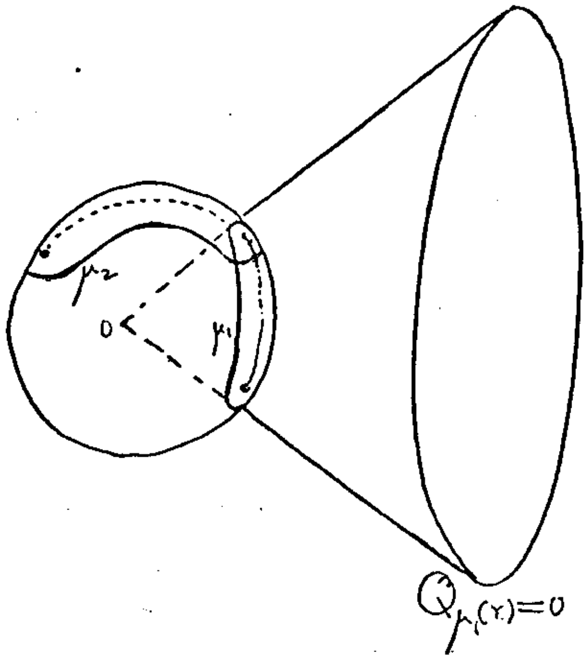
This formula determines x_j up to sign. For points with $x_1,x_2,\dots,x_n \neq 0$ one sees that the zeros of $Q_z(x)$ interlace the poles so that we can order the μ_j so that

$$\alpha_1 < \mu_1 < \alpha_2 < \dots < \mu_{n-1} < \alpha_n \ .$$

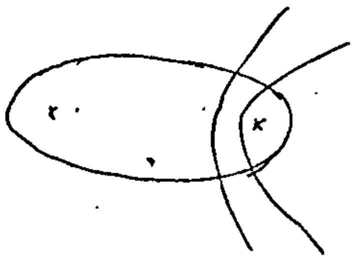
To interpret these coordinates geometrically on the two-sphere we take n = 3 and note that

$$Q_{\mu_j}(x) = 0 \quad , \qquad j = 1,2,$$

represents a cone and the μ_j-curves on the sphere are the intersections of this cone and the sphere $|x| = 1$.



The μ_1-curves and the μ_2-curves intersect orthogonally and form a pattern similar to the familiar confocal ellipses and hyperbolae of the elliptic coordinates in the plane.



For $\mu_1 = \alpha_2$ one gets a singular cone:

$$x_2 = 0 , \quad \frac{x_1^2}{\alpha_2-\alpha_1} + \frac{x_3^2}{\alpha_2-\alpha_3} = 0$$

defining two lines, the analogue of the focal points.

Also in higher dimensions the $\mu_1, \mu_2, \dots, \mu_{n-1}$ define orthogonal coordinates as is evident from the expression for the length element

$$(5.9) \quad \begin{cases} ds^2 = \sum_{j=1}^{n} dx_j^2 = \sum_{i=1}^{n-1} g_i \, d\mu_i^2 , \\ g_i = - \dfrac{m'(\mu_i)}{4q(\mu_i)} . \end{cases}$$

To prove this we deduce from (5.8)

$$2 \frac{dx_j}{x_j} = \sum_{i=1}^{n-1} \frac{d\mu_i}{\mu_i-\alpha_j} ,$$

hence

$$ds^2 = \sum_{j=1}^{n} dx_j^2 = \sum g_{ij} \, d\mu_i \, d\mu_j$$

with

$$g_{ij} = \frac{1}{4} \sum_{k=1}^{n} \frac{x_k^2}{(\mu_i-\alpha_k)(\mu_j-\alpha_k)} .$$

For $i \neq j$ this expression agrees with

$$\frac{1}{4(\mu_j-\mu_i)} \left(\sum_k \frac{x_k^2}{\mu_i-\alpha_k} - \sum_k \frac{x_k^2}{\mu_j-\alpha_k} \right) = 0 ,$$

since $Q_{\mu_i}(x) = 0$. For $i = j$ we get

$$g_i = g_{ii} = \frac{1}{4} \sum_{k=1}^{n} \frac{x_k^2}{(\mu_i - \alpha_k)^2} = -\frac{1}{4} \frac{\partial}{\partial z} Q_z(x) \Big|_{z=\mu_i} = -\frac{1}{4} \frac{m'(\mu_i)}{m(\mu_i)} .$$

We describe the variational principle describing the motion on the sphere in terms of these variables: The kinetic energy and the potential energy become by (5.9)

$$T = \frac{1}{2} |\dot{x}|^2 = \frac{1}{2} \sum_{j=1}^{n-1} g_j \dot{\mu}_j^2$$

$$U = \frac{1}{2} \langle Ax, x \rangle = \frac{1}{2} \sum_{j=1}^{n} (\alpha_j - \mu_j) , \quad \mu_n = 0.$$

The last relation can be read off from the asymptotic expansion of (5.7) for $z \to \infty$:

$$Q_z(x) = \frac{|x|^2}{z} + \frac{\langle Ax, x \rangle}{z^2} + \cdots = \frac{1}{z} + \frac{\sum_{j=1}^{n} (\alpha_j - \mu_j)}{z^2} + \cdots .$$

The equations of motion are the Euler equations of Lagrange's variational principle

$$\delta \int (T(\dot{\mu}) - U(\mu)) \, dt = 0 .$$

Introducing the variables π_j by the Legendre transformation (Chapter I, 2b)

$$\pi_j = \frac{\partial T}{\partial \dot{\mu}_j} = g_j \dot{\mu}_j , \qquad j = 1,2,\ldots n-1,$$

and the Hamiltonian

(5.10) $$H(\mu, \pi) = T + U = \frac{1}{2} \sum_{j=1}^{n-1} \left(g_j^{-1} \pi_j^2 - \mu_j \right) ,$$

the differential equation takes the canonical form

$$\dot{\mu}_j = H_{\pi_j} , \quad \dot{\pi}_j = -H_{\mu_j} , \qquad j = 1,2,\ldots,n-1;$$

on the right-hand side of (5.10) we have dropped the unessential term $\frac{1}{2} \sum_{j=1}^{n} \alpha_j$.

(c) The Hamilton-Jacobi Equation.

We seek to introduce new variables ξ_j, η_j ($j=1,2,\ldots,n-1$) by a canonical transformation, such that the Hamiltonian takes a particularly simple form, say equal to $\Phi(\xi,\eta) = \eta_1$. Then the differential equation becomes

$$\dot{\xi} = e_1 , \quad \dot{\eta} = 0$$

and the flow takes place along straight lines parallel to the ξ_1-axis.

We introduce such a canonical transformation by a generating function $S = S(\mu,\eta)$:

$$\pi_j = S_{\mu_j} , \quad \xi_j = S_{\eta_j} , \qquad j = 1,2,\ldots,n-1 .$$

Then the new Hamiltonian $\Phi(\xi,\eta)$ is related to $H = H(\mu,\pi)$ by

$$H(\mu, S_\mu) = \Phi(S_\eta, \eta)$$

and in the special case that $\Phi = \frac{1}{2} \eta_1$ by

(5.11) $$H(\mu, S_\mu) = \frac{1}{2} \eta_1 .$$

This is the Hamilton Jacobi equation for which we have to find a complete solution $S(\mu,\eta)$ depending besides η_1 on the n-2 parametrss $\eta_2,\ldots,\eta_{n-1}$ such that the Hessian $S_{\mu\eta}$ is nonsingular. The remarkable fact is that this equation can be solved by separation of variables in our situation. This was observed by C. Neumann in 1859 already, at least for n = 3.

We write the Hamilton-Jacobi equation more explicitly, using (5.9-11):

$$\sum_{j=1}^{n-1} \frac{1}{m'(\mu_j)} \left[4a(\mu_j)\left(\frac{\partial S}{\partial \mu_j}\right)^2\right] + \sum_{j=1}^{n-1} \mu_j + \eta_1 = 0 . \tag{5.12}$$

Here the notation is somewhat misleading: The function $m'(\mu_j)$ is not a function of μ_j alone but depends on all μ_k. The clue to the separation of variables is that the last two terms are rewritten as given by the following lemma.

Lemma 5.1. If we define

$$p(z) = z^{n-1} + \eta_1 z^{n-2} + \eta_2 z^{n-3} + \cdots + \eta_{n-1} \tag{5.13}$$

then

$$\sum_{j=1}^{n-1} \frac{p(\mu_j)}{m'(\mu_j)} = \eta_1 + \sum_{j=1}^{n} \mu_j$$

provided the $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are distinct.

Proof: The left-hand side can be viewed as the residue sum of the integral

$$\frac{1}{2\pi i} \int \frac{p(z)}{m(z)}\, dz$$

taken over a large circle containing the μ_k. On the other hand this integral agrees with

$$\frac{1}{2\pi i} \int \frac{p(z)-m(z)}{m(z)}\, dz$$

where the numerator is a polynomial of degree n-2 with leading coefficient $\eta_1 + \sum_{j=1}^{n-1} \mu_j$. Therefore if we let the radius of the circle tend to infinity we obtain the result.

Using this lemma the equation (5.12) can be rewritten as

$$\sum_{j=1}^{n-1} \frac{1}{m'(\mu_j)} \left\{-4a(\mu_j)\left(\frac{\partial S}{\partial \mu_j}\right)^2 + p(\mu_j)\right\} = 0$$

where p(z) is the polynomial (5.12). Now the variables are "separated"; the above equation holds if

$$\left(\frac{\partial S}{\partial \mu_j}\right)^2 = -\frac{p(\mu_j)}{4a(\mu_j)}$$

is satisfied for each $j = 1, 2, \ldots, n-1$. But each of these equations depends on one variable μ_j only and can be solved by

$$S(\mu, \eta) = \sum_{j=1}^{n-1} \int^{\mu_j} \sqrt{\frac{p(z)}{-4a(z)}}\, dz \tag{5.14}$$

where the integration is to be taken over some paths on the Riemannian surface given by $w^2 = -a(z)p(z)$.

The desired canonical transformation is now given by

$$\xi_k = S_{\eta_k} = \frac{1}{4} \sum_{j=1}^{n-1} \int^{\mu_j} \frac{z^{n-k-1}\, dz}{\sqrt{-a(z)p(z)}}; \quad \pi_k = \sqrt{\frac{p(\mu_k)}{-4a(\mu_k)}} . \tag{5.15}$$

(d) The Jacobi Inverse Problem.

We have reduced the solution of our problem to the integration of the n-1 integrals in (5.15). They are well studied integrals, generalizing the elliptic integrals

$$\int^{z} \frac{dz}{\sqrt{P(z)}}$$

where P(z) is any cubic polynomial. In our case the denominator contains the polynomial a(z)p(z) of degree 2n-1. For n = 2 we are indeed dealing with elliptic integrals. In general, one has to study the Riemann manifold R of the algebraic function

$$w^2 = - a(z)\ p(z)$$

which is called a hyperelliptic curve of genus n-1. Without going into the extensive theory (see, e.g., H. Weyl, Die Idee der Riemannschen Fläche, Teubner 1955, or C. L. Siegel, Topics in complex function theory II, Wiley-Interscience 1971) we mention that

$$\omega_k = \frac{z^{n-k-1}\, dz}{\sqrt{-a(z)p(z)}}, \qquad k = 1,2,\ldots,n-1,$$

form a basis of the differentials of the first kind on our Riemann surface, and for any set of n-1 points $\mu_1, \mu_2, \ldots, \mu_{n-1}$ one defines the mapping of this set into the (n-1)-vectors with the components

$$s_k = \sum_{j=1}^{n-1} \int^{\mu_j} \omega_k$$

over appropriate paths. This defines a mapping of the symmetric product of n-1 copies of R into C^{n-1}. This mapping is not well defined since closed paths of integration on the right-hand side need not give the zero vector. The possible values for

the above integrals (so-called Abelian sum) for closed paths on the Riemann surface are called periods. They form, in general, an Abelian group Γ with 2n-2 linearly independent generators, and the above mapping actually takes the symmetric product of n-1 copies of R into the torus $C^n/\Gamma = T$ where $\dim_R T = 2 \dim_C T = 2(n-1)$. This torus is called the Jacobi variety of R, and the Jacobi inverse problem asks for the inverse mapping, which has been fully solved, see the above references.

In these variables s_k on the Jacobi variety, we have a straight line motion as follows from (5.15); indeed $\xi_k = \frac{1}{4} s_k$ up to a constant. The linear structure on the Jacobi variety is intimately related to Abel's theorem for hyperelliptic integrals, the generalization of the addition theorem for elliptic functions.

We summarize: The integration of the Neumann problem can be reduced to the above hyperelliptic integrals. As integrals in involution we can take $\eta_1, \eta_2, \ldots, \eta_{n-1}$.

§ 6. Constrained Vectorfields

In this and the following section we will give a different derivation of the integrable character of the geodesic flow on the ellipsoid and the other example of the previous section and obtain explicit expressions for the integrals in question. For this purpose we will construct matrices whose eigenvalues (or their symmetric functions) are the desired integrals so that these matrices undergo iso-spectral deformations, as it was discussed in Section 4. But first we extend the problem for the flow (5.1), (5.2) which is given on the 2n-2 dimensional tangent bundle on the sphere, to a flow in R^{2n}, in such a way that the extended system is also an integrable Hamiltonian system. We will then construct the integrals for the extended problem and show that the constrained integrals are the desired integrals. For this purpose we first discuss how to constrain a Hamiltonian system to a symplectic manifold.

(a) Constrained Hamiltonian Systems

We consider a Hamiltonian system X_H on a symplectic manifold (N,ω) which will be $(R^{2n}, \sum_{j=1}^{n} dy_j \wedge dx_j)$ in our applications. We consider a symplectic submanifold M in N and ask for the "constrained vectorfield" X^* of X_H to the manifold M. In particular, we will require that X^* is tangential to M and gives rise to a Hamiltonian flow on M. One way to do this

is the following: If we denote by j the so-called inclusion map of M into N then $j^*\omega = \omega_M$ is the induced differential form on M. It is nondegenerate, since we assume M to be symplectic. The Hamiltonian function H in N can be restricted to M by $H \cdot j = H_M$, and as usual defines a vectorfield $X^* = X_{H_M}$ in TM by

$$\omega_M(X^*, \cdot) = dH_M .$$

This X^* is the desired "constrained vectorfield" of X_H.

Note that, in general, the vectorfield X_H is not tangent to the submanifold $M \subset N$, i.e. $X_H|M \notin TM$ and therefore $X^* \neq X_H|M$. Only in the special case, where $X_H \in TM$, we have $X^* = X_H$, since then $j^*\omega(X_H,\cdot) = d(H \circ j) = \omega_M(X_H, \cdot)$.

For actual calculation this definition is not very helpful and we give a different description - and which leads to the same result. It suffices to do this locally, and we describe the points in a neighborhood $U \subset N$ by x,y and ω by $\sum_{j=1}^{n} dy_j \wedge dx_j$. The manifold M will locally be give by

$$(6.1) \qquad M \cap U = \{(x,y) \in U \,|\, F_1(x,y) \dots = F_s(x,y) = 0\}$$

where $dF_1, dF_2, \dots, dF_s$ are assumed to be linearly independent.

<u>Lemma 6.1</u> M is symplectic in U if and only if

$$(6.2) \qquad \det(\{F_i, F_j\}) \neq 0 \quad i,j = 1, 2, \dots, s .$$

Proof: We have to check that $E = T_pM$ at any point $p = (x,y) \in M \cap U$ - which we fix - is a symplectic subspace of (R^{2n}, ω), i.e. that

$$E \cap E^{\perp} = (0) ,$$

where $E^{\perp}$ is the orthogonal complement with respect to ω, i.e.

$$E^{\perp} = \{X \in R^{2n} \mid \omega(X,E) = 0\} .$$

From the symmetry of the above condition E is symplectic if and only if $E^{\perp}$ is symplectic. Now $E = TM$ is given by

$$E = \{X \in R^{2n} \mid dF_1(X) = 0, \dots, dF_s(X) = 0\}.$$

If we use the definition of X_{F_j}:

$$\omega(X_{F_j}, Y) = dF_j(Y) = Y(F_j) ,$$

then we see that the X_{F_j} belong to $E^{\perp}$ and since they are linearly independent they span $E^{\perp}$. Thus $E^{\perp}$ is symplectic if and only if the form

$$\omega\left(\sum_{i=1}^{s} \xi_i X_{F_i} , \sum_{j=1}^{s} \eta_j X_{F_j}\right)$$

is nondegenerate, which is the same as the nondegeneracy of the matrix

$$\omega(X_{F_i}, X_{F_j}) = \{F_i , F_j\} .$$

This finishes the proof of the Lemma. Note that (6.2) implies that s is even, say s = 2r.

We use the decomposition

$$R^{2n} = TN = TM \oplus (TM)^{\perp}$$

to decompose X_H into

$$X_H = X_1 + X_2 , \quad X_1 \in TM, \quad X_2 \in (TM)^{\perp}$$

and naturally call X_1 the projection of X_H in TM. It is easily seen that X_1 agrees with X^* described before. In fact, for every $Y \in T_pM$ we have

$$dH(Y) = \omega(X_H, Y) = \omega(X_1 + X_2, Y) = \omega(X_1, Y) ,$$

since $\omega(X_2, Y) = 0$. Therefore, with $\omega_M = j^*\omega$ and $H_M = H \circ j$, hence $dH_M = j^*dH$, we have

$$dH_M(Y) = \omega_M(X_1 , Y)$$

for every $Y \in T_pM$, and since ω_M is nondegenerate we conclude that indeed $X_1 = X^*$.

Since $(TM)^{\perp}$ is spanned by $X_{F_1}, X_{F_2}, \dots, X_{F_s}$, we set

$$X_2 = \sum_{j=1}^{s} \lambda_j X_{F_j}$$

so that

$$X_1 = X_H - \sum_{j=1}^{s} \lambda_j X_{F_j}$$

is tangental to M, i.e.

$$0 = X_1(F_i) = X_H(F_i) - \sum_{j=1}^{s} \lambda_j X_{F_j}(F_i)$$

or

$$(6.3) \qquad 0 = \{H, F_i\} - \sum_{j=1}^{s} \lambda_j \{F_j , F_i\} .$$

These equations uniquely define the functions λ_j on M because of (6 ~~not only on M but~~ in a small neighborhood of M in N. ~~because of (6.2).~~

(We use the equations to define the λ_j even

Therefore, we obtain an effective calculation of the constrained vectorfield X^* on M as follows. If H is the given Hamiltonian, and if the symplectic submanifold $M \subset N$ is represented by

$$M = \{z \in N \mid F_1(z) = 0, \ldots, F_s(z) = 0\}$$

with dF_j linearly independent on M, then we set

$$H^* = H - \sum_{j=1}^{s} \lambda_j F_j \quad , \tag{6.4}$$

where the functions λ_j are ~~uniquely~~ determined by (6.3) in a small neighborhood of M. Then the Hamiltonian vectorfield X_{H^*} defined by $\omega(X_H, \cdot) = dH^*(\cdot)$, and given by

$$X_{H^*} = X_H - \sum_{j=1}^{s} \lambda_j X_{F_j} - \sum_{j=1}^{s} F_j X_{\lambda_j} \quad ,$$

is tangental to the submanifold M, and, on M, it is the desired constrained vectorfield of X_H :

$$X_{H^*} = X^* = X_1 \quad \text{on } M \quad .$$

The function H* and the vectorfield X_{H^*} are not only defined on M but in a neighborhood of M, but X_{H^*} has the desired meaning as a constrained vectorfield only on M.

It is also good to notice that H* and H agree on M,

$$H^* = H \quad \text{on} \quad M \; ,$$

but not their gradients. In fact

$$dH^* = dH - \sum_{j=1}^{s} \lambda_j dF_j - \sum_{j=1}^{s} F_j \, d\lambda_j \quad ,$$

hence if $X \in T_p N$ at a point $p \in M$, then

$$dH^*(X) = dH(X) - \sum_{j=1}^{s} \lambda_j \, dF_j(X) \quad ,$$

since $F_j(p) = 0$. However, if X lies in the tangentspace to the submanifold M, i.e. $X \in T_p M$, then $dH^*(X) = dH(X)$, since now $dF_j(X) = 0$.

~~Clearly outside of M one has the freedom to choose H* as long as $dH^*(X_{F_j}) = 0$ on M, i.e. X_{H^*} is tangental to M, and $dH^*(X) = dH(X)$ for all $X \in TM$.~~

If F is a function on the symplectic manifold $(N = R^{2n}, \omega)$ we denote its restriction onto the symplectic submanifold (M, ω_M) by

$$F_M = F \circ j \quad ,$$

where j is the inclusion mapping $M \to N$. We shall prove

Lemma 6.2

Let (M, ω_M) be a symplectic submanifold of N. Then we have for two functions F, G on N

$$\{F,G\} \circ j = \{F \circ j, G \circ j\}_{\omega_M} = \{F_M , G_M\}_{\omega_M} \quad ,$$

if either X_F or X_G is tangental to the submanifold M.

Proof: Assume X_F is tangential to M, $X_F \in TM$, then $X_F = X_{F_M}$ on M. Furthermore one M we have the unique ~~opening~~

$$X_G = X_G^1 + X_G^2 \quad :$$

$$X_G^1 = X_{G \circ j} \in TM \; , \quad X_G^2 \in (TM)^{\perp} \quad .$$

Hence

$$(F,G)\circ j = \omega(X_F, X_G)\circ j$$

$$= \omega(X_{F_M}, X_{G_M} + X^2_{G'}) = \omega(X_{F_M}, X_{G_M})$$

$$= \{F_M, G_M\}_{\omega_M}$$

as we have claimed.

If H^* is the constrained vectorfield of H, which is tangential to M, we conclude from the Lemma that

$$X_{H^*}(G)\circ j = \{H^*, G\}\circ j =$$

$$\{H^*\circ j, G\circ j\}_{\omega_M} = \{H\circ j, G\circ j\}_{\omega_M}$$

for every function G on N.

(b) Three Examples:

Example 1: Consider a submanifold M_1 in R^n which is given locally by

$$M_1 \cap U = \{x \in U \mid f(x) = 0\},$$

where $df(x) \neq 0$. Then the tangent bundle of M_1 is given by

$$f(x) = 0 \quad , \langle f_x(x), y\rangle = 0$$

which we consider as a submanifold of R^{2n}, with coordinates x, y in R^{2n}. It is a symplectic submanifold with respect to $\omega = \sum dy_j \wedge dx_j$. Indeed, if we set

$$F_1(x) = f(x) \quad , \quad F_2(x) = \langle f_x, y\rangle$$

then

$$\{F_1, F_2\} = |f_x|^2 > 0,$$

and Lemma 6.1 shows that $TM_1 = M$ is a symplectic submanifold of (R^{2n}, ω).

We suggest to constrain the "straight line motion" given by the Hamiltonian

$$H = \frac{1}{2}|y|^2$$

to $TM_1 = M$. We set

$$H^* = H - \lambda_1 F_1 - \lambda_2 F_2$$

and compute λ_1, λ_2 via (6.3). Using

$$\{F_1, H\} = \langle f_x, y\rangle = F_2 = 0 \quad \text{in } M$$

$$\{F_2, H\} = \langle f_{xx}y, y\rangle$$

we obtain

$$\lambda_1 = \frac{\{F_2, H\}}{\{F_2, F_1\}} = -\langle f_{xx}y, y\rangle |f_x|^{-2}, \quad \lambda_2 = 0.$$

Therefore the constrained differential equations become

$$\dot{x} = H^*_y = y \quad ; \quad \dot{y} = -H^*_x = +\lambda_1 f_x$$

or

$$\ddot{x} = \lambda_1 f_x.$$

This equation describes the geodesics on M_1 or the motion of a particle on M_1 without external force. Thus we see that constraining $H = \frac{1}{2}|y|^2$ to the tangent bundle of M_1 gives rise to the geodesic flow on M_1 with respect to the standard metric $|dx|^2$.

Example 2: We constrain

$$(6.5)\qquad H = \frac{1}{2}\left(|x|^2|y|^2 - \langle x,y\rangle^2 + \langle Ax,x\rangle\right)$$

to the tangent bundle M of the unit sphere

$$|x|^2 = 1 \quad , \quad \langle x,y\rangle = 0.$$

We set

$$(6.6)\qquad F_1 = \frac{1}{2}(|x|^2-1) \quad , \quad F_2 = \langle x,y\rangle$$

and, as above,

$$H^* = H - \lambda_1 F_1 - \lambda_2 F_2 \qquad .$$

Then we calculate on M

$$\{F_1,H\} = 0,\ \{F_2,H\} = -\langle Ax,x\rangle \ ,\ \{F_1,F_2\} = |x|^2 = 1$$

giving

$$\lambda_1 = \langle Ax,x\rangle \ ,\ \lambda_2 = 0 \quad .$$

The constrained differential equations are

$$\dot{x} = H^*_y = y,\quad \dot{y} = -H^*_x = -Ax + (-|y|^2 + \lambda_1)\,x$$

or

$$\ddot{x} = -Ax + (-|y|^2 + \lambda_1)x \ .$$

This agrees precisely with the problem (5.1), (5.2) of the previous Section. In other words the mechanical problem of Neumann can be extended to the system with Hamiltonian (6.5) so that constraining of (6.5) to the tangent bundle M of the sphere gives the desired problem. To be sure H is <u>not</u> uniquely determined by this requirement; for example $\langle x,y\rangle^2$ could be dropped without harm since it vanishes with its derivative on M. But we

included this term because then the system (6.5) is an integrable system in (R^{2n}, ω) as we shall see.

Example 3 With a positive definite symmetric matrix A we introduce the bilinear form

$$(6.7)\qquad Q(x,y) = -\langle A^{-1}x,y\rangle \ ,\ Q(x) = Q(x,y)$$

and use the Hamiltonian

$$(6.8)\qquad H = \frac{1}{2}\left(1 + Q(x)\right)Q(y) - Q^2(x,y))$$

and the constraints

$$(6.9)\qquad F_1 = \frac{1}{2}(|y|^2 - 1) = 0,\quad F_2 = Q(x,y) = 0.$$

The last two equations define a symplectic submanifold since

$$\{F_1\ ,\ F_2\} = -Q(y) > 0) \quad .$$

We set

$$H^* = H - \lambda_1 F_1 - \lambda_2 F_2$$

with

$$\lambda_1 = |A^{-1}y|^2(1 + Q(x))\,Q(y)^{-1} - |A^{-1}x|^2 \ ,\ \lambda_2 = 0 \ .$$

The constrained differential equations become

$$\dot{x} = H^*_y = -(1 + Q(x))A^{-1}y - \lambda_1 y$$

$$\dot{y} = -H^*_x = \quad Q(y)\,A^{-1}x \quad .$$

If we restrict ourselves to the energy surface H = 0 then (6.9) implies $1 + Q(x) = 0$ and so the differential equations simplify to

$$\dot{x} = -\lambda_1 y \qquad ,\quad \lambda_1 = -|Ax|^2$$

$$\dot{y} = -Q(y)\,A^{-1}x.$$

Finally, if we use as new parameter s with $\frac{ds}{dt} = -\lambda_1$ then the equations become

$$\frac{dx}{ds} = y \ , \ \frac{dy}{ds} = \lambda_1^{-1} Q(y) A^{-1}x$$

which agrees with the geodesic flow on the ellipsoid (5.3), (5.4), restricted to the unit ~~cotangent bundle $|y|^2 = 1$.~~ tangent bundle of the ellipsoid.

Thus the geodesic flow on the ellipsoid is obtained from (6.8) by the restriction to (6.9) and H = 0, and reparameterization. This extension is somewhat artificial, but we chose it because we will recognize (6.8) as an integrable Hamiltonian.

(c) Constraining an integrable system.

If an integrable system X_H is constrained, then the constrained system is generally not integrable. ~~Included,~~ otherwise, as example 1 shows the geodesic flow on any manifold would be integrable. It is a very special property of a manifold for the geodesic flow to be integrable. We will give sufficient conditions under which the integrable character is preserved under constraining.

For this purpose we begin with a family $\mathcal{F}$ of functions $F \in C^\infty(R^{2n})$ which are in involution and are closed under composition i.e. if F_1, $F_2 \in \mathcal{F}$ then also $\phi(F_1,F_2)$ belongs to $\mathcal{F}$ where ϕ is any C^∞ function *). Moreover, we assume that $\mathcal{F}$ contains n functions, say $F_1, F_2, \ldots F_n$ for which $dF_1, dF_2, \ldots, dF_n$

*) One could consider narrower classes of functions like linear functions, retained functions, etc.

are linearly independent on an open and dense set D of R^{2n}. Then any Hamiltonian $H = H(F_1, F_2, \ldots, F_n)$ belongs to $\mathcal{F}$ and defines an integrable vectorfield X_H on D with $F_1, F_2, \ldots, F_n$ as integrals. More generally any function of $\mathcal{F}$ is an integral of X_H.

Now we restrict X_H to a submanifold

(6.10) $M = \{(x,y) \in R^{2n} \mid F_1 = F_2 = \ldots = F_r = 0, G_1 = G_2 = \ldots G_r = 0\}$

where we assume that

$$(6.11) \quad \begin{cases} \text{(i)} & F_1, F_2, \ldots, F_r \in \mathcal{F} \\ \text{(ii)} & \det\{F_i, G_j\}_{i,j=1,\ldots r} \neq 0 \text{ on } M \end{cases}$$

Clearly this is a special case of the previous situation. In fact, if we set for the moment $\phi_i = F_i$, and $\phi_{i+r} = G_i$, $i = 1,2,\ldots, r$ (note that the functions G_i do not belong to $\mathcal{F}$) then

$$\det\{\phi_i,\phi_j\}_{i,j = 1,2,\ldots 2r} = \left(\det\{F_i,G_j\}_{i,j = 1,2,\ldots,r}\right)^2 > 0$$

on M. Hence the dF_i, dG_j, $i,j = 1,2,\ldots r$ are linearly independent on M and by Lemma 6.1, M is a symplectic submanifold. N is of dimension 2(n-r).

By $\mathcal{F}_M$ we shall denote the family of functions obtained by restriction of the functions in $\mathcal{F}$ onto the submanifold M, i.e.

$$\mathcal{F}_M = \{ F \circ j = F_M \mid F \in \mathcal{F} \} \ ,$$

where j is the inclusion mapping $M \to R^{2n}$. The following Lemma provides a simple device of constraining an integrable system to obtain a new integrable system.

Lemma 6.3 Let M be a symplectic submanifold given by (6.10) satisfying the assumptions (6.11). Then the functions in $\mathcal{F}_M$ are pairwise in involution on (M, ω_M), i.e.

$$\{H_M, G_M\}_{\omega_M} = 0 \text{ , for } H, G \in \mathcal{F}.$$

In particular, if $H \in \mathcal{F}$, then the constrained vectorfield X_{H_M} has $\mathcal{F}_M$ as integrals. Moreover X_{H_M} is an integrable system on (M, ω_M) provided M is contained in D, ~~on which~~ the functions $F_1, \ldots, F_n$ are by assumption linearly independent.

Proof: If $H \in \mathcal{F}$, then the constrained Hamiltonian (6.4) takes the form

$$H^* = H - \sum_{j=1}^{r} (\lambda_j F_j + \mu_j G_j) \quad ,$$

where by (6.3)

$$0 = \{H, F_i\} = - \sum_{j=1}^{r} \mu_j \{G_j , F_i\} ,$$

since both F and F_i $(i = 1,2,\ldots,r)$ are in $\mathcal{F}$. By our assumption (6.11) (ii) we conclude $\mu_j = 0$, so that

$$X_{H^*} = X_{H_M} = X_H - \sum_{j=1}^{r} \lambda_j X_{F_j} .$$

Therefore, if $G \in \mathcal{F}$, then $\{H^*,G\} = 0$ and by Lemma 6.2

$$0 = \{H^*,G\} \circ j = \{H^* \circ j, G \circ j\}_{\omega_M} = \{H_M , G_M\}_{\omega_M} .$$

If $F_1,\ldots,F_r, F_{r+1},\ldots, F_n \in \mathcal{F}$ are linearly independent on D, and if $M \subset D$, then it follows from the above expressions for the constrained vectorfields, that the vectorfields on M:

$$X_{F_k \circ j} \quad , \quad k = r+1,\ldots, n$$

are linearly independent, hence $(F_k \circ j)$; $k = r+1,\ldots, n$ are $(n-r)$ linearly independent functions in $\mathcal{F}_M$, which are in involution, and the Lemma is proved.

Example 4. For $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ we set

$$(6.12) \qquad F_k(x,y) = x_k^2 + \sum_{j=1}^{n}{}' \frac{(x_k y_j - x_j y_k)^2}{\alpha_k - \alpha_j}$$

where the prime indicates that j=k is to be omitted in the summation. We will see in the next section that

$$\{F_k , F_j\} = 0$$

and we can take for $\mathcal{F}$ the class of functions generated by $F_1, F_2 \ldots F_n$.

For instance the functions

$$|x|^2 = \sum_{k=1}^{n} F_k$$

$$|x|^2 |y|^2 - \langle x,y\rangle^2 + \sum_{k=1}^{n} \alpha_k x_k^2 = \sum_{k=1}^{n} \alpha_k F_k$$

belong to $\mathcal{F}$. Hence the Hamiltonian (6.5) with $A = \text{diag}(\alpha_1 , \alpha_2,\ldots, \alpha_n)$ defines an integrable vectorfield X_H with the integrals (6.12). Clearly the dF_k are linearly independent in a punctured neighborhood of the origin, and therefore the set of x and y where one has linear dependence is of lower dimension.

Next we restrict this vectorfield X_H to the tangent bundle M of the unit sphere given by

$$F = \frac{1}{2}(|x|^2 - 1) = 0 \quad , \quad G = \langle x,y \rangle = 0 .$$

Since F, H both belong to $\mathcal{F}$ it follows from Lemma 6.2 that the constrained vectorfield X_{H^*} on M is also integrable and that $F_k|_M$ are integrals in involution for X_{H^*}. Of course, these restricted functions are dependent, since

$$\sum_{k=1}^{n} F_k|_M = |x|^2 = 1 .$$

This shows that the mechanical problem (5.1), (5.2) is integrable with the above rational integrals. In particular we see that the level manifolds

$$\{x,y \mid F_k = c_k\}$$

are algebraic manifolds.

7. Isospectral Deformations

In this section we construct a matrix $L = L(x,y)$ depending on $x,y \in R^n$ such that its eigenvalues remain fixed when $x, y = \dot{x}$ travel on orbits of the problem (5.1), (5.2) of Section 5 or on the geodesics (5.3) of the ellipsoid. Thus these matrices undergo isospectral deformations corresponding to these problems — similarly as we saw previously for the Toda lattice.

(a) Deformation of the Spectrum

Consider the matrix $A = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. For $x,y \in R^n$ we denote the n by n matrix $(x_i y_j)$ by $x \otimes y$, the tensor product of x and y. If $|x| = 1$ the matrix

$$(7.1) \qquad P_x = I - x \otimes x$$

is the projection into the orthogonal complement of x. Indeed, for any $\phi \in R^n$

$$P_x \phi = \phi - x\langle x,\phi \rangle .$$

If x is not normalized we set

$$P_x = P_\xi \quad \text{if} \quad x = \lambda\xi, \quad |\xi| = 1.$$

Now we consider the symmetric matrix

$$(7.2) \qquad L = L(x,y) = P_x(A - y \otimes y)\, P_x ,$$

which depends on $x,y \in R^n$, $x \neq 0$. Clearly $\lambda = 0$ is an eigenvalue of L belonging to the eigenvector x. The remaining eigenvalues will be denoted by $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}$ and

$$\ell_{n-1}(z) = \prod_{j=1}^{n-1} (z-\lambda_j)$$

so that

$$\det (z - L) = z\, \ell_{n-1}(z) .$$

We also set

$$\det (z - A) = a_n(z) = \prod_{j=1}^{n} (z - \alpha_j) .$$

In order to determine the spectrum of L in terms of x,y we introduce the bilinear form

$$(7.3) \qquad Q_z(x,y) = \langle (z - A)^{-1}x,y\rangle \ ; \qquad Q_z(x) = Q_z(x,x)$$

for $z \neq \alpha_j$ and prove

Lemma 7.1. The eigenvalues λ_j of L are determined by the identity

$$(7.4) \quad \frac{|x|^2}{z}\frac{\det (z-L)}{\det (z-A)} = |x|^2 \frac{\ell_{n-1}(z)}{a_n(z)} = Q_z(x)\bigl(1 + Q_z(y)\bigr) - Q_z(x,y)^2$$

i.e. the eigenvalues $\lambda_1,\ldots,\lambda_{n-1}$ are the zeros of the rational function on the right-hand side of (7.4), its poles are the eigenvalues $\alpha_1,\alpha_2,\ldots,\alpha_n$ of A.

Proof: If $\lambda \neq 0$ is an eigenvalue of L and $\phi \perp x$ the corresponding eigenvector then we have $P_x\phi = \phi$ and

$$(L-\lambda)\phi = P_x\bigl(A-\lambda-y \otimes y\bigr)\phi = 0$$

or

$$(A-\lambda)\phi - y\langle y,\phi\rangle = cx$$

Hence,

$$(A-\lambda)\phi + ax + by = 0$$

with some constants a,b where $b = -\langle y,\phi\rangle$. If we assume that $\lambda \neq \alpha_j$ we see that

$$\phi = a(\lambda-A)^{-1}x + b(\lambda-A)^{-1}y$$

and the two conditions

$$\langle x,\phi\rangle = 0$$
$$\langle y,\phi\rangle + b = 0$$

give two linear equations for a,b. In order that a nontrivial solution (a,b) exists we need that

$$\det \begin{pmatrix} Q_\lambda(x) & Q_\lambda(x,y) \\ Q_\lambda(x,y) & 1+Q_\lambda(y) \end{pmatrix} = 0$$

which means that the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}$ are the zeros of this rational function. Its poles are the eigenvalues $\alpha_1,\alpha_2,\ldots,\alpha_n$ of A, so that this determinant is a multiple of

$$\frac{\ell_{n-1}(z)}{a_n(z)} .$$

The coefficient is then easily determined by the behavior as $|z| \to \infty$. This argument is valid only for those x,y for which the λ_j are distinct and different from the α_k. This is the case however on an open dense set and this suffices to identify the two rational functions as we will show after the proof of the following lemma.

Lemma 7.2. The function

$$\Phi_z(x,y) = Q_z(x)\left(1 + Q_z(y)\right) - Q_z(x,y)^2$$

has the partial fraction expansion

$$\Phi_z = \sum_{j=1}^{n} \frac{F_j(x,y)}{z - \alpha_j} \tag{7.5}$$

where $F_j(x,y)$ are the functions (6.12).

Proof. This is a simple calculation: use

$$\Phi_z(x,y) = \sum_{i,j} (z-\alpha_i)^{-1}(z-\alpha_j)^{-1}\left(x_i^2(1+y_j^2) - x_i y_i x_j y_j\right)$$

and write

$$(z-\alpha_i)^{-1}(z-\alpha_j)^{-1} = \frac{1}{\alpha_i-\alpha_j}\left(\frac{1}{z-\alpha_i} - \frac{1}{z-\alpha_j}\right)$$

to get the result.

Now it is easy to see that the $\lambda_j = \lambda_j(x,y)$ are distinct from the α_k on an open and dense set. Indeed otherwise the rational function Φ_z of z would haveless than n poles and at least one of the F_j in (7.5) would vanish. This takes place however only on a lower dimensional set of R^{2n}. Also the $\lambda_j = \lambda_j(x,y)$ are distinct for x,y in an open and dense set of R^{2n}. In fact, if we choose $x = e_n$ and $|y|$ small then the eigenvalues λ_j are close to $\alpha_1,\alpha_2,\ldots,\alpha_{n-1}$, hence distinct.

The basic observation is that the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}$ of $L(x,y)$ together with $|x|^2$ are in involution, provided they are distinct They will serve as the integrals of the isospectral deformations of $L(x,y)$. Actually it is cumbersome to work with the eigenvalues of $L(x,y)$ since they are only algebraic functions which are not well defined if one has multiple eigenvalues and it is more convenient to work with their symmetric functions. From Lemmas 7.1 and 7.2 it is clear that $F_1,F_2,\ldots,F_n$ are expressible in terms of symmetric functions of the eigenvalues of $|x|^2$. Then our statement namely that the λ_j are in involution is equivalent to the previous not yet proven claim that the $F_1,F_2,\ldots,F_n$ are in involution. The proof of this statement will follow from the following consideration.

(b) Iso-spectral Deformations of $L(x,y)$

We consider a Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} \beta_j F_j(x,y) \tag{7.6}$$

with the above F_j defined in (6.12) and some constants $\beta_1,\beta_2,\ldots,\beta_n$. We claim that the vector field X_H defines an isospectral deformation of $L(x,y)$. More precisely, we have

Theorem 7.1. The Hamiltonian system

$$\dot{x} = H_y, \qquad \dot{y} = -H_x$$

with H given by (7.6) can be written in the matrix form

(7.7) $$\frac{d}{dt} L = [B, L] ,$$

where L is given by (4.2) and where

(7.8) $$B = -\left(\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j}\left(x_i y_j - x_j y_i\right)\right) ;$$

here the diagonal elements are to be set equal to 0.

Corollary. The $F_j(x,y)$ are in involution.

Proof of the corollary: First we note that

(7.9) $$X_H(|x|^2) = \{|x|^2, H\} = 0 ,$$

which is equivalent to the fact that $X_\phi(H) = 0$ for $\phi = \frac{1}{2}|x|^2$, which is obvious. In fact the vector field X_ϕ defines the flow $(x,y) \to (x, y+tx)$ and it is evident that the F_k and hence H are invariant under this flow.

Second, we recall that the eigenvalues of L are preserved under the flow for (7.7), hence by the theorem they are preserved under the flow for the Hamiltonian system H, therefore

$$\frac{d}{dt} \ell_{n-1}(z) = \{\ell_{n-1}(z), H\} = X_H(\ell_{n-1}(z)) = 0 ,$$

and hence, with (7.9), for every $z \neq \alpha_j$

$$X_H\left(|x|^2 \frac{\ell_{n-1}(z)}{a_n(z)}\right) = \frac{\ell_{n-1}(z)}{a_n(z)} X_H\left(|x|^2\right) + \frac{|x|^2}{a_n(z)} X_H\left(\ell_{n-1}(z)\right) = 0 .$$

By Lemma 7.1 this agrees with

$$X_H(\phi_z) = 0 ,$$

where $\phi_z = \phi_z(x,y)$ is introduced in Lemma 7.2. By (7.5) this yields

$$X_H(F_j) = \{H, F_j\} = 0 , \quad j = 1,2,\ldots,n,$$

or

$$\sum_{i=1}^{n} \beta_i \{F_i, F_j\} = 0 ,$$

for any choice of β_i, and so $\{F_i, F_j\} = 0$ for $i,j = 1,2,\ldots,n$. This verifies the claim of the corollary.

Proof of Theorem 7.1: This requires a calculation. We write out the differential equation for the Hamiltonian (7.6)

$$\dot{x}_i = H_{y_i} = -\sum_{k=1}^{n} \frac{\beta_i-\beta_k}{\alpha_i-\alpha_k} (x_i y_k - x_k y_i) x_k$$

or

$$\dot{x} = Bx$$

Similarly we compute

$$\dot{y} = -H_x = -\beta x + By , \quad \beta = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n) .$$

In order to derive the matrix equation we compute

$$\frac{d}{dt} x \otimes x = (Bx) \otimes x + x \otimes Bx$$
$$= B(x \otimes x) + (x \otimes x) B^T$$

and since $B^T = -B$

(7.10) $$\frac{d}{dt} x \otimes x = [B, x \otimes x]$$

Similarly, one computes

(7.11) $\frac{d}{dt} y \otimes y = [B, y \otimes y] - \beta x \otimes y - y \otimes \beta x$.

We rewrite this with the help of

(7.12) $[B,A] = \left((\beta_i-\beta_j)(x_i y_j - x_j y_i)\right) = \beta x \otimes y + y \otimes \beta x -(x\otimes\beta y + \beta y\otimes x)$
$- [\beta,(x\otimes y - y \otimes x)]$

as

$$\frac{d}{dt} y \otimes y = [B, y \otimes y - A] - (x \otimes \beta y + \beta y \otimes x) .$$

If we set $A - y \otimes y = N_y$ then we get from (7.11)

(7.13) $\frac{d}{dt} N_y = [B,N_y] + x \otimes \beta y + \beta y \otimes x$.

$$\frac{d}{dt} P_x = - \frac{d}{dt} (x \otimes x) = - [B, x \otimes x] = [B,P_x] .$$

With these identities we can get the differential equation for $L = P_x N_y P_x$:

$$\frac{d}{dt} L = [B,P_x]N_y P_x + P_x [B,N_y] P_x + P_x N_y [B,P_x]$$

In the middle term we used (7.13) and the fact that the last two terms of (7.13) get _illed by the projection P_x. The last equation can be rewritten as

$$\frac{dL}{dt} = [B,P_x N_y P_x] = [B,L]$$

since the commutator acts like a derivation. This verifies Theore 7.1.

It is easy to generalize Theorem 7.1 to a Hamiltonian of the form

$$H = H(F_1,F_2,\dots,F_n)$$

where F_k are defined by (6.12). Indeed since

$$X_H = \sum_{j=1}^{n} \frac{\partial H}{\partial F_j} X_{F_j}$$

and F_j are integrals of the motion the $\partial H/\partial F_j$ are constants along the orbit. Therefore if $H = H(F_1,\dots,F_n)$, then the vector field X_H corresponds to the isospec-ral deformation (7.7) and (7.8) where now

$$\beta_j = 2 \frac{\partial H}{\partial F_j} .$$

(c) Summary

Any Hamiltonian of the form $H = H(F_1,F_2,\dots,F_n)$, in particular,

$$H = \frac{1}{2} \sum_{j=1}^{n} \alpha_j F_j = \frac{1}{2} \left(|x|^2|y|^2 - \langle x,y\rangle^2 + \langle Ax,x\rangle\right)$$

is integrable, if F_j are the functions defined by (6.12). If we denote by $\mathcal{F}$ the family of rational functions generated by $F_1,F_2,\dots, F_n$ the $\mathcal{F}$ is involutory. Moreover, the above Hamiltonian as well as

$$|x|^2 = \sum_{j=1}^{n} F_j$$

belong to $\mathcal{F}$. Therefore we can apply Lemma 6.2 and see that the vector field $X_{H\circ j}$ obtained from X_H by constraining it to the unit tangent bundle M:

$$|x|^2 = 1 , \quad \langle x,y\rangle = 0$$

is integrable and the restriction of the F_j to M form a set of integrals. This fact could, of course, be verified by direct calculation, but we wanted to show the connection of these integrals with the eigenvalues of $L = L(x,y)$.

The geodesic flow on the ellipsoid can be treated in a similar fashion. We return to Example 3 of Section 5. We observe that the Hamiltonian (6.8) agrees with

$$H = \frac{1}{2} \Phi_z(y,x)\Big|_{z=0} = -\frac{1}{2} \sum_{j=1}^{n} \alpha_j^{-1} F_j(y,x)$$

where we had to exchange x and y. Denoting by F^1 the family of polynomials generated by $F_j(y,x)$ we see that $H \in F^{1}$ and also

$$\frac{1}{2} (|y|^2 - 1) \in F^{1}$$

prime

so that the vector field X_{HT} constrained to the symplectic manifold (6.9) is also integrable, with F^{1}/M as integrals. Thus the geodesic flow on the ellipsoid admits the integrals $F_j(y,x)$ when restricted to M and $H = 0$, and is therefore an integrable system. This flow also admits an isospectral deformation with the matrix $L = L(y,x)$ and $\beta_j = -\alpha_j^{-1}$.

(d) The Elliptical Billiard Problem

The geodesic problem, the three-axial ellipsoid,

$$\frac{x_1^2}{\alpha_1} + \frac{x_2^2}{\alpha_2} + \frac{x_3^2}{\alpha_3} = 1 , \quad 0 < \alpha_1 < \alpha_2 < \alpha_3$$

goes into the billiard problem on an elliptical table

(7.14) $$x_1 = 0 , \quad \frac{x_2^2}{\alpha_2} + \frac{x_3^2}{\alpha_3} \le 1$$

if we let α_1 tend to zero. This fact was pointed out previously. We use it here to interpret the integrals

$$F_k(y,x) = y_k^2 + \sum_j \frac{(x_k y_j - x_j y_k)^2}{\alpha_k - \alpha_j}$$

Since $x_1 = 0$ the first integral tends to zero and we are left with

$$F_2 = y_2^2 - \frac{(x_2 y_3 - x_3 y_2)^2}{\alpha_3 - \alpha_2}$$

$$F_3 = y_3^2 + \frac{(x_2 y_3 - x_3 y_2)^2}{\alpha_3 - \alpha_2}$$

so that $F_2 + F_3 = y_2^2 + y_3^2$. Since $x_j = y_j$ the sum of these integrals is twice the kinetic energy which we can normalize to one:

$$F_2 + F_3 = y_2^2 + y_3^2 = 1 .$$

Instead of F_2 or F_3 we consider the integral

(7.15) $$I = \alpha_3 F_2 + \alpha_2 F_3 = \alpha_3 y_2^2 + \alpha_2 y_3^2 - (x_2 y_3 - x_3 y_2)$$

to whose interpretation we turn.

Theorem 7.2. The set $\{x,y \mid I(x,y) = \text{const.} = y_2^2 + y_3^2 = 1\}$ describes the locus of lines

$$x = ty + x(0)\ , \quad x = (x_2,x_3)\ , \quad y = (y_2,y_3)$$

in the plane which are tangential to the confocal ellipse

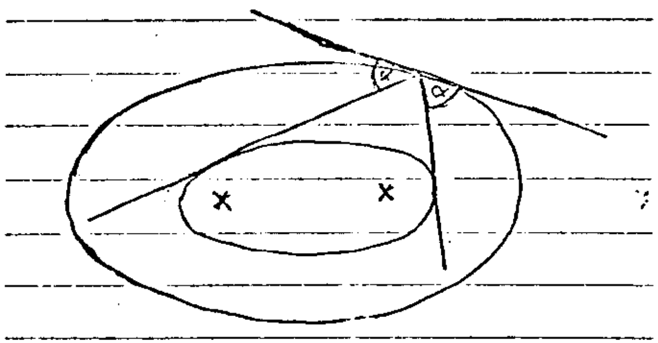
$$\frac{x_2^2}{\alpha_2-\lambda} + \frac{x_3^2}{\alpha_3-\lambda} = 1 \tag{7.16}$$

for $\lambda \neq \alpha_2,\alpha_3$.

This theorem contains the elementary geometrical result that two tangents of a confocal conic section (7.15) intersecting on the ellipse

$$\frac{x_2^2}{\alpha_2} + \frac{x_3^2}{\alpha_3} = 1$$

form the same angles with the tangent to that ellipse.



Proof: We set

$$Q_\lambda(x,y) = \frac{x_2y_2}{\lambda-\alpha_2} + \frac{x_3y_3}{\lambda-\alpha_3}\ ; \quad Q_\lambda(x) = Q_\lambda(x,x)$$

If $x^* = t^*y + x(0)$ is a point of tangency of our line with the quadric $Q_\lambda(x) + 1 = 0$ then we have

$$Q_\lambda(x^*,y) = 0\ , \quad Q_\lambda(x^*) + 1 = 0\ . \tag{7.17}$$

We used the first equation to eliminate t^* setting $x^* = x + t^*y$ to get

$$Q_\lambda(x,y) + t^*Q_\lambda(y) = 0$$

Inserting this into the second equation of (7.17) we obtain (assuming $Q_\lambda(y) \neq 0$) that

$$Q_\lambda(x)\ Q_\lambda(y) - Q_\lambda^2(x,y) + Q_\lambda(y) = 0$$

which, after a short calculation becomes

$$\frac{(x_2y_3 - x_3y_2)^2}{(\lambda-\alpha_2)(\lambda-\alpha_3)} + \frac{y_2^2}{\lambda-\alpha_2} + \frac{y_3^2}{\lambda-\alpha_3} = 0$$

or

$$(x_2y_3 - x_3y_2)^2 - \alpha_3y_2^2 - \alpha_2y_3^2 + \lambda(y_2^2 + y_3^2) = 0$$

or

$$I = \lambda(y_2^2 - y_3^2) = \lambda,$$

as we wanted to show.

(e) Linear Dependence of the $d\omega_j$

So far we assumed that the gradient of the integrals of our system are linearly independent. In that case the level manifolds turned out to be tori in the compact case and the solutions quasiperiodic functions of t. What happens when the gradients of the integrals become linearly dependent? This gives rise to singularities in the foliation of tori and also the quasiperiodic character of the solutions get lost

We will not make a study of this question in generality but give examples of solutions on such singular sets. It turns out that the more remarkable solutions can be found on these singular sets.

We illustrate the different behavior on level sets where the dF_j are linear dependent for the Neumann problem. This admits n-1 integrals, say $F_1, F_2, \ldots, F_{n-1}$ while F_n is determined from

$$\sum_{j=1}^{n} F_j = \sum x_j^2 = 1 .$$

From the form of the functions F_j (see (6.12)) we see that

$$dF_j = 0 \quad \text{if} \quad x_j = y_j = 0 ,$$

so that the gradients are linearly dependent on the subspace

$$E_j = \{x,y \mid x_j = y_j = 0\}$$

All solutions are periodic except those for which

$$2H = \alpha_2, \alpha_1 .$$

For $2H = \alpha_1$ we have a minimum, taken on for $\theta = \pm \frac{\pi}{2} \pmod{2\pi}$ which are stable equilibria.

For $2H = \alpha_2$ we have saddle points at $\theta = 0, \pi \pmod{2\pi}$ and the curves

$$\dot{\theta} = \pm \sqrt{\alpha_2 - \alpha_1} \sin \theta$$

The solution of this equation is not almost periodic. It is easily integrated if one introduces $\tau = \tan \frac{\theta}{2}$. Then

$$x_1 = \sin \theta = \frac{2\tau}{1+\tau^2}$$

$$x_2 = \cos \theta = \frac{1-\tau^2}{1+\tau^2}$$

$$\frac{d\tau}{dt} = \dot{\theta}\, \frac{1+\tau^2}{2} = \pm \sqrt{\alpha_2 - \alpha_1}\, \tau .$$

This gives for the solution

$$x_1 = \cosh^{-1} \sqrt{\alpha_2 - \alpha_1}\, t , \qquad x_2 = \pm \tanh \sqrt{\alpha_2 - \alpha_1}\, t$$

which approach $(x_1, x_2) \to (\pm 1, 0)$ as $t \to \pm \infty$.

We determine the equilibrium solutions of the Neumann problem. They are given by the points on the sphere where the right-hand side of (5.1) vanishes with $\dot{x} = 0$, i.e. where

$$Ax = \langle Ax,x\rangle x \quad \text{on} \quad |x| = 1 .$$

These points are given by $x = \pm e_k$, $k = 1,2,\dots,n$.

In order to determine their stability behavior we use at $x = \pm e_k$ the coordinates x_j, y_j for $j \neq k$ and obtain x_k, y_k from

$$x_k^2 = \sum_{j\neq k} x_j^2 , \quad y_k = - x_k^{-1} \sum_{j\neq k} x_j y_j .$$

Linearization of the differential equations at $\pm e_k$ gives

$$\ddot{x}_j = - (\alpha_j - \alpha_k) x_j \quad \text{for} \quad j \neq k,$$

so that the characteristic exponents at $\pm e_k$ are

$$\pm \sqrt{\alpha_k - \alpha_j} \quad \text{for} \quad j = 1,2,\dots,k-1$$
$$\pm i\sqrt{\alpha_j - \alpha_k} \quad \text{for} \quad j = k+1,\dots,n .$$

This shows that only $\pm e_1$ are stable equilibria. Incidentally for all other $\pm e_k$ $(k \neq 1)$ we have asymptotic orbits connecting e_k with $-e_k$. On the equilibrium points $\pm e_k$ we have

$$F_k = 1 , \quad F_j = 0 \quad \text{for} \quad j \neq k ,$$

and it is interesting to investigate the singular manifolds

$$S_k = \left\{x,y \mid F_k = 1,\ F_j = 0\ (j \neq k),\ \langle x,y\rangle = 0\right\}$$

which contain not only $\pm e_k$ but all orbits asymptotic to these. It turns out that all solutions on S_k can be expressed in terms of exponential functions.

We conclude by characterizing S_k by the spectrum of $L = L(x,y)$, namely S_n is characterized by the property that the nontrivial eigenvalues of L are

$$\lambda_j = \alpha_j \quad \text{for} \quad j = 1,2,\dots,n$$

Indeed, by Lemmas 7.1 and 7.2

$$\frac{\prod_{k=1}^{n-1} (z-\lambda_k)}{\prod_{1}^{n} (z-\alpha_j)} = \frac{\ell_{n-1}(z)}{a_n(z)} = \sum_{j=1}^{n} \frac{F_j(x,y)}{z-\alpha_j} = \frac{1}{z-\alpha_n}$$

which gives the statement

Chapter 3, Section 7 EXERCISES

-1. The matrix $L = L(x,y)$ which is isospectrally deformed by our flow is by no means uniquely determined. We give another example for such a matrix for the Hamiltonian (7.12).

1. Show that

$$X_H(x \otimes y - y \otimes x) = [B, x \otimes y - y \otimes x] + [\beta, x \otimes x]$$

where $\beta = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)$.

2. Using Exercise 1 and Theorem 7.1 show that the matrix

$$L_\varepsilon(x,y) = A - \varepsilon(x \otimes y - y \otimes x) - \varepsilon^2 x \otimes x, \quad \varepsilon \neq 0$$

satisfies the differential equation

$$\frac{d}{dt} L_\varepsilon = [B_\varepsilon, L_\varepsilon] \quad \text{where} \quad B_\varepsilon = B + \varepsilon^{-1}\beta$$

if $\dot{x} = H_y$, $\dot{y} = -H_x$ where H is given by (7.12).

3. Show that

$$\frac{\det(z - L_\varepsilon)}{\det(a - A)} = 1 - \varepsilon^2 \Phi_z(x,y)$$

where $\Phi_z(x,y)$ is the same function as in Lemma 7.2.

This exercise shows that the eigenvalues of L_ε are defined as the roots of

$$\Phi_z(x,y) = \frac{1}{\varepsilon^2}$$

while the eigenvalues of $L = L_0$ are the zeros of $\Phi_z(x,y)$.

4. Verify directly that

$$\{F_i, F_j\} = 0$$

for the function F_j defined in (6.12).

Hint: This is lengthier to calculate, first show that

$$G_k = \sum_j c_{kj}(x_k y_j - x_j y_k)^2$$

are in involution if the constants c_{kj} satisfy

$$c_{ij}c_{kj} + c_{jk}c_{ik} + c_{ki}c_{ji} = 0.$$

which holds for $c_{ij} = (\alpha_i - \alpha_j)^{-1}$. Second, show that

$$x_k^2 + G_k$$

are in involution, if the c_{ij} satisfy in addition $c_{ij} + c_{ji} = 0$.

5. If we set $\alpha_j = \varepsilon^{n-j+1}$ for $j = 1, 2, \ldots, n$, and let $\varepsilon \to 0$, show that

$$\alpha_k F_k \quad \sum_{j=1}^{k-1} (x_k y_j - x_j y_k)^2 = G_k^2 - G_{k-1}^2$$

where G_k stands for the function defined in Section 2.

This proves again that the G_k are in involution.