Generalized Averaging Principle and Proper Elements for NEAs

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Abstract. We present a review of the results concerning a generalization of the classical averaging principle suitable to deal with orbit crossings, that make singular the Newtonian potential at the values of the anomalies corresponding to collisions. These methods have been applied to study the secular evolution of Near Earth Asteroids and to define proper elements for them, that are useful to study the possibility of impact between these asteroids and the Earth.

1 Introduction

The averaging principle is a powerful tool to study the qualitative behavior of the solutions of Ordinary Differential Equations. It consists in solving averaged equations, obtained by an integral average of the original equations over some angular variables; if some conditions are satisfied the solutions of the averaged equations remain close to the solutions of the original equations for a long time span. A review of the classical results on averaging methods in perturbation theory can be found in [1].

These methods have been used to study the secular evolution of the Main Belt Asteroids (MBAs) starting from [26], see [11],[22],[15] .

On the other hand in the case of Near Earth Asteroids (NEAs) the intersections between the orbits of the asteroid and those of the planets generate singularities in the Newtonian potential corresponding to the collision values of the phases on their orbits: in this case the averaged equations have no meaning.

In 1998 Gronchi and Milani [8] have defined piecewise differentiable solutions that can be regarded as solutions of the averaged equations in a weak sense: they solve slightly modified averaged equations in which an inversion of the integral and differential operators occurs. These equations correspond to the classical averaged equations when there are no crossings between the orbits.

The eccentricity and the inclination of the Solar System planets are not considered in this framework: this simplification gives rise to some nice properties, like the periodicity of the solutions of the averaged equations with respect to the perihelion argument, and it allows to prove a stability property [9].

Using the generalized averaging principle Gronchi and Milani [10] computed proper elements and proper frequencies for all the known NEAs using the NEODyS database of orbits (`http://newton.dm.unipi.it/neodys/`). The related catalog is continuously updated according to the discovery of new asteroids and to the changes in the orbits of the known ones: it can be found at the web address

`http://newton.dm.unipi.it/neodys/propneo/catalog.tot`.

The reliability of these solutions has been tested by a comparison with the outputs of pure numerical integrations [7] and the results are quite satisfactory.

Recently this generalized averaging theory has been extended to the eccentric/inclined case for the planets [5]; this will allow to define more reliable crossing times between the orbits, that are useful to detect the possibility of collisions.

In this paper we shall review the classical averaging principle for non crossing orbits and we shall describe in all details the generalization of the principle when a crossing occurs in the case with the planets on circular coplanar orbits. Then we shall present an application of the generalized principle to compute the secular evolution of NEAs and proper elements for them. Finally two short sections are devoted to discussions on the reliability of the averaged orbits and to the recent work that extends the averaging theory including the eccentricity and inclination of the planets.

2 The classical averaging principle

First we shall write canonical equations of motion to compute the time evolution of the orbit of an asteroid. Then we shall describe the averaged equations for the evolution of asteroids that do not cross the orbits of the planets.

2.1 The full equations of motion

Let us consider a Solar System model with the Sun, $N-2$ planets and an asteroid: we assume that the mass of the asteroid is negligible, so that we have a *restricted problem*. We also suppose that the masses of the planets are small if compared with the mass of the Sun, so that we have $N-2$ small perturbative parameters $\mu_i, i = 1 \dots N-2$, corresponding to the ratio of the mass of each planet with the mass of the Sun.

We assume that the motion of the planets is completely determined and that there are no collisions among them or with the Sun. With these assumptions we write the full equations of motion for the asteroid in Hamiltonian form.

We use heliocentric Delaunay's variables for the asteroid, defined by

$$\begin{cases} L = k\sqrt{a} \\ G = k\sqrt{a(1-e^2)} \\ Z = k\sqrt{a(1-e^2)}\cos I \end{cases} \qquad\qquad \begin{cases} \ell = n(t-t_0) \\ g = \omega \\ z = \Omega \end{cases}$$

where $\{a, e, I, \omega, \Omega, \ell\}$ is the set of the Keplerian elements, k is Gauss's constant, n is the mean motion and t_0 is the time of passage at perihelion.

Delaunay's variables, like the Keplerian elements, describe the evolution of the osculating orbit of the asteroid, that is of the trajectory that the asteroid would describe in a heliocentric reference frame, given its position and velocity at a time t, if only the Sun were present. For negative values of the Keplerian energy of the asteroid the osculating orbits are ellipses; we shall consider only such cases.

The Hamiltonian can be written as

$$H = -\frac{k^2}{2L^2} - R$$

where $-k^2/(2L^2)$ is the *unperturbed term*, describing the two body motion of the asteroid around the Sun, and R is the *perturbing function* defined by

$$R = \sum_{i=1}^{N-2} \mu_i R_i \, ; \quad R_i = k^2 \left[\frac{1}{|x - x_i|} - \frac{< x, x_i >}{|x_i|^3} \right] ; \quad i = 1 \ldots N-2 \qquad (1)$$

in which $\langle \, , \rangle$ is the Euclidean scalar product and x and x_i are the position vectors of the asteroid and of all the planets in a heliocentric reference frame.

Note that each R_i is the sum of a *direct term* $k^2/|x - x_i|$, due to the direct interaction between the planet i and the asteroid, and an *indirect term* $-k^2 < x, x_i > /|x_i|^3$, representing the effects on the motion of the asteroid caused by the interaction between the Sun and the planet i.

If we set $\mathfrak{E}_{\mathcal{D}} = (L, G, Z, \ell, g, z)$ we can write Hamilton's equations as

$$\dot{\mathfrak{E}}_{\mathcal{D}} = \mathfrak{J} \left(\nabla_{\mathfrak{E}_{\mathcal{D}}} H \right)^t \qquad (2)$$

where the *dot* means derivative with respect to time, $\mathfrak{J}$ is the 6×6 matrix

$$\begin{bmatrix} \mathcal{O} & -\mathcal{I}_3 \\ \mathcal{I}_3 & \mathcal{O} \end{bmatrix}$$

composed by 3×3 zero and identity matrixes, and

$$\left(\nabla_{\mathfrak{E}_{\mathcal{D}}} H \right)^t = \left(\frac{\partial H}{\partial \mathfrak{E}_{\mathcal{D}}} \right)^t$$

is the transposed vector of the partial derivatives of the Hamiltonian H with respect to $\mathfrak{E}_{\mathcal{D}}$.

2.2 The averaged equations

The classical averaging principle consists in solving equations obtained by the integral average of the right hand side of (2) over the mean anomalies $\ell, \ell_1, .., \ell_{N-2}$ of the asteroid and the planets.

This method can be applied to study the qualitative behavior of the orbits of the MBAs, that do not cross the orbits of the Solar System planets during their evolution, assuming that *no mean motion resonances with low order occur* between the asteroid and the planets in the model. This means that there exists $\epsilon > 0$ not too small and a positive integer M not too large such that for each pair $(a(t), a_i(t))$, composed by the semimajor axes of the osculating orbits of the asteroid and the planet i ($i = 1 \dots N-2$) we have

$$\left| \, p \, [a(t)]^{3/2} - q \, [a_i(t)]^{3/2} \, \right| > \epsilon$$

for each pair of positive integers $p, q \leq M$ and for each t in the considered time span.

Remark 1. As our purpose is not a study of the structure of the mean motion resonances we shall not give further details or any estimates on the size of ϵ and M.

In the expression of the perturbing function (1) the effect of each planet is independently taken into account: each R_i is a function of the coordinates and the masses of the asteroid and one planet only. We shall study the case of only one perturbing planet and we shall use a prime for the quantities related to this planet: the perturbation of all the planets, up to the first order in the perturbing masses μ_i, will be obtained by the sum of the contribution of each planet.

If we consider the reduced set of Delaunay's variables $E_{\mathcal{D}} = (G, Z, g, z)$ the averaged equations for the asteroid can be written in the following form:

$$\dot{\widetilde{E}}_{\mathcal{D}} = -\mathcal{J} \, \overline{\nabla_{E_{\mathcal{D}}} R}^{\,t} \tag{3}$$

where $\widetilde{E}_{\mathcal{D}} = (\tilde{G}, \tilde{Z}, \tilde{g}, \tilde{z})$ are averaged Delaunay's variables, $\mathcal{J}$ is the 4×4 matrix

$$\begin{bmatrix} \mathcal{O} & -\mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{O} \end{bmatrix}$$

composed by 2×2 zero and identity matrixes, and $\overline{\nabla_{E_{\mathcal{D}}} R}^{\,t}$ is the transposed vector of the integral average over (ℓ, ℓ') of the partial derivatives of the perturbing function R with respect to $E_{\mathcal{D}}$

$$\overline{\nabla_{E_{\mathcal{D}}} R} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \nabla_{E_{\mathcal{D}}} R \, d\ell \, d\ell' \; ; \qquad \nabla_{E_{\mathcal{D}}} R = \frac{\partial R}{\partial E_{\mathcal{D}}} \; .$$

Remark 2. As we are considering non–crossing orbits, the derivatives of R with respect to Delaunay's variables are regular functions and we can use the *theorem of differentiation under the integral sign* [3] to exchange the derivatives and the integrals in (3); then the averaged equations take the form

$$\dot{\widetilde{E}}_{\mathcal{D}} = -\mathcal{J} \, (\nabla_{E_{\mathcal{D}}} \overline{R})^{\,t} \tag{4}$$

where

$$\overline{R} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} R \, d\ell \, d\ell' = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\mu k^2}{|x - x'|} \, d\ell \, d\ell' \qquad (5)$$

(μ is the ratio between the mass of the planet and the mass of the Sun) because the average of the indirect term of the perturbing function is zero (see [26]).

Remark 3. We shall skip the 'tilde' over the averaged variables in the following to avoid the use of heavy notations.

We stress that the solutions of (3) are representative of the solutions of the full equations of motion only if there are no mean motion resonances with low order between the asteroid and the planet.

2.3 Difficulties arising with crossing orbits

We say that an asteroid is *planet crossing* if its orbit crosses the orbit of some planet during its secular evolution.

When we consider a planet crossing asteroid at the time of intersection of the orbits, the averaged perturbing function $\overline{R}$ is the integral of an unbounded function that is convergent because $1/|x - x'|$ has a first order polar singularity in the values $\overline{\ell}, \overline{\ell}'$ corresponding to a collision. The derivatives at the right hand side of (3) have second order polar singularities in $\overline{\ell}, \overline{\ell}'$, hence equations (3) do not make sense in this case because the integrals over ℓ, ℓ' of these derivatives are divergent and *the classical averaging principle cannot be applied.*

3 Generalized averaging principle in the circular coplanar case

We present the ideas of the generalization of the averaging principle to the case of crossing orbits, assuming that all the planets in the model have *circular and coplanar orbits* (see [13]) and that no low order mean motion resonances are present.

The natural choice for a heliocentric reference frame is then a system $Oxyz$ with the (x, y)-plane corresponding to the common orbital plane of all the planets, oriented in such a way that the planets have positive z component of the angular momentum with respect to the origin O.

3.1 Geometry of the node crossing

We assume that *the inclination between the osculating orbit of the asteroid with respect to the orbital plane of the planets is different from zero during its* *whole evolution*; then it is possible to define, for all times, the *mutual nodal line*, representing the intersection of the two orbital planes of the asteroid and the planets.

Let us consider one planet at a time: we give the following

Definition 1. We call *mutual node* each pair of points on the mutual nodal line, one belonging to the orbit of the asteroid and the other to the one of the planet, that lie on the same side of the mutual nodal line with respect to the common focus of the two conics. For each planet in this model there are two mutual nodes, the *ascending* and the *descending* one: they differ in the change of sign of the z component along the asteroid orbit (negative to positive in the first case and vice-versa in the second).

We say that an *ascending (resp. descending) node crossing* occurs when the orbit of the asteroid intersects the orbit of a planet at the ascending (resp. descending) mutual node, that in this case becomes a set of two coinciding points.

Unless the inclination of the asteroid never vanishes, the only way to have orbital intersection is a node crossing. In the following we give a description of the possible geometric configurations of node crossings in the plane $(e\cos\omega, e\sin\omega)$.

Recall that an ascending and descending node crossing with a planet whose orbit has semimajor axis a'_i is characterized by the vanishing of the following expressions respectively:

$$d^+_{nod}(i) = \frac{a(1-e^2)}{1+e\cos\omega} - a'_i\,; \qquad d^-_{nod}(i) = \frac{a(1-e^2)}{1-e\cos\omega} - a'_i \tag{6}$$

that are called *nodal distances* (and can be negative).

In the averaged problem with the planets on circular coplanar orbits we have three integrals of motions: the semimajor axis a, the *Kozai integral* $\overline{H} = H_0 - \overline{R}$ (that is the averaged Hamiltonian) and the *z-component of the angular momentum* $Z = k\sqrt{a(1-e^2)}\cos I$. The Z integral allows to determine the evolution of $I(t)$ if we know $e(t)$; if we also know $\omega(t)$ we can determine $\Omega(t)$ by a simple quadrature of $\partial\overline{R}/\partial Z$, that does not depend on Ω. From the expression of the integral Z we deduce the maximum value of the averaged inclination and eccentricity:

$$I_{max} = I\big|_{e=0} = \arccos\frac{Z}{k\sqrt{a}}\,; \qquad e_{max} = e\big|_{I=0} = \frac{\sqrt{k^2a - Z^2}}{k\sqrt{a}}\,.$$

For a given value of the semimajor axis a we can represent the level lines of the averaged Hamiltonian, on which the averaged solutions evolve, in the plane $(\xi,\eta) := (e\cos\omega, e\sin\omega)$. We define the *Kozai domain*

$$W = \{(\xi,\eta) : \xi^2 + \eta^2 \le e^2_{max}\}\,,$$

where the averaged dynamics is confined.

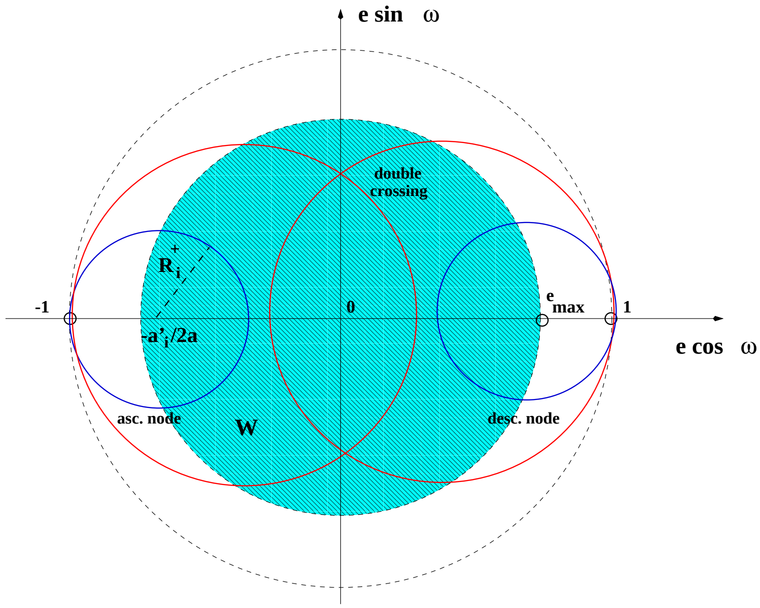

Fig. 1. The Kozai domain $\{(e\cos\omega, e\sin\omega) : 0 \leq e \leq e_{max}, \omega \in \mathbb{R}\}$ is represented in the figure by the set W. We plot also the four circles corresponding to ascending and descending node crossing with two planets (they have their centers shifted respectively on the left and on the right). An additional exterior circle corresponding to the boundary for closed orbits ($e = 1$) is drawn.

In the (ξ, η) reference plane the node crossing lines with the planets are circles: they are defined by

$$\Gamma^+(i) = \{(\xi,\eta) : d^+_{nod}(i) = 0\}\,; \qquad \Gamma^-(i) = \{(\xi,\eta) : d^-_{nod}(i) = 0\}$$

where i is the index of the planet.

At the ascending node crossing with the planet i the equation to be considered is

$$1 - \xi^2 - \eta^2 = \frac{a'_i}{a}(1+\xi)\ .$$

After the coordinate change $X = \xi + a'_i/(2a);\ Y = \eta$ we obtain

$$X^2 + Y^2 = \left(1 - \frac{a'_i}{2a}\right)^2\ ,$$

that is, in the (ξ,η)-plane, the equation of a circle of radius $R^+_i = 1 - a'_i/(2a)$, with center in $(\xi_+, \eta_+) = \big(-a'_i/(2a), 0\big)$ (see Figure 1).

By the previous calculations we have

$$\begin{cases} d^+_{nod}(i) > 0 & \text{inside } \Gamma^+(i) \\ d^+_{nod}(i) < 0 & \text{outside } \Gamma^+(i)\ . \end{cases}$$

In a similar way we can prove that the equation $d^-_{nod}(i) = 0$ represents a circle of radius $R^-_i = R^+_i = 1 - a'_i/(2a)$, with center in $(\xi_-, \eta_-) = \big(+a'_i/(2a), 0\big)$.

Definition 2. A *double (node) crossing* is a crossing between the orbit of the asteroid and the orbit of a planet at both the ascending and descending node.

By the symmetry of the circles $\{d^+_{nod}(i) = 0\}$ and $\{d^-_{nod}(i) = 0\}$, for each index i, we can deduce that a double crossing is possible only when $\omega = \pi/2$ or $\omega = 3\pi/2$ (see Figure 1). We obtain the following condition on the ratio of the semimajor axes a, a'_i:

$$-\frac{a'_i}{2a} \geq -\frac{1}{2} \qquad \text{that is} \qquad a \geq a'_i \ , \tag{7}$$

and in particular we obtain that *there cannot exist Aten asteroids* (see the article by A. Celletti, Chapter????) *that have a double crossing with the Earth.*

Definition 3. A *simultaneous crossing* is a crossing of the orbit of the asteroid and the orbits of two planets at the same time.

We note that if we call a'_1, a'_2 the semimajor axes of the orbits of two different planets, we cannot have a simultaneous crossing at the ascending node of both planets (this would imply $a'_1 = a'_2$ in this model). By a similar argument we cannot have a simultaneous crossing at the descending node of both planets. On the other hand we can have a simultaneous crossing at the ascending node with one planet and at descending node with the other one if

$$a'_1 = \frac{a(1-e^2)}{1+e\cos\omega} \qquad \text{and} \qquad a'_2 = \frac{a(1-e^2)}{1-e\cos\omega} \ ,$$

that is if

$$\frac{a'_1}{a'_2} = \frac{1-e\cos\omega}{1+e\cos\omega} \ .$$

In this framework we cannot have crossings of different type from the ones presented above (like *triple crossing*, etc.).

3.2 Description of the osculating orbits

We consider a model with three bodies only: Sun, planet, asteroid. We set the x axis along the line of the nodes, pointing towards the ascending mutual node. The equations defining the osculating orbits $P(u) = (p_1(u), p_2(u), p_3(u))$ and $P'(u') = (p'_1(u'), p'_2(u'), p'_3(u'))$ of the asteroid and the planet are

$$\begin{cases} p_1 = a[(\cos u - e)\cos\omega - \beta\sin u\sin\omega] \\ p_2 = a[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\cos I \\ p_3 = a[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\sin I \end{cases} \qquad \begin{cases} p'_1 = a'\cos u' \\ p'_2 = a'\sin u' \\ p'_3 = 0 \end{cases} \tag{8}$$

where u, u' are the eccentric anomalies and $\beta = \sqrt{1-e^2}$. These orbits are respectively an ellipse and a circle.

The distance between a point on an orbit and a point on the other one, appearing at the denominator of the direct term of the perturbing function, is defined by its square as

$$\begin{aligned}\mathfrak{D}^2(u,u') &= (p_1 - p_1')^2 + (p_2 - p_2')^2 + (p_3 - p_3')^2 = \\ &a^2(1-e\cos u)^2 + a'^2 - 2aa'\{\cos u'[(\cos u - e)\cos\omega - \beta\sin u\sin\omega] + \\ &+ \sin u'\cos I[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\}\ .\end{aligned}$$

We introduce the function $\mathrm{D}(\ell,\ell')$, which is implicitly defined by

$$\mathrm{D}\left(\ell(u), \ell'(u')\right) = \mathfrak{D}(u,u') \tag{9}$$

and by Kepler's equations

$$\ell = u - e\sin u\,; \qquad\qquad \ell' = u' \tag{10}$$

for the asteroid and the planet (the latter has a simpler form because the orbit of the planet is circular).

We define the values of the anomalies $\overline{u}, \overline{u}'$ corresponding to the mutual ascending node: we immediately notice that $\overline{u}' = 0$, while from

$$a(1 - e\cos\overline{u}) = \frac{a\beta^2}{1 + e\cos\omega} \tag{11}$$

we obtain

$$\cos\overline{u} = \frac{\cos\omega + e}{(1 + e\cos\omega)}; \qquad \sin\overline{u} = -\frac{\beta\sin\omega}{(1 + e\cos\omega)}$$

(the sign of $\sin\overline{u}$ has been chosen in such a way that it is opposite to the sign of $\sin\omega$).

The equations defining the anomalies $\overline{u}_1, \overline{u}_1'$, corresponding to the mutual descending node, are

$$\overline{u}_1' = \pi\,; \qquad \cos\overline{u}_1 = \frac{e - \cos\omega}{(1 - e\cos\omega)}\,; \qquad \sin\overline{u}_1 = \frac{\beta\sin\omega}{(1 - e\cos\omega)}\ .$$

In the following we shall study only ascending node crossings, but the same methods are suitable to deal also with the descending ones and even with double crossings (see [6],[10]).

3.3 Weak averaged solutions

The idea of the generalization of the averaging principle comes from Remark 2: if there are no crossings between the orbits, then the averaged equations of motion (3) are equivalent to equations (4).

We write equations (4) in a more explicit form:

$$\begin{cases} \dot{\overline{G}} = \dfrac{\partial \overline{R}}{\partial g} \\ \dot{\overline{Z}} = \dfrac{\partial \overline{R}}{\partial z} = 0 \end{cases} \qquad \begin{cases} \dot{\overline{g}} = -\dfrac{\partial \overline{R}}{\partial G} \\ \dot{\overline{z}} = -\dfrac{\partial \overline{R}}{\partial Z} \,; \end{cases} \tag{12}$$

the equation $\dot{\overline{Z}} = 0$ holds because the equations of the orbits do not depend on the longitude of the node Ω.

We shall prove that when the orbits intersect each other it is possible to define piecewise smooth solutions of equations (12), that we call *weak averaged solutions*, and we shall see that the loss of regularity corresponds exactly to the crossing configurations of the orbits: in fact we shall give a twofold meaning to the right hand sides of (12) at the node crossing, corresponding to the two limit values of the derivatives coming from inside and outside the circle representing the ascending node crossing with the planet in the plane (ξ, η).

Note that the weak averaged solutions correspond to the classical averaged solutions as far as their trajectories in the reduced phase space (ξ, η) do not pass through a node crossing line.

We also observe that the exchange of the differential and integral operators in (12) is not essential for a theoretical definition of the weak solutions (they could anyway be defined as the limits of the solutions of (3) coming from both sides of the node crossing lines) but, as we shall see, this operation is necessary to obtain analytic formulas for the discontinuity of the average of the derivatives of R, that are not defined on the node crossing lines, and to define the semianalytic procedure to compute the weak solutions.

3.4 The Wetherill function

Let $\{P(\overline{u}), P'(\overline{u}')\}$ be the ascending mutual node. We consider the two straight lines $\mathrm{r}(\ell)$ and $\mathrm{r}'(\ell')$, tangent in $P(\overline{u})$ and $P'(\overline{u}')$ to the orbits of the asteroid and of the planet (see Figure 2); they can be parametrized by the mean anomalies ℓ, ℓ' so that $P(u(t))$ and $\mathrm{r}(\ell(t))$ have the same velocities (derivatives with respect to t) in $P(\overline{u})$ and $P'(u'(t))$ and $\mathrm{r}'(\ell'(t))$ have the same velocities in $P'(\overline{u}')$:

$$\begin{cases} r_1 = \overline{x} - \mathcal{F}(\ell - \overline{\ell}) \\ r_2 = \overline{y} + \mathcal{G}\cos I(\ell - \overline{\ell}) \\ r_3 = \overline{z} + \mathcal{G}\sin I(\ell - \overline{\ell}) \end{cases} \qquad \begin{cases} r_1' = \overline{x}' \\ r_2' = \overline{y}' + a'(\ell' - \overline{\ell}') \\ r_3' = \overline{z}' \end{cases} \tag{13}$$

where $\overline{\ell}, \overline{\ell}'$ are the values of the mean anomalies corresponding to $\overline{u}, \overline{u}'$ (so that $\overline{\ell}' = 0$). We have used the following notations

$$\mathcal{F} = \frac{ae\sin\omega}{\beta}\,; \qquad \mathcal{G} = \frac{a(1 + e\cos\omega)}{\beta}$$

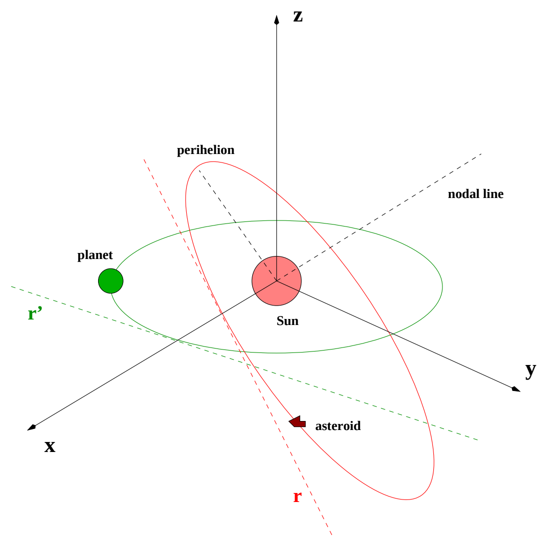

Fig. 2. The straight lines r, r′ represent Wetherill's approximation at the ascending node for the two osculating orbits of the asteroid and the planet.

and

$$\overline{x} = \frac{a\beta^2}{1 + e\cos\omega}; \qquad \overline{x}' = a'; \qquad \overline{y} = \overline{z} = \overline{y}' = \overline{z}' = 0\ .$$

Definition 4. We call *Wetherill function* the approximated distance function d, whose square is defined by

$$\begin{aligned} \mathrm{d}^2(\ell, \ell') &= (r_1 - r_1')^2 + (r_2 - r_2')^2 + (r_3 - r_3')^2 = \\ &= a'^2 k'^2 + \frac{a^2(1 + 2e\cos\omega + e^2)}{\beta^2} k^2 - 2kk'[\mathcal{G}a'\cos I] - 2d_{nod}^{+}\mathcal{F}k + (d_{nod}^{+})^2 \end{aligned}$$

with $k = \ell - \overline{\ell}, k' = \ell'$.

Note that d^2 is a quadratic form in the variables k, k': it is homogeneous when there is a crossing at the ascending node. We can write it more concisely as

$$\mathrm{d}^2(\ell, \ell') = d^2(\kappa) = \kappa^t \mathcal{A}\, \kappa + B^t \kappa + (d_{nod}^{+})^2$$

where

$$\kappa = (k', k)\ ; \qquad B = 2(B_1, B_2)\ ; \qquad \mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\ ;$$

with components

$$\begin{cases} B_1 = 0 \\ B_2 = -d_{nod}^{+}\mathcal{F} \end{cases} \qquad \begin{cases} A_{11} = a'^2 \\ A_{12} = A_{21} = -\mathcal{G}a'\cos I \\ A_{22} = [\mathcal{F}^2 + \mathcal{G}^2]\ . \end{cases}$$

For later use we define

$$\mathfrak{d}^2(u,u') = \mathrm{d}^2(\ell(u),\ell'(u'))\ .$$

The geometry of Wetherill's straight lines is strictly related to the degeneracy of the matrix $\mathcal{A}$, in fact we have

Lemma 1. *The matrix $\mathcal{A}$ is always* positive definite *if $I > 0$. If $I = 0$ we have degeneracy of $\mathcal{A}$ if and only if the straight lines* r, r' *are parallel: in this case $\mathcal{A}$ is* positive semi-definite.

Proof. $\mathcal{A}$ is a symmetric 2×2 matrix and it is positive definite if and only if its principal invariants, the trace $tr(\mathcal{A})$ and the determinant $det(\mathcal{A})$, are positive. By a direct computation we have

$$\begin{cases} tr(\mathcal{A}) = a'^2 + a^2\dfrac{(1+2e\cos\omega+e^2)}{1-e^2} \\ det(\mathcal{A}) = \dfrac{a^2a'^2}{(1-e^2)}\Big\{(1+e\cos\omega)^2\sin^2 I + e^2\sin^2\omega\Big\}\ . \end{cases}$$

From the above expressions we deduce that $tr(\mathcal{A}) > 0$ (we are considering only bounded orbits, so that $0 \le e < 1$); furthermore

$$det(\mathcal{A}) = 0 \quad \Longleftrightarrow \quad \begin{cases} I = 0 \\ e\sin\omega = 0\ , \end{cases}$$

that corresponds to the straight lines r, r' being parallel.

Definition 5. We call *tangent crossings* the crossing orbital configurations for which $det(\mathcal{A}) = 0$.

The assumption that the inclination I of the asteroid is different from zero during its whole time evolution implies that *no tangent crossings occur.*

3.5 Kantorovich's method

We shall describe Kantorovich's method of singularity extraction (see [2]) that allows to improve the stability of the numerical computation of the integrals when the integrand function $f_1(x)$ is unbounded in the neighborhood of one or more points.

Kantorovich's method consists in searching for a function $f_2(x)$ whose primitive has an analytic expression in terms of elementary functions and such that the difference $f_1(x) - f_2(x)$ is more regular than $f_1(x)$ (for example it is bounded or even continuous).

It is then convenient to split the computation as follows

$$\int f_1(x)\,dx = \int [f_1(x) - f_2(x)]\,dx + \int f_2(x)\,dx$$

so that the singularity has moved to the second term, that can be better handled.

This method can help us to study the regularity properties of the averaged perturbing function $\overline{R}$ defined in (5); we shall use the inverse of the Wetherill function $1/\mathrm{d}$ to extract the principal part from the direct term of the perturbing function.

The function D is 2π-periodic in both variables ℓ, ℓ' and this property can be used to shift the integration domain

$$\mathbb{T}^2 = \{(\ell,\ell') : -\pi \leq \ell \leq \pi, -\pi \leq \ell' \leq \pi\}$$

in a suitable way, so that the crossing values $(\overline{\ell}, 0)$ will be always internal points of this domain.

We shall prove that in computing the derivatives of $\overline{R}$ with respect to the variables $E_{\mathcal{D}}$, for instance the G-derivative, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2}\frac{\partial}{\partial G}\overline{R} = \int_{\mathbb{T}^2}\frac{\partial}{\partial G}\left[\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}}\right] d\ell\, d\ell' + \frac{\partial}{\partial G}\int_{\mathbb{T}^2}\left[\frac{1}{\mathrm{d}}\right] d\ell\, d\ell'; \tag{14}$$

namely we shall prove the validity of the hypotheses of the theorem of differentiation under the integral sign to exchange the symbols of integral and derivative in front of the *remainder function* $1/\mathrm{D} - 1/\mathrm{d}$. The average of the remainder function is then differentiable as it is derivable with continuity with respect to all the variables $E_{\mathcal{D}}$. Therefore we shall need only to study the regularity properties of the last term of the sum in (14), which is easier to handle.

Note that we use Kantorovich's method of singularity extraction in a wider extent: the derivatives of the remainder function still have a polar singularity in $(\overline{\ell}, 0)$, but it is of order one, so that the integrals over ℓ, ℓ' of these derivatives are convergent.

3.6 Integration of $1/d$

We shall discuss the analytic method to integrate $1/d$ over the unshifted domain $\mathbb{T}^2 = \{(\ell,\ell') : -\pi \leq \ell \leq \pi, -\pi \leq \ell' \leq \pi\}$, assuming that $(\overline{\ell}, 0)$ is an internal point of this domain.

We move the ascending node crossing point $(\overline{\ell}, 0)$ to the origin of the reference system by the variable change

$$\tau_{\overline{\ell},0} : \ (\ell,\ell') \longrightarrow (k,k') \tag{15}$$

and we set

$$\overline{\mathbb{T}}^2 = \tau_{\overline{\ell},0}\left[\mathbb{T}^2\right] = \{(\ell - \overline{\ell}, \ell') : (\ell,\ell') \in \mathbb{T}^2\} \ .$$

Then we perform another variable change to eliminate the linear terms in the quadratic form $d^2(\kappa)$ defined by (14). The inverse of the transformation used for this purpose is

$$\Xi^{-1} : \psi \longrightarrow \kappa = \mathcal{T}\,\psi + S \tag{16}$$

where $S = (S_1, S_2) \in \mathbb{R}^2$, $\psi = (y', y) \in \mathbb{R}^2$ are the new variables and $\mathcal{T}$ is a 2×2 real-valued invertible matrix.

Setting to zero the coefficients of the linear terms of the quadratic form in the new variables ψ we obtain the equations

$$2\,\mathcal{A}\,S + B = 0 \tag{17}$$

whose solutions are

$$S_1 = \frac{B_2 A_{12}}{det(\mathcal{A})}\,; \qquad\qquad S_2 = -\frac{B_2 A_{11}}{det(\mathcal{A})}\ .$$

We can choose the matrix $\mathcal{T}$ such that

$$\mathcal{T}^t \mathcal{A} \mathcal{T} = \mathcal{I}_2$$

($\mathcal{I}_2$ is the 2×2 identity matrix) by setting

$$\mathcal{T} = \begin{pmatrix} 1/\tau & -\sigma/\tau\rho \\ 0 & 1/\rho \end{pmatrix}$$

with

$$\tau = \sqrt{A_{11}}; \qquad \rho = \sqrt{\frac{det(\mathcal{A})}{A_{11}}}; \qquad \sigma = A_{12}\sqrt{\frac{1}{A_{11}}}\ .$$

The coordinate change

$$\Xi : \kappa \longrightarrow \psi = \mathcal{R}\,[\kappa - S]\ , \tag{18}$$

where

$$\mathcal{R} = \mathcal{T}^{-1} = \begin{pmatrix} \tau & \sigma \\ 0 & \rho \end{pmatrix}\ ,$$

brings $d^2(\kappa)$ into the form

$$d^2\left(\Xi^{-1}(\psi)\right) = y^2 + y'^2 + (d^+_{min})^2$$

in the new variables ψ, with

$$d^+_{min} = |d^+_{nod}| \left\{ 1 - \frac{a'^2 \mathcal{F}^2}{det(\mathcal{A})} \right\}^{1/2}\ . \tag{19}$$

The domain $\overline{\mathbb{T}}^2$ is transformed into a parallelogram with two sides parallel to the y' axis (see Figure 3).

Remark 4. Note that d^+_{min} is the minimal distance between the straight lines r and r'.

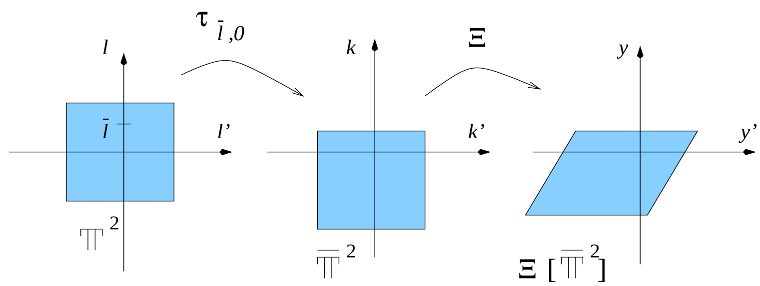

Fig. 3. Description of the transformations of the integration domain $\mathbb{T}^2$ with the two coordinate changes (15), (16) used to bring the squared Wetherill function $\mathrm{d}^2(\ell,\ell')$ into the form $y^2+y'^2+(d^+_{min})^2$ in the new variables (y',y). Note that $\overline{\ell}'=0$ implies that $\overline{\mathbb{T}}^2$ is symmetric with respect to the k axis.

Using the variable changes (15), (18) and the transformation to polar coordinates, whose inverse is

$$\Pi^{-1}:\begin{pmatrix} r\\ \theta\end{pmatrix}\longrightarrow\begin{pmatrix} y'\\ y\end{pmatrix}=\begin{pmatrix} r\cos\theta\\ r\sin\theta\end{pmatrix}$$

we obtain

$$\begin{aligned}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell\,d\ell' &= \frac{1}{\sqrt{det(\mathcal{A})}}\int_{\Xi[\overline{\mathbb{T}}^2]}\frac{1}{\sqrt{y^2+y'^2+(d^+_{min})^2}}\,dy\,dy' = \\ &= \frac{1}{\sqrt{det(\mathcal{A})}}\int_{\mathfrak{T}}\frac{r}{\sqrt{r^2+(d^+_{min})^2}}\,dr\,d\theta\end{aligned}\tag{20}$$

where $\Xi^{-1}\left[\Pi^{-1}(\mathfrak{T})\right]=\overline{\mathbb{T}}^2$.

Let us describe the domain $\mathfrak{T}$ in details. We define the straight lines that bound the integration domain $\Xi\left[\overline{\mathbb{T}}^2\right]$ as

$$\mathrm{r}_1=\{(y,y'):y'=\frac{\sigma}{\rho}y+\tau(\pi-S_1)\}\,;\qquad \mathrm{r}_2=\{(y,y'):y=\rho(\pi-\overline{\ell}-S_2)\}\,;$$
$$\mathrm{r}_3=\{(y,y'):y'=\frac{\sigma}{\rho}y-\tau(\pi+S_1)\}\,;\qquad \mathrm{r}_4=\{(y,y'):y=-\rho(\pi+\overline{\ell}+S_2)\}\,.$$

The intersections of these lines with the y axis are

$$y_1=-\rho(\pi+\overline{\ell}+S_2)\,;\qquad y_2=\rho(\pi-\overline{\ell}-S_2)\,;$$

while the intersections with the y' axis are

$$\begin{aligned}y'_1&=\lambda_3(y_1)=-\sigma(\pi+\overline{\ell}+S_2)-\tau(\pi+S_1)\\ y'_2&=\lambda_3(0)=-\tau(\pi+S_1)\end{aligned}$$

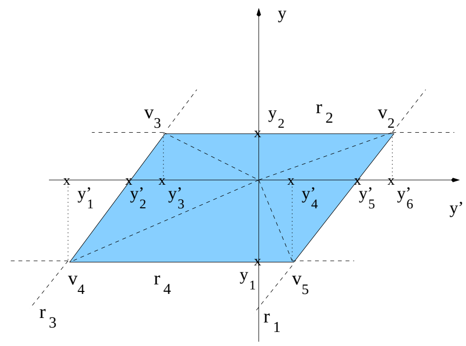

Fig. 4. We show the decomposition of the integration domain used to compute the last integral in (20) in polar coordinates

$$\begin{aligned} y_3' &= \lambda_3(y_2) = \sigma(\pi - \overline{\ell} - S_2) - \tau(\pi + S_1) \\ y_4' &= \lambda_1(y_1) = -\sigma(\pi + \overline{\ell} + S_2) + \tau(\pi - S_1) \\ y_5' &= \lambda_1(0) = \tau(\pi - S_1) \\ y_6' &= \lambda_1(y_2) = \sigma(\pi - \overline{\ell} - S_2) + \tau(\pi - S_1) \end{aligned}$$

where $\lambda_1(y) = (\sigma/\rho)y + \tau(\pi - S_1)$ and $\lambda_3(y) = (\sigma/\rho)y - \tau(\pi + S_1)$.

We can then decompose the domain $\mathfrak{T}$ into four parts (see Figure 4)

$$\mathfrak{T} = \bigcup_{j=1}^{4} \big\{(r,\theta) \in \mathbb{R}^2 \ : \ \theta_j \leq \theta \leq \theta_{j+1} \text{ and } \ 0 \leq r \leq r_j(\theta)\big\}$$

where $r_j(\theta)$, with $j = 1 \ldots 4$, represent the lines r_j delimiting $\Xi[\overline{\mathbb{T}}^2]$ in polar coordinates:

$$r_1(\theta) = \frac{\rho\tau(\pi - S_1)}{\rho\cos\theta - \sigma\sin\theta}\,; \qquad r_2(\theta) = \frac{\rho(\pi - \overline{\ell} - S_2)}{\sin\theta}\,;$$

$$r_3(\theta) = \frac{-\rho\tau(\pi + S_1)}{\rho\cos\theta - \sigma\sin\theta}\,; \qquad r_4(\theta) = \frac{-\rho(\pi + \overline{\ell} + S_2)}{\sin\theta}\,;$$

while $\theta_1 = \theta_5 - 2\pi$ and θ_l, with $l = 2 \ldots 5$, are the counter-clockwise angles between the y' axis and the vertexes v_l seen from the origin of the axes (see Figure 4):

$$0 < \theta_2 < \theta_3 < \pi < \theta_4 < \theta_5 < 2\pi\,;$$

$$\tan\theta_2 = \frac{\rho(\pi - \overline{\ell} - S_2)}{\sigma(\pi - \overline{\ell} - S_2) + \tau(\pi - S_1)}; \ \ \tan\theta_3 = \frac{\rho(\pi - \overline{\ell} - S_2)}{\sigma(\pi - \overline{\ell} - S_2) - \tau(\pi + S_1)};$$

$$\tan\theta_4 = \frac{\rho(\pi + \overline{\ell} + S_2)}{\sigma(\pi + \overline{\ell} + S_2) + \tau(\pi + S_1)}; \ \ \tan\theta_5 = \frac{\rho(\pi + \overline{\ell} + S_2)}{\sigma(\pi + \overline{\ell} + S_2) - \tau(\pi - S_1)}\ .$$

Using the previous decomposition for $\mathfrak{T}$ and integrating in the r variable the last expression in (20) we obtain

$$\int_{\mathbb{T}^2} \frac{1}{\mathrm{d}}\, d\ell\, d\ell' = \frac{1}{\sqrt{det(\mathcal{A})}} \cdot \left\{ \sum_{j=1}^{4} \int_{\theta_j}^{\theta_{j+1}} \sqrt{\left(d^+_{min}\right)^2 + r_j^2(\theta)}\, d\theta - 2\pi d^+_{min} \right\} . \tag{21}$$

Note that the integrals in (21) are elliptic and the integrand functions are bounded so that these integrals are differentiable functions of the orbital elements. We shall see that the loss of regularity of the averaged perturbing function is due only to the term d^+_{min}.

3.7 Boundedness of the remainder function

When there is a crossing at the ascending node, then from the equations of the orbits (8) and from Kepler's equations (10) we deduce that Taylor's development of $\mathcal{D}^2(\kappa) = \mathrm{D}^2(\ell, \ell')$ in a neighborhood of $\kappa = (0, 0)$ is given by

$$\mathcal{D}^2(\kappa) = d^2(\kappa) + O(|\kappa|^3) \tag{22}$$

where $O(|\kappa|^3)$ is an infinitesimal of the same order as $|\kappa|^3$ for $|\kappa| \to 0$. We prove the following

Lemma 2. *If there is an ascending node crossing between the orbits, there exist a neighborhood $\mathcal{U}_0$ of $\kappa = (0, 0)$ and two positive constants B_1, B_2 such that*

$$B_1 d^2(\kappa) \le \mathcal{D}^2(\kappa) \le B_2 d^2(\kappa) \qquad \forall \kappa \in \mathcal{U}_0.$$

Proof. First we notice that for $d^+_{nod} = 0$ we have $d^2(\kappa) = \kappa^t \mathcal{A} \kappa$, where $\mathcal{A}$ is positive definite, hence there exist two positive constants C_1, C_2 such that

$$C_1 |\kappa|^2 \le \kappa^t \mathcal{A} \kappa \le C_2 |\kappa|^2 \qquad \forall \kappa \in \mathbb{R}^2. \tag{23}$$

Using the relations (22) and (23) we obtain

$$\lim_{|\kappa| \to 0} \frac{\mathcal{D}^2(\kappa)}{d^2(\kappa)} = 1\,,$$

that implies the existence of the neighborhood $\mathcal{U}_0$ and of the constants B_1, B_2 as in the statement of the lemma.

We prove the following result:

Proposition 1. *The remainder function* $1/\mathrm{D} - 1/\mathrm{d}$ *is bounded even if there is an ascending node crossing.*

Proof. If there are no crossings between the orbits the remainder function is trivially bounded, in fact $\mathrm{D}(\ell,\ell')>0$ for each $(\ell,\ell')\in\mathbb{T}^2$ and the minimum value of $\mathrm{d}(\ell,\ell')$ is d^+_{min} that, for $I\neq 0$, can be zero only if $d^+_{nod}=0$ (see equation (19)).

If there is a crossing at the ascending node we have to investigate the local behavior of the remainder function in a neighborhood of $(\ell,\ell')=(\overline{\ell},0)$, where both D and d can vanish. The boundedness of the remainder function can be shown using the previous lemma: we know that there exists a neighborhood $\mathcal{U}_0$ and a positive constant B_1 such that the relation

$$\mathcal{D}(\kappa)\geq\sqrt{B_1}d(\kappa)$$

holds for each $\kappa\in\mathcal{U}_0$. It follows that in this neighborhood the remainder function can be bounded in the following way:

$$\left|\frac{1}{\mathcal{D}(\kappa)}-\frac{1}{d(\kappa)}\right|=\frac{|d^2(\kappa)-\mathcal{D}^2(\kappa)|}{d(\kappa)\mathcal{D}(\kappa)[d(\kappa)+\mathcal{D}(\kappa)]}\leq$$
$$\leq\frac{1}{\sqrt{B_1}[1+\sqrt{B_1}]}\cdot\frac{|d^2(\kappa)-\mathcal{D}^2(\kappa)|}{|\kappa|^3}\cdot\frac{|\kappa|^3}{d^3(\kappa)}\ .$$

We observe that $|d^2(\kappa)-\mathcal{D}^2(\kappa)|=O(|\kappa|^3)$ and that by (23) there is a positive constant C_1 such that $d^2(\kappa)\geq C_1|\kappa|^2$, so that there exists a constant $L>0$ such that

$$\left|\frac{1}{\mathcal{D}(\kappa)}-\frac{1}{d(\kappa)}\right|\leq L\qquad\forall\kappa\in\mathcal{U}_0\ .$$

Remark 5. Although the remainder function $1/\mathrm{D}-1/\mathrm{d}$ is bounded, it is not continuous in $(\ell,\ell')=(\overline{\ell},0)$ when there is a crossing at the ascending node.

3.8 The derivatives of the averaged perturbing function $\overline{R}$

Kantorovich's method is used to describe the singularities of the derivatives of the averaged perturbing function with respect to Delaunay's variables appearing in equations (12).

Note that by the *chain rule* we can write

$$\frac{\partial\overline{R}}{\partial E_{\mathcal{D}}}=\frac{\partial\overline{R}}{\partial E_{\mathcal{K}}}\frac{\partial E_{\mathcal{K}}}{\partial E_{\mathcal{D}}}$$

where $E_{\mathcal{K}}=\{e,I,\omega,\Omega\}$ is a subset of the Keplerian elements of the asteroid and

$$\frac{\partial E_{\mathcal{K}}}{\partial E_{\mathcal{D}}}=\begin{bmatrix}\mathcal{M} & \mathcal{O}\\ \mathcal{O} & I_2\end{bmatrix}$$

in which I_2 and $\mathcal{O}$ are the 2×2 identity and zero matrixes, and

$$\mathcal{M}=-\frac{1}{k\sqrt{a}}\begin{bmatrix}\beta/e & 0\\ -\mathrm{cotan}I/\beta & 1/(\beta\sin I)\end{bmatrix}\ .$$

Hence we can do the computation using the derivatives of $\overline{R}$ with respect to the Keplerian elements e, I, ω ($\overline{R}$ does not depend on Ω).

We shall not need to perform the splitting of Kantorovich's method to compute the derivative with respect to the inclination I; in fact the derivative of $1/\mathfrak{D}$ with respect to I can be bounded by a function with a first order polar singularity in $\overline{u}, \overline{u}'$, so it is Lebesgue integrable over $\mathbb{T}^2$.

In the following we shall first prove that the derivatives of the remainder function $1/\mathrm{D} - 1/\mathrm{d}$ are always Lebesgue integrable over $\mathbb{T}^2$, even if the two orbits intersect each other, so that the average of the remainder function is differentiable: indeed its derivatives can be computed by exchanging the position of the integral and differential operators as in (14). Then we shall see that if there is an ascending node crossing, then a discontinuous term appears in the derivatives of the average of $1/\mathrm{d}$ and this is responsible of the discontinuity of the derivatives of $\overline{R}$. These derivatives admit two limit values at crossings (coming from the regions defined by $d^+_{nod} > 0$ and $d^+_{nod} < 0$).

As the properties we intend to prove are invariant by coordinate changes, we shall show them using the coordinates (u, u') instead of (ℓ, ℓ').

The derivatives of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$.

Let us set $\upsilon = (u, u')$ and $\nu = (v, v') = (u - \overline{u}, u' - \overline{u}')$. We apply Taylor's formula with the integral remainder to the vector functions $P(u), P'(u')$:

$$\begin{cases} P(u) = P(\overline{u}) + P_u(\overline{u})\, v + \displaystyle\int_{\overline{u}}^{u} (u - s) P_{ss}(s)\, ds \\ P'(u') = P'(\overline{u}') + P'_{u'}(\overline{u}')\, v' + \displaystyle\int_{\overline{u}'}^{u'} (u' - t) P'_{tt}(t)\, dt \ . \end{cases}$$

The functions defining the straight lines $r(u) = \mathrm{r}(\ell(u))$ and $r'(u') = \mathrm{r}'(\ell'(u'))$ have the same Taylor's development, up to the first order in $|\nu| = \sqrt{v^2 + v'^2}$, as $P(u)$ and $P'(u')$ respectively, so that we can write

$$\begin{cases} r(u) = P(\overline{u}) + P_u(\overline{u})\, v + \displaystyle\int_{\overline{u}}^{u} (u - s) r_{ss}(s)\, ds \\ r'(u') = P'(\overline{u}') + P'_{u'}(\overline{u}')\, v' + \displaystyle\int_{\overline{u}'}^{u'} (u' - t) r'_{tt}(t)\, dt \ . \end{cases}$$

We prove the following

Theorem 1. *If there is an ascending node crossing at $(u, u') = (\overline{u}, \overline{u}')$, the derivatives of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$ with respect to e, ω can be bounded by functions having a first order polar singularity in $\overline{u}, \overline{u}'$, so they are Lebesgue integrable over $\mathbb{T}^2$.*

Proof. We shall consider only the derivatives with respect to e: the proof for the other derivatives is similar. First we note that

$$\frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{D}(\upsilon)}\right] = -\frac{1}{2\mathfrak{D}^3(\upsilon)}\frac{\partial}{\partial e}\left[\mathfrak{D}^2(\upsilon)\right] \ ; \qquad \frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{d}(\upsilon)}\right] = -\frac{1}{2\mathfrak{d}^3(\upsilon)}\frac{\partial}{\partial e}\left[\mathfrak{d}^2(\upsilon)\right] \ .$$

Let us write $\langle\, ,\rangle$ for the Euclidean scalar product. We have

$$\frac{\partial}{\partial e}\left[\mathfrak{D}^2(v)\right] = \mathfrak{D}^2_{e,0}(v) + \mathfrak{D}^2_{e,1}(v) + \mathfrak{D}^2_{e,2}(v) \tag{24}$$

where

$$\mathfrak{D}^2_{e,0}(v) = 2\left\langle \frac{\partial}{\partial e}\left[P(u) - P'(u')\right], P(\overline{u}) - P'(\overline{u}')\right\rangle$$

$$\mathfrak{D}^2_{e,1}(v) = 2\left\langle \frac{\partial}{\partial e}\left[P(u) - P'(u')\right], P_u(\overline{u})\, v - P'_{u'}(\overline{u}')\, v'\right\rangle$$

$$\mathfrak{D}^2_{e,2}(v) = 2\left\langle \frac{\partial}{\partial e}\left[P(u) - P'(u')\right], \int_{\overline{u}}^{u}(u-s)P_{ss}(s)\, ds - \int_{\overline{u}'}^{u'}(u'-t)P'_{tt}(t)\, dt\right\rangle$$

and

$$\frac{\partial}{\partial e}\left[\mathfrak{d}^2(v)\right] = \mathfrak{d}^2_{e,0}(v) + \mathfrak{d}^2_{e,1}(v) + \mathfrak{d}^2_{e,2}(v) \tag{25}$$

where

$$\mathfrak{d}^2_{e,0(v)} = 2\left\langle \frac{\partial}{\partial e}\left[r(u) - r'(u')\right], P(\overline{u}) - P'(\overline{u}')\right\rangle$$

$$\mathfrak{d}^2_{e,1}(v) = 2\left\langle \frac{\partial}{\partial e}\left[r(u) - r'(u')\right], P_u(\overline{u})\, v - P'_{u'}(\overline{u}')\, v'\right\rangle$$

$$\mathfrak{d}^2_{e,2}(v) = 2\left\langle \frac{\partial}{\partial e}\left[r(u) - r'(u')\right], \int_{\overline{u}}^{u}(u-s)r_{ss}(s)\, ds - \int_{\overline{u}'}^{u'}(u'-t){r'}_{tt}(t)\, dt\right\rangle .$$

If we set the *crossing conditions* $P(\overline{u}) = P'(\overline{u}')$ we obtain

$$\mathfrak{D}^2_{e,0}(v) = \mathfrak{d}^2_{e,0}(v) = 0$$

and, in particular, the constant terms in Taylor's developments of $\partial\mathfrak{D}^2/\partial e$ and $\partial\mathfrak{d}^2/\partial e$ vanish.

The terms defined by $\mathfrak{D}^2_{e,2}$ and $\mathfrak{d}^2_{e,2}$ are at least infinitesimal of the second order with respect to $|\nu|$ as $v \to (\overline{u}, \overline{u}')$, so that the first order terms in $|\nu|$ at crossing can be given only by $\mathfrak{D}^2_{e,1}$ and $\mathfrak{d}^2_{e,1}$.

Using the theorems on the integrals depending on a parameter we obtain

$$\frac{\partial}{\partial e}\left[\int_{\overline{u}}^{u}(u-s)P_{ss}(s)\, ds - \int_{\overline{u}'}^{u'}(u'-t)P'_{tt}(t)\, dt\right] =$$

$$\int_{\overline{u}}^{u}(u-s)\frac{\partial P_{ss}}{\partial e}(s)\, ds - \frac{\partial\overline{u}}{\partial e}P_{uu}(\overline{u})\, v - \int_{\overline{u}'}^{u'}(u'-t)\frac{\partial P'_{tt}}{\partial e}(t)\, dt + \frac{\partial\overline{u}'}{\partial e}P'_{u'u'}(\overline{u}')\, v'$$

$$\frac{\partial}{\partial e}\left[\int_{\overline{u}}^{u}(u-s)r_{ss}(s)\, ds - \int_{\overline{u}'}^{u'}(u'-t){r'}_{tt}(t)\, dt\right] =$$

$$\int_{\overline{u}}^{u}(u-s)\frac{\partial r_{ss}}{\partial e}(s)\, ds - \frac{\partial\overline{u}}{\partial e}r_{uu}(\overline{u})\, v - \int_{\overline{u}'}^{u'}(u'-t)\frac{\partial {r'}_{tt}}{\partial e}(t)\, dt + \frac{\partial\overline{u}'}{\partial e}{r'}_{u'u'}(\overline{u}')\, v'$$

so that these two expressions are at least infinitesimal of the first order with respect to $|\nu|$. As these terms are multiplied by first order terms in the expressions of $\mathfrak{D}^2_{e,1}$ and $\mathfrak{d}^2_{e,1}$, they give rise to at least second order terms.

We can conclude that the first order terms in the expressions (24) and (25) are equal and they are given by

$$2\left\langle \frac{\partial}{\partial e}\left[P(\overline{u}) - P'(\overline{u}')\right] - \left[\frac{\partial \overline{u}}{\partial e}P_u(\overline{u}) - \frac{\partial \overline{u}'}{\partial e}P'_{u'}(\overline{u}')\right], P_u(\overline{u})\,v - P'_{u'}(\overline{u}')\,v' \right\rangle \;;$$

therefore the asymptotic developments of the e-derivatives of $\mathfrak{D}^2(v)$ and $\mathfrak{d}^2(v)$ in a neighborhood of $v = (\overline{u}, \overline{u}')$ are

$$\frac{\partial}{\partial e}\left[\mathfrak{D}^2(v)\right] = \alpha\, v + \beta\, v' + \mathfrak{r}_{\mathfrak{D}}(v); \qquad \frac{\partial}{\partial e}\left[\mathfrak{d}^2(v)\right] = \alpha\, v + \beta\, v' + \mathfrak{r}_{\mathfrak{d}}(v)$$

where α, β are independent on u, u' and $\mathfrak{r}_{\mathfrak{D}}(v)$, $\mathfrak{r}_{\mathfrak{d}}(v)$ are infinitesimal of the second order with respect to $|\nu|$ as $v \to (\overline{u}, \overline{u}')$.

Using the decomposition

$$\left[\frac{1}{\mathfrak{D}^3} - \frac{1}{\mathfrak{d}^3}\right] = \left[\frac{1}{\mathfrak{D}} - \frac{1}{\mathfrak{d}}\right]\left[\frac{1}{\mathfrak{D}^2} + \frac{1}{\mathfrak{D}\mathfrak{d}} + \frac{1}{\mathfrak{d}^2}\right] \;,$$

the boundedness of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$ and lemma 2 (that also hold in the (u, u') coordinates), we conclude that there exist two constants $L_1, L_2 > 0$ such that

$$\left|\frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{D}(v)}\right] - \frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{d}(v)}\right]\right| = \frac{1}{2}\left|\left\{\left[\frac{1}{\mathfrak{D}^3(v)} - \frac{1}{\mathfrak{d}^3(v)}\right](\alpha\, v + \beta\, v') + \right.\right.$$
$$\left.\left. + \frac{1}{\mathfrak{D}^3(v)}\mathfrak{r}_{\mathfrak{D}}(v) - \frac{1}{\mathfrak{d}^3(v)}\mathfrak{r}_{\mathfrak{d}}(v)\right\}\right| \le L_1 \frac{1}{|v|} + L_2$$

in a neighborhood of $v = (\overline{u}, \overline{u}')$ and the theorem is proven.

Singularities of the $\{e, \omega\}$-derivatives of the average of $1/\mathrm{d}$.

As $det(\mathcal{A}) > 0$ and $(\overline{\ell}, 0)$ is in the interior part of $\mathbb{T}^2$, we have $d^2_{min} + r^2_j(\theta) > 0$ for each $\theta \in [\theta_j, \theta_{j+1}]$ and for each $j = 1 \ldots 4$. Then we can use again the theorem of differentiation under the integral sign and compute, for instance, the derivative of the average of $1/\mathrm{d}$ with respect to e as

$$\frac{\partial}{\partial e}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell\, d\ell' = \frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\sqrt{\left(d^+_{min}\right)^2 + r^2_j(\theta)}\,d\theta - 2\pi d^+_{min}\right\} +$$
$$+\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\frac{\frac{\partial}{\partial e}[(d^+_{min})^2 + r^2_j(\theta)]}{\sqrt{(d^+_{min})^2 + r^2_j(\theta)}}\,d\theta - 2\pi\frac{\partial}{\partial e}d^+_{min}\right\} \;. \tag{26}$$

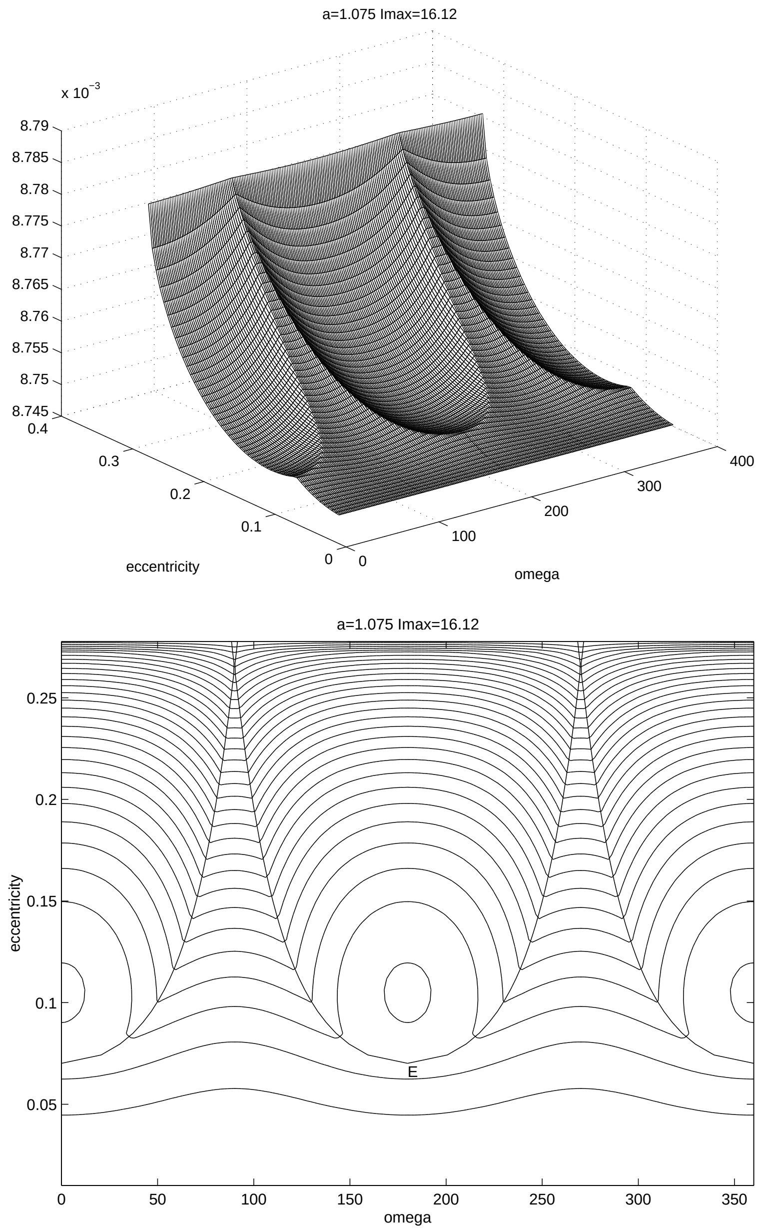

Fig. 5. We draw the graphic of the averaged perturbing function (top) and its level lines (bottom) in the plane (ω, e) for the Near Earth Asteroid 2000 CO_{101} (ω is in degrees in the figure). The loss of regularity at the node crossing lines with the Earth is particularly evident for this object.

We have similar formulas for the derivatives with respect to ω, obtained simply by substitution of the partial derivative operators.

The discontinuities present in the terms

$$\frac{\partial}{\partial e} d^+_{min} \, ; \qquad \frac{\partial}{\partial \omega} d^+_{min} \, ;$$

are responsible of the discontinuities in the derivatives of the averaged perturbing function that produce a sort of *crests* in the surfaces representing this function (see Figure 5) and cause the loss of regularity in its level lines, where the weak averaged solutions lie. The detailed analytical formulas for the discontinuities in the derivatives of $\overline{R}$ can be found in [10], [6].

4 Secular evolution theory

The generalized averaging principle has been used in [10] to define a method to compute the secular evolution of the NEAs in the framework of a Solar System with the planets on circular coplanar orbits. We shall review this method in the following of this section and we shall describe some features of the secular dynamics of NEAs.

First we note that the averaged Hamiltonian $\overline{H}$ *is invariant under the symmetries*

$$\begin{aligned} \omega &\longrightarrow -\omega \\ \omega &\longrightarrow \pi - \omega \\ \omega &\longrightarrow \pi + \omega \; ; \end{aligned}$$

this allows to draw the level lines of the averaged Hamiltonian, on which the solution curves are confined, simply knowing their shape in a subset of the reduced phase space (ω, e) of the form

$$\{(\omega, e) : k\,\pi/2 \leq \omega \leq (k+1)\,\pi/2 \; , \; 0 \leq e \leq e_{max}\}$$

with $k \in \mathbb{Z}$.

4.1 The secular evolution algorithm

The numerical method used in [10] to solve Hamilton's equations (12) is an implicit Runge-Kutta-Gauss algorithm of order 6, which is also symplectic (see [23]).

Note that Kantorovich's method, used to study the regularity of the averaged perturbing function, gives us analytical formulas for the discontinuities of the derivatives at the right hand sides of (12).

The Runge-Kutta-Gauss methods use sub-steps which include neither the starting point nor the final point of the step being computed, this allows to avoid the computation of the values of the derivatives of $\overline{R}$ at the node crossing points, where they are defined in a twofold way. We resort to the following procedure (which had already been used in [18]): every time the asteroid orbit is close enough to a node crossing line, the standard iteration scheme known as *regula falsi* is used to set the second extreme point of the step exactly on that line; as the computation of the right hand sides of (12) is performed only at the intermediate points of the integration step, we avoid the computation at node crossings.

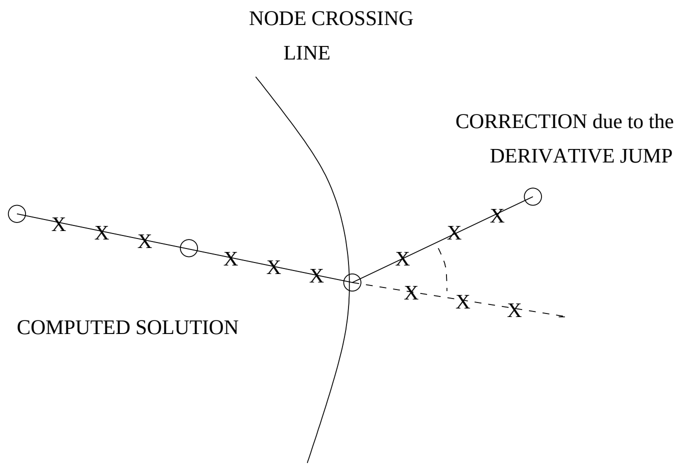

Fig. 6. Graphical description of the algorithm employed in this numerical integration: it is an implicit Runge-Kutta-Gauss method, symplectic, of order 6. In an integration step, delimited in the figure by two consecutive small circles, we compute the derivatives of $\overline{R}$ only at the intermediate points marked with crosses.

When the node crossing point is reached within the required precision, then a correction given by the explicit formulas for the discontinuities of the derivatives of $\overline{R}$ is applied before restarting the integration (see Figure 6).

An additional regula falsi is used to compute the value of the solutions of the averaged equations exactly at the symmetry lines in the plane (ω, e) (that is at the lines of the form $\omega = k\pi/2$ with $k \in \mathbb{Z}$); the computation of the secular evolution of a NEA requires to compute the solution of the equations (12) between two successive crossings of the symmetry lines. The complete evolution is then obtained by means of the symmetries of $\overline{R}$.

Kantorovich's decomposition of the integrals (like in (14)) is used not only when the final point of the integration step is on a node crossing line, but also when the solution is very close to a node crossing. This allows to stabilize the computation when the nodal distance is small and the integrand functions can be bounded only by very large constants.

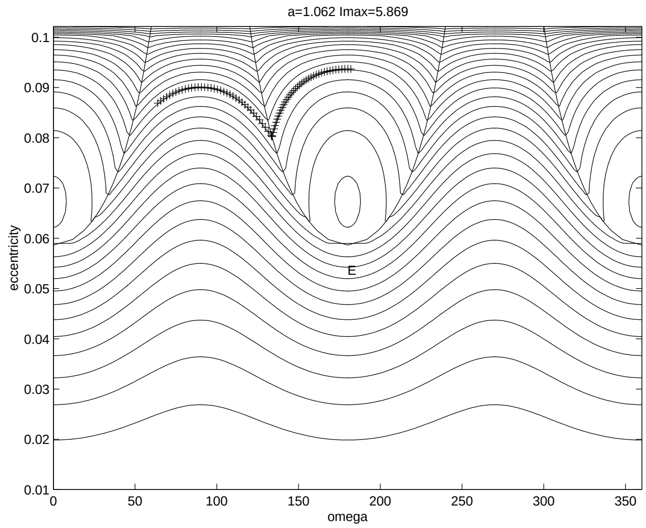

Fig. 7. Secular evolution figure for 2001 QJ_{142} on the background of the level lines of the averaged perturbing function. The node crossing lines with the Earth (E) are also drawn.

4.2 Different dynamical behavior of NEAs

We describe the secular evolution of the Near Earth Asteroids 2001 QJ_{142} and 1999 AN_{10}: these celestial objects show two very different kinds of dynamical behavior.

In Figure 7 we can see that the perihelion argument ω of 2001 QJ_{142} circulates; the loss of regularity of the solution curve, corresponding to a crossing at the ascending node with the Earth, is particularly enhanced in this figure.

In Figure 8 we have ω-libration for 1999 AN_{10}. Note that it starts its secular evolution with a double crossing with the orbit of the Earth, that gives it two possibilities to approach the Earth for each revolution and makes this object particularly dangerous. This asteroid has been intensively studied as a possible Earth impactor for the years 2039 (see [16]) and 2044 (see [17]).

In addition to this kind of libration, symmetric with respect to the lines $\omega = k\pi/2$ ($k \in \mathbb{Z}$), it is possible to have a sort of asymmetric librations, as it is shown by some of the level lines of Figure 9 in which the evolution of the asteroid (2100) Ra Shalom is shown. It is possible to choose initial values for e and ω, defining a fictitious object with the same value of the integral a as (2100) Ra Shalom, such that an asteroid starting with those values is constrained into a very narrow asymmetric libration.

We remark that asymmetric librations have a very low probability to occur: all the known NEAs examined so far do not show this kind of behavior.

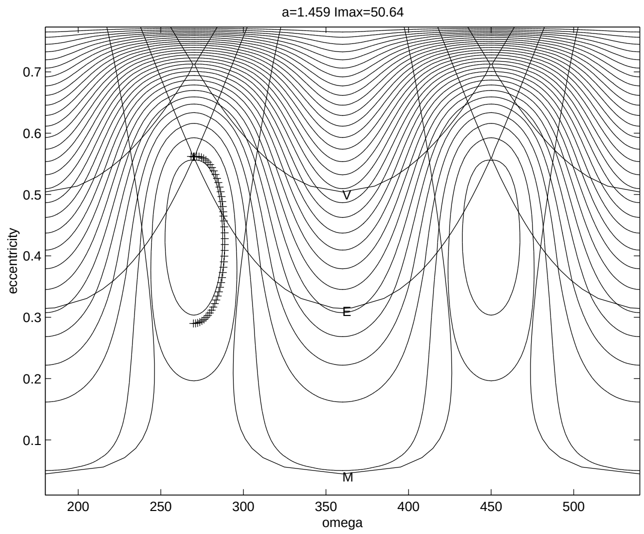

Fig. 8. Secular evolution figure for 1999 AN_{10}. The node crossing lines with the Earth (E) and Venus (V) are drawn.

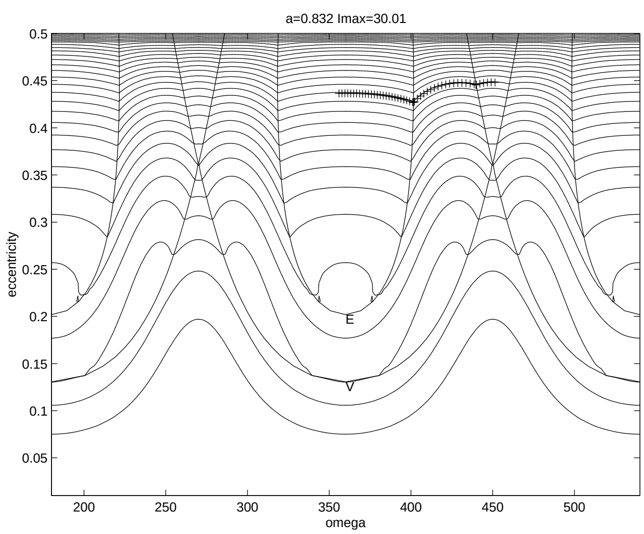

Fig. 9. Secular evolution figure for (2100) Ra Shalom. The four asymmetric libration regions are so small that they cannot be seen in the figure, but their presence can be detected by seeing the shape of the level lines near $e = 0.25$ and $\omega = 240°, 300°, 420°, 480°$.

5 Proper elements for NEAs

In the study of orbit dynamics a very important role is played by the integrals of the motion, that is by quantities that are constant during the time evolution of a dynamical system.

When the dynamics is non integrable, as it is the case for the N-body problem ($N \geq 3$), it is also useful to compute quantities that are nearly constant during the motion. We give the following definition:

Definition 6. The *proper elements* are quasi–integrals of the motion, that is quantities that change very slowly with time and can be considered approximatively constant over time spans not too long.

The first to employ the concept of proper elements was Hirayama [12]: he defined a linear theory to identify asteroid families in the main belt. The identification of families together with the understanding of the dynamical structure of the asteroid belt (e.g. the relevance of secular resonances) are two important reasons for the computation of the proper elements for MBAs.

There are presently three different possible methods to compute these quantities:

1. An analytical theory by Milani and Kneẑević [19],[20],[21] based on series expansion in eccentricity and inclination, particularly suitable for orbits with low eccentricity and low inclination ($< 17°$). The proper elements computed in [20],[21] have been proven to be stable over time scales of the order of 10^7 years.
2. A semianalytic theory by Lemaître and Morbidelli [15], which is more appropriate for the orbits with either large eccentricities or large inclinations. This method is based on the classical averaging method in a revisited version by Henrard [11]. A similar method had already been used by Williams [26],[27],[28] to obtain a set of proper elements that led to the understanding of the secular resonances resulting from two secular frequencies being equal.
3. A synthetic theory by Kneẑević and Milani [14] based on the computation of the asteroid orbits by pure numerical integration; the short periodic perturbations are then removed by a filtering process performed during the integration. This recent theory allows to obtain a high accuracy of the elements; on the other hand it requires CPU times longer than the two preceding methods.

Almost all the NEAs are planet crossing, and the singularities coming from the possibility of collisions result in the strong divergence of the series used in the analytical theory and in the divergence of the integrals of the classical averaging principle used in the semianalytic theory. Furthermore the strong chaoticity of crossing orbits and the very short integration steps to be chosen in this case make the synthetic theory inapplicable on large scale: in fact we have to use several values of the initial phase on the orbit for each NEA and in this case it would require very long computational times.

On the other hand the reasons to compute proper elements are very different in the case of NEAs and the required stability times are only of the order of 10 000 to 100 000 years. Over longer time spans the dynamics is dominated by large changes in the orbital elements, including the semimajor axis, resulting from the close approaches with the planets and from the effects of secular resonances.

We wish to compute proper elements for NEAs mainly for the following reasons:

1. to detect the possibility of collision of Earth-crossing objects and to compute its probability;
2. to identify the objects whose long term evolution is controlled by one or more of the main secular resonances;
3. to identify meteor streams (sets of very small objects, that can be observed only when they are crossing the orbit of the Earth) and to give a criterion to be used in the identification of their parent bodies.

We give the definition of proper elements for Near Earth Asteroids and we explain how to compute them using the generalized averaging principle explained in section 3.

Given the osculating orbit of a NEA, represented by its Keplerian elements $(a, e, I, \omega, \Omega)$ at a given time t, the following quantities are constant during the averaged motion in the framework of the circular coplanar case:

$$a, e_{min}, e_{max}, I_{min}, I_{max} \ ;$$

they are respectively the semimajor axis and the minimum and maximum value of the averaged eccentricity and the averaged inclination.

We can consider as set of proper elements either

$$\{a, e_{min}, I_{max}\} \qquad \text{or} \qquad \{a, e_{max}, I_{min}\} \ .$$

Remark 6. If we consider an ω-librating orbit we can also define

$$\omega_{min}, \omega_{max},$$

that are the minimum and maximum value of the averaged perihelion argument. These quantities are also constant during the averaged motion and can give additional informations to understand the dynamics of the objects that we are studying.

We can also use a set of proper elements in which the extreme values of the eccentricity and inclination are substituted by the secular frequencies of the longitude of perihelion g and the longitude of the node s in case of

ω-circulation. If ω is librating we can use the libration frequency l_f in place of g.

The computation of these proper frequencies requires some additional comments:

Proposition 2. *If ω is* circulating, *then let t_0 and t_1 be the times of passage at* 0 *and $\pi/2$ and let $\Omega_{t_0}, \Omega_{t_1}$ be the corresponding values of the longitude of the node. We have the following formulas to compute the secular frequencies of the argument of perihelion $g - s$ and of the longitude of the node s:*

$$g - s = \frac{2\pi}{4(t_1 - t_0)}; \qquad s = \frac{\Omega_{t_1} - \Omega_{t_0}}{t_1 - t_0} .$$

If ω is symmetrically librating, *then let τ_0 and τ_1 be two consecutive times of passage at the same integer multiple of $\pi/2$ and let $\Omega_{\tau_0}, \Omega_{\tau_1}$ be the corresponding values of the longitude of the node. We can compute the frequency of the longitude of the node s and the libration frequency l_f by the following formulas:*

$$s = \frac{\Omega_{\tau_1} - \Omega_{\tau_0}}{\tau_1 - \tau_0}; \qquad l_f = \pm\frac{2\pi}{2(\tau_1 - \tau_0)} ,$$

where the sign has to be chosen negative for clockwise libration.

Proof. The formula for $g - s$ is an immediate consequence of the fact that the period of circulation of ω must be four times the time interval required by an increase by $\pi/2$ of ω.

The proper frequency s has to be computed taking into account that the node has a secular precession, but also long periodic oscillations controlled by the argument 2ω.

The averaged Hamiltonian does not contain Ω because of the invariance with respect to rotation around the z axis; according to D'Alembert's rules (see [24]) it contains only the cosine of 2ω, thus the perturbative equation of motion for Ω is

$$\frac{d\Omega}{dt} = -\frac{\partial \overline{R}}{\partial Z}(2\omega) .$$

If ω is circulating it is possible to change (at least locally) variable and write the equation

$$\frac{d\Omega}{d\omega} = \frac{\partial \overline{R}/\partial Z}{\partial \overline{R}/\partial G} = F(\cos(2\omega))$$

with a right hand side containing only $\cos(2\omega)$. The solution for Ω as a function of ω is:

$$\Omega(\omega) = \alpha\ \omega + f(\sin(2\omega))$$

and the function $f(\sin(2\omega))$ is zero for $\omega = 0, \pi/2$, thus

$$\Omega_{t_1} - \Omega_{t_0} = \alpha\ \pi/2$$

identifies the secular part of the evolution of Ω, whose frequency can be computed by

$$s = \frac{\Omega_{t_1} - \Omega_{t_0}}{t_1 - t_0} \ .$$

The symmetric libration cases are different because ω returns to the same multiple of $\pi/2$ after a time interval $\tau_1 - \tau_0$, while in the same time span Ω has changed by $\Omega_{\tau_1} - \Omega_{\tau_0}$. Then $g - s$ is not a proper frequency (its secular part is by definition zero), and by arguments that are similar to the ones above, the libration frequency l_f and the secular frequency of the node s can be computed by

$$l_f = \pm\frac{2\pi}{2(\tau_1 - \tau_0)}; \qquad s = \frac{\Omega_{\tau_1} - \Omega_{\tau_0}}{\tau_1 - \tau_0} \ .$$

We agree that the negative sign for l_f is chosen for clockwise librations.

6 Reliability Tests

A numerical test on the reliability of the weak averaged solutions and of the proper elements for NEAs obtained with the generalized averaging principle can be found in [7]. A sample of orbits has been taken into account and the weak averaged solutions for the objects of this sample have been compared with their corresponding quantities obtained by pure numerical integrations using initial conditions that correspond to circular coplanar orbits for the planets.

The results of this comparison are satisfactory: the cases in which the secular evolution theory fails are generally the ones for which it is not valid *a priori*, that is low order mean motion resonances and close approaches with one or more planets, that could change the value of the semimajor axis, which is assumed to be constant in the averaging theory.

	(433) *Eros*		(1981) *Midas*	
Prop. El.	AT	NI	AT	NI
e_{min}	0.22285	0.22272	0.34872	0.34978
e_{max}	0.23307	0.23289	0.65075	0.65029
I_{min}	10.06653	10.06472	39.78069	39.79393
I_{max}	10.82960	10.83039	51.49418	51.43053
ω_{min}	$-\infty$	$-\infty$	248.61698	248.51999
ω_{max}	$+\infty$	$+\infty$	291.38302	291.56100

Table 1. Proper elements table for the asteroids (433) Eros and (1981) Midas: the angles are given in degrees. In case of ω-circulation we define $\omega_{min} = -\infty, \omega_{max} = +\infty$. AT means Averaged Theory, while NI means Numerical Integrations.

	(433) *Eros*		(1981) *Midas*	
Prop. Freq.	AT	NI	AT	NI
g	14.675	14.876	−25.625	−25.650
s	−21.101	−21.136	−25.625	−25.650
l_f	0	0	−45.672	−45.422

Table 2. Proper frequencies table for (433) Eros and (1981) Midas: the units are arc-seconds per year. In case of ω-circulation we define $l_f = 0$ while in case of ω-libration we define $g = s$.

The agreement of the elements for the sample selected in [7] is of the order of 5×10^{-3} both for the proper eccentricity and the proper inclination (if considered in radians) over a time span of the order of 10^4 yrs.

In Tables 1,2 we present the set of proper elements and proper frequencies of two objects of the sample: 433 (Eros) and (1981) Midas, that are respectively ω-circulating and ω-librating. The orbital elements used for these asteroids are in Table 3.

Body	a (AU)	ω (°)	Ω (°)	e	I (°)
(433) Eros	1.45823	178.640	304.411	0.222863	10.829
(1981) Midas	1.77611	267.720	357.097	0.650113	39.831

Table 3. Orbital elements used in the comparison for (433) Eros and for (1981) Midas.

In [7] there is also a comparison of the previous results with full numerical integrations starting with the actual eccentricity and inclination of the planets: we observe that the difference between the proper elements obtained in both ways for the selected sample is not dramatic (at least for the time scale of this integration), but the crossing times between the orbits are not reliable at all if we do not take into account the eccentricity and inclination of the planets and the computation of these times is useful for several applications such as the study of the possibility of collision. Thus the need of a more accurate averaging theory is evident.

7 Generalized averaging principle in the eccentric–inclined case

Recently we have proven that the generalized averaging theory, defined in Section 3, can be extended including the eccentricities and the inclinations of the planets. We give in this section only the main idea of this generalization; the reader interested can found a complete explanation of the theory in [5],[6].

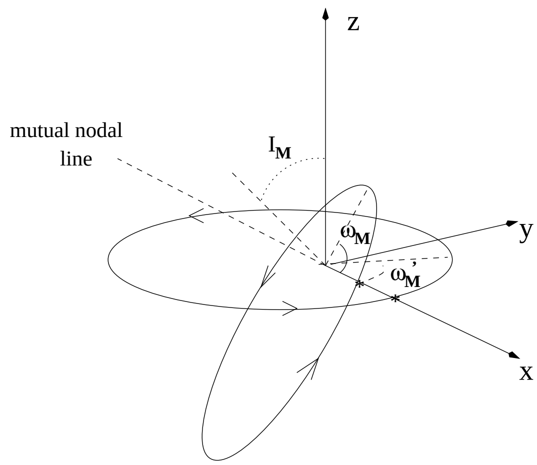

Fig. 10. The mutual reference frame: the ascending mutual node is marked with asterisks.

Also in this framework we can study the case of only one perturbing planet and we can obtain the total perturbation by the sum of the contribution of each planet in the model. We can write the averaged perturbing function $\overline{R}$ as a function of a particular set of variables, called *mutual elements*, that are almost everywhere regular functions of Delaunay's variables and are defined by the mutual position of the osculating orbits of the asteroid and the planet. Then the equations of motion for the asteroid become

$$\begin{cases} \dot{\tilde{G}} = \dfrac{\partial \overline{R}}{\partial E_{\mathcal{M}}} \dfrac{\partial E_{\mathcal{M}}}{\partial g} \\ \dot{\tilde{Z}} = \dfrac{\partial \overline{R}}{\partial E_{\mathcal{M}}} \dfrac{\partial E_{\mathcal{M}}}{\partial z} \end{cases} \qquad \begin{cases} \dot{\tilde{g}} = -\dfrac{\partial \overline{R}}{\partial E_{\mathcal{M}}} \dfrac{\partial E_{\mathcal{M}}}{\partial G} \\ \dot{\tilde{z}} = -\dfrac{\partial \overline{R}}{\partial E_{\mathcal{M}}} \dfrac{\partial E_{\mathcal{M}}}{\partial Z} \end{cases} \tag{27}$$

where $E_{\mathcal{M}}$ is a suitable subset of the mutual elements.

First note that the definitions of *mutual nodal line* and *mutual nodes* make sense even in the elliptic case; but we observe that we have generally a different mutual nodal line for each planet while in the circular coplanar case it is the same for all of them.

7.1 The mutual reference frame

We give a short description of the mutual elements. Let us consider two elliptic non-coplanar osculating orbits of an asteroid and a planet with a common focus: we give the following

Definition 7. We call *mutual reference frame* a system $Oxyz$ (see Figure 10) such that the x axis is along the mutual nodal line and is directed towards the mutual ascending node; the y axis lies on the planet orbital plane, so that the orbit of the planet lies on the (x, y) plane. We shall use the further convention that the positive z axis is oriented as the angular momentum of the planet.

Let ω_M, ω'_M be the mutual pericenter arguments (the angles between the x axis and the pericenters) of the orbit of the asteroid and of the planet respectively, and let I_M be the mutual inclination between the two conics. We define as *mutual elements* the set of variables

$$\{a, e, a', e', \omega_M, \omega'_M, I_M\}$$

and we set $E_{\mathcal{M}} = \{a, e, \omega_M, \omega'_M, I_M\}$.

Note that we can express the mutual variables ω_M, ω'_M, I_M as functions of the Keplerian elements ω,Ω,I, ω',Ω',I'; furthermore the derivatives of the mutual elements with respect to Delaunay's variables appearing in equations (27) can be easily computed by means of the Keplerian elements $E_{\mathcal{K}}$ of the asteroid:

$$\frac{\partial E_{\mathcal{M}}}{\partial E_{\mathcal{D}}} = \frac{\partial E_{\mathcal{M}}}{\partial E_{\mathcal{K}}} \frac{\partial E_{\mathcal{K}}}{\partial E_{\mathcal{D}}} .$$

8 Conclusions

The generalized averaging principle and the related results reviewed in this paper have been applied to the search for parent bodies of meteor streams (using appropriate variables, like the ones in [25]) and to the computation of the secular evolution of the MOID (using algebraic methods as in [4]). We think that the extension of the theory including the eccentricity and inclination of the planets will be very useful to improve the accuracy of the results of such applications. There are additional possible applications of this theory, that we have still to investigate, to the computation of the collision probability between a NEA and the Earth.

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