Global observables and infinite mixing

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June 12, 2015

The problem

Definition finite mixing

If T is a measure-preserving map of the probability space $(\mathcal{M}, \mu)$, the dynamical system $(\mathcal{M}, \mu, T)$ is called mixing if, for all measurable $A, B \subseteq \mathcal{M}$,

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

Equivalently, for all $f, g \in L^2(\mathcal{M}, \mu)$,

$$\lim_{n\to\infty} \mu((f \circ T^n)g) = \mu(f)\mu(g)$$

(abuse of notation: $\mu(f) = \int_{\mathcal{M}} f \, d\mu$, etc.).

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Intrinsically probabilistic notion.

What if $\mu(\mathcal{M}) = \infty$?

Some history

Hopf 1937

- Considers $(\mathcal{M}, \mu, T)$ with $\mu(\mathcal{M}) = \infty$
  ($\mathcal{M}$ = half-infinite strip in $\mathbb{R}^2$, $\mu = \mathrm{Leb}_{\mathcal{M}}$)
- Proves $\exists \{\rho_n\}_{n\in\mathbb{N}}$, $\rho_n \nearrow \infty$, such that

$$\lim_{n\to\infty} \rho_n \, \mu(T^{-n} A \cap B) = \mu(A)\mu(B)$$

  for all squarable sets $A, B \subset \mathcal{M}$ (i.e., $\mu(\partial A) = \mu(\partial B) = 0$)
- Calls $(\mathcal{M}, \mu, T)$ an example of a mixing system

Some history

Krickeberg 1967

Turns Hopf's example into a definition:

Definition Kr-mixing

Let $\mathcal{M}$ be a completely regular topological space with a Borel measure μ. Let $\{H_k\}_{k\in\mathbb{N}}$ make μ σ-finite. Let T be a μ-preserving homeomorphism mod μ. $(\mathcal{M}, \mu, T)$ is called mixing if $\exists\{\rho_n\}_{n\in\mathbb{N}}$, $\rho_n \nearrow \infty$, such that

$$\lim_{n\to\infty} \rho_n \, \mu(T^{-n}A \cap B) = \mu(A)\mu(B)$$

for all squarable sets $A, B \subset H_k$ (some k).

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The Good: Very natural definition. Used by many (e.g., *Thaler, Isola, Melbourne-Theresiu, Arbieto-Markarian-Pacifico-Soares*), in some cases independently (ρ_n nowadays called scaling rate)

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The Ugly: Only sees finite-measure sets

Krengel & Sucheston 1969

More measure-theoretic approach:

Definition — KS-(complete) mixing

Let $T : \mathcal{M} \longrightarrow \mathcal{M}$ be non-singular w.r.t. μ. $(\mathcal{M}, \mu, T)$ is called

1. mixing if $\{T^{-n}A\}_{n\in\mathbb{N}}$ is semiremotely trivial $\forall A$, $\mu(A) < \infty$
2. completely mixing if $\{T^{-n}A\}_{n\in\mathbb{N}}$ is semiremotely trivial $\forall A$

$\{A_n\}$ semiremotely trivial iff $\exists\{n_k\}$ s.t. $\bigcap_j \sigma(A_{n_j}, A_{n_{j+1}}, \ldots)$ trivial

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When $\mu(\mathcal{M}) = 1$, both definitions coincide with the classical one *(Sucheston 1963)*

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Too weak! (Cf. translations in $\mathbb{R}^d$)

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For invertible measure-preserving DS.'s, KS-complete mixing incompatible with ergodicity.
($\exists\, \mu_0 \ll \mu$, $\mu_0(\mathcal{M}) = 1$, invariant and mixing)
Too strong!

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How about letting go of a *a priori* universal definition?

Also, the previous attempts involved (mostly) finite-measure sets; equivalently integrable, or local, observables ("local-local mixing")

Seeking notion of mixing that uses global observables

Setup

Dynamical system: $(\mathcal{M}, \mathscr{A}, \mu, T^t)$

- $(\mathcal{M}, \mathscr{A}, \mu)$ σ-finite measure space
- $\mu(\mathcal{M}) = \infty$
- $T^t : \mathcal{M} \longrightarrow \mathcal{M}$ (semi)group of transformations preserving μ
  ($t \in \mathbb{G} = \mathbb{N}, \mathbb{Z}$ or $\mathbb{R}$)

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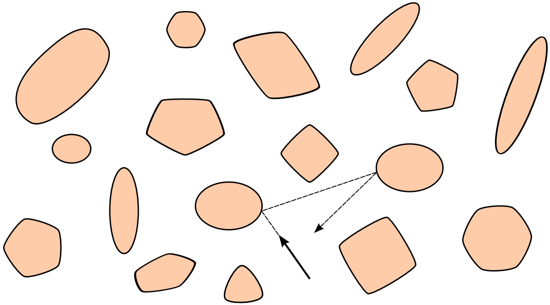
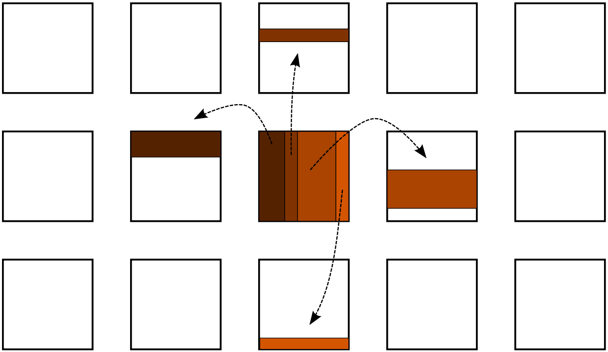
Have in mind systems with:

1. "extended chaoticity", or
2. "localized chaoticity"

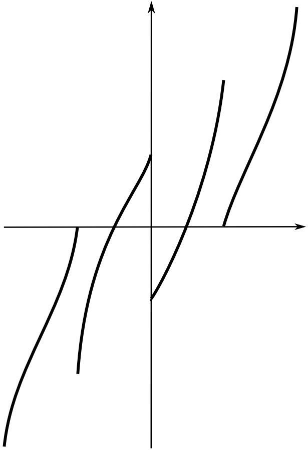
Lorentz gas: Flow on $(\mathbb{R}^2 \setminus \textit{scatterers}) \times S^1$, preserves Liouville



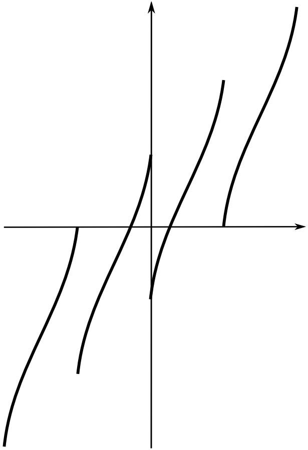
Random walk: Map on $\mathbb{Z}^d \times [0,1]^2$, preserves Lebesgue



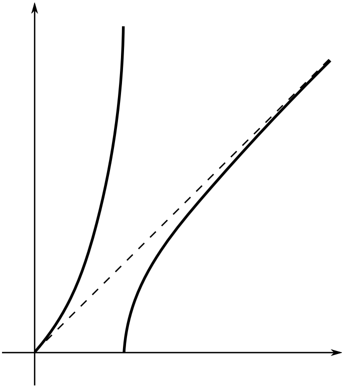
Expanding Markov map: $\mathbb{R} \longrightarrow \mathbb{R}$



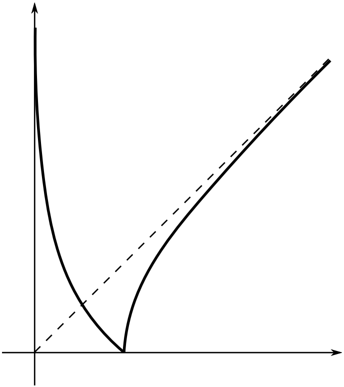
Quasi-lift of S^1-expanding map: $\mathbb{R} \longrightarrow \mathbb{R}$, $\mathbb{Z}$-invariant



Pomeau-Manneville maps: (via conjugation $\phi : (0,1] \longrightarrow \mathbb{R}^+$)



Farey map: (via conjugation –log : $(0,1] \longrightarrow \mathbb{R}^+$)



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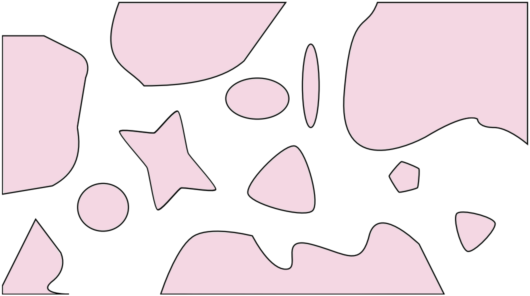

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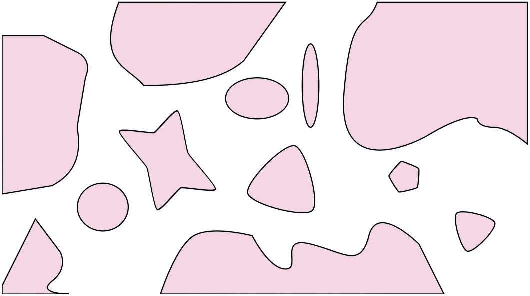

Infinite-volume limit

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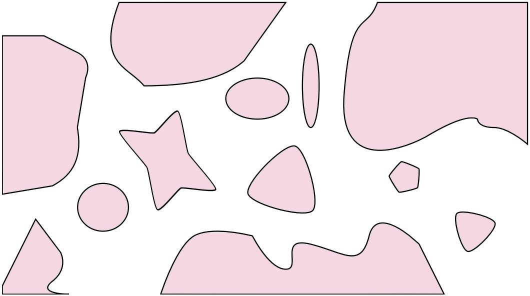

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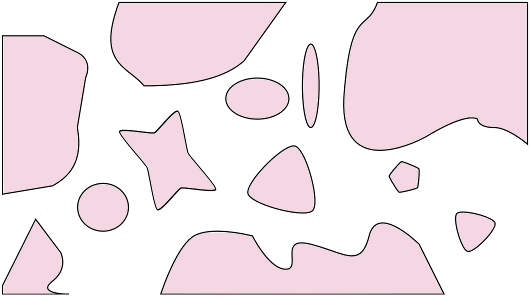

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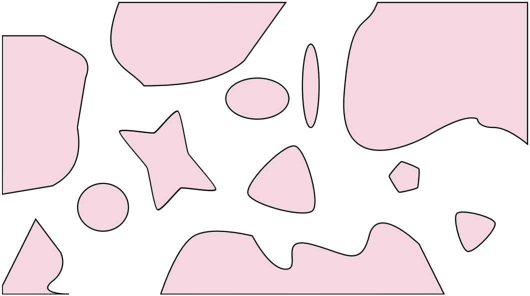

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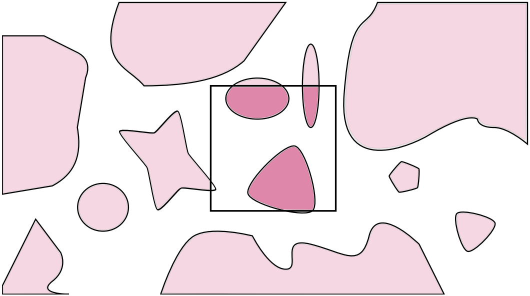

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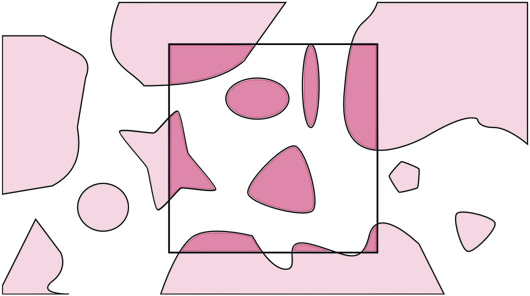

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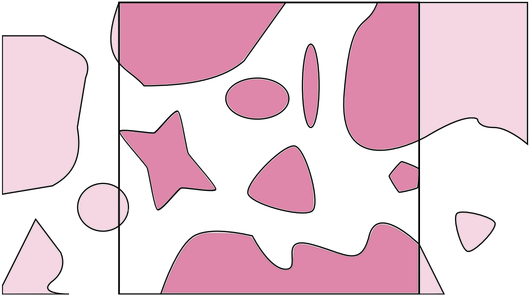

perhaps...

$$\text{“Probability” of } A = \lim_{r \to \infty} \frac{\mathrm{Leb}(A \cap [-r, r]^2)}{4r^2}$$

Mathematical formulation

The class of measurable sets $\mathscr{V}$ is called exhaustive if

1. $\forall V \in \mathscr{V}, \quad \mu(V) < \infty$,
2. $\exists \{V_n\}$ such that $V_n \nearrow \mathcal{M} \quad (V_n \subseteq V_{n+1}, \bigcup_n V_n = \mathcal{M})$.

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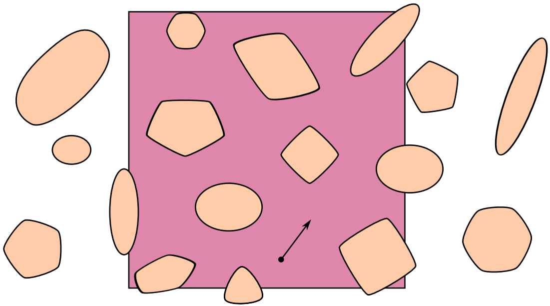
Definition — infinite-volume limit

We call infinite-volume limit (μ-uniform along $\mathscr{V}$) the limit

$$\lim_{V \nearrow \mathcal{M}} \left(\cdots \right) = \lim_{\substack{\mu(V) \to \infty \\ V \in \mathscr{V}}} \left(\cdots \right).$$

Examples of exhaustive classes

Example for Lorentz gas: $V = ([-r, r]^2 \setminus scatterers) \times S^1$ $(r > 0)$



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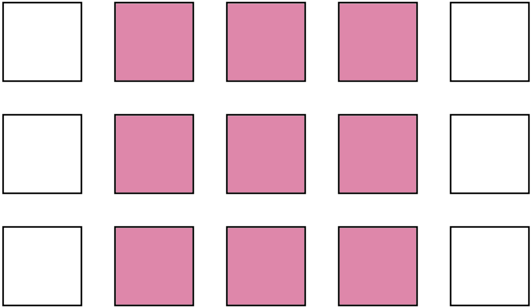
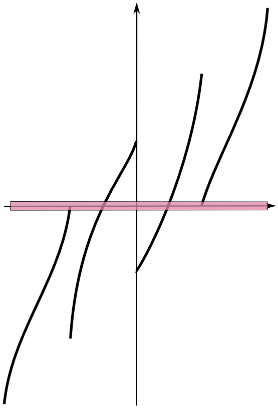
Example for random walk: $V = \{-k, \ldots, k\}^d \times [0,1]^2$ $(k \in \mathbb{Z}^+)$



Example for expanding Markov map: $V = [-k, k]$ $(k \in \mathbb{Z}^+)$



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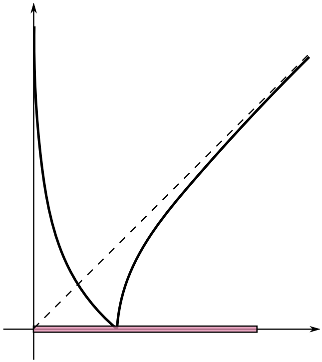
Example for Farey map: $V = [0, r]$ $(r > 0)$



Assumption:

(A1) For fixed t, $\mu(T^{-t} V \triangle V) = o(\mu(V)) \quad (V \nearrow \mathcal{M})$

Global observables

Global observables $\mathcal{G}$: $F : \mathcal{M} \longrightarrow \mathbb{R}$ supported (more or less) throughout the phase space

E.g., if a translation is defined on $\mathcal{M}$, periodic, quasiperiodic functions; in general, functions that look alike in different regions of $\mathcal{M}$.

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Proposition

(A1)-(A3) $\Longrightarrow \quad \overline{\mu}(F) = \overline{\mu}(F \circ T^t) \quad \forall F \in \mathcal{G}, \forall t \in \mathbb{G}$

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Definition **(GGM2)**

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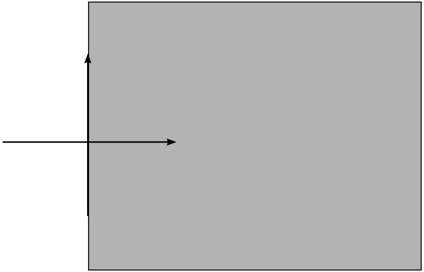
$$F(x,y) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \qquad F \circ T^n(x,y) = \begin{cases} 0, & \text{white area} \\ 1, & \text{gray area} \end{cases}$$



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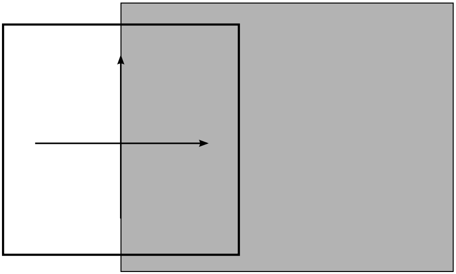

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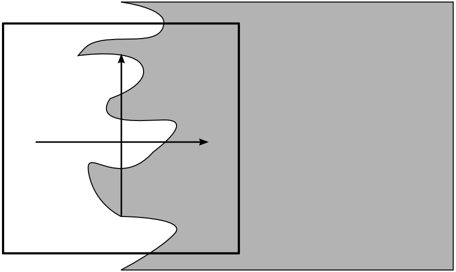

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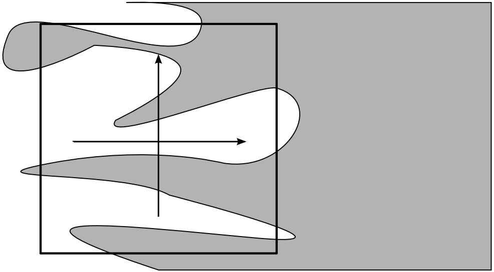

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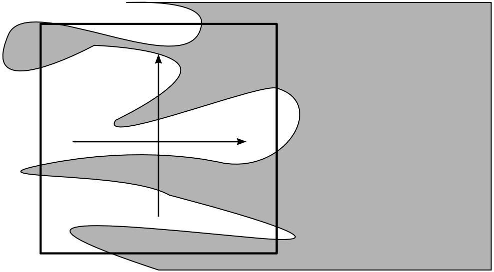

$$\forall n \in \mathbb{N}, \qquad \overline{\mu}((F \circ T^n)F) = \frac{1}{2} \neq [\overline{\mu}(F)]^2 = \frac{1}{4}$$

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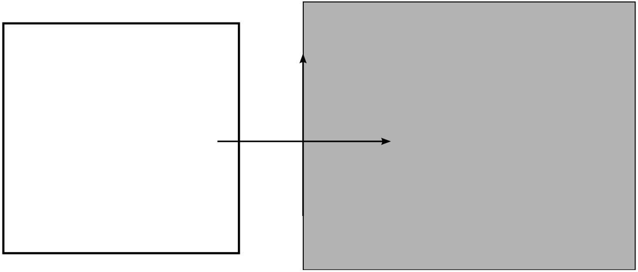

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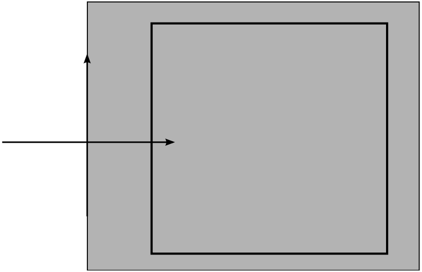

All these (foregoing and following) definitions crucially depend on $\mathscr{V}$ and $\mathcal{G}$!

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Introducing...

Local observables $\mathcal{L}$: Localized $f : \mathcal{M} \longrightarrow \mathbb{R}$

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Minimal requirements:

(A4) $\mathcal{L} \subset L^1(\mathcal{M}, \mu)$

The choice of $\mathcal{L}$ is less crucial than those of $\mathscr{V}$ and $\mathcal{G}$.
$\mathcal{L} = L^1$ works well in many cases. (As new definitions are mostly continuous in the L^1-norm. Occasionally one might require compact support, or additional regularity, etc.)

Global-local mixing

Definition **(GLM2)**

$\forall F \in \mathcal{G}, \forall g \in \mathcal{L},$

$$\lim_{t \to \infty} \mu((F \circ T^t) g) = \overline{\mu}(F) \mu(g)$$

Global-local mixing

Interpretation **(GLM2)**

$\forall F \in \mathcal{G}, \forall \mu_g = \mu(\cdot\, g), \;\; (g \in \mathcal{L},\ g \geq 0,\ \mu(g) = 1)$

$$\lim_{t \to \infty} T_*^t \mu_g(F) = \overline{\mu}(F)$$

Global-local mixing

Definition (GLM1)

$\forall F \in \mathcal{G}, \forall g \in \mathcal{L}$ with $\mu(g) = 0$,
$$\lim_{t\to\infty} \mu((F \circ T^t)g) = 0$$

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Global-local mixing

Interpretation **(GLM1)**

$\forall F \in \mathcal{G}, \forall \mu_g, \mu_h,$ (g, h densities $\in \mathcal{L}$)

$$\lim_{t \to \infty} \left(T_*^t \mu_g(F) - T_*^t \mu_h(F) \right) = 0$$

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Definition (GLM3)

$\forall F \in \mathcal{G}$,

$$\lim_{t\to\infty} \sup_{g\in\mathcal{L}\setminus 0} \frac{1}{\mu(|g|)} \left| \mu((F \circ T^t)g) - \overline{\mu}(F)\mu(g) \right| = 0$$

Proposition

Assuming **(A1)-(A4)**,

$$\textbf{(GLM3)} \implies \textbf{(GLM2)} \implies \textbf{(GLM1)}$$

On the other hand, if, $\forall F, G \in \mathcal{G}$, $\exists\, \overline{\mu}((F \circ T^t)G)$ for t large enough, then

$$\textbf{(GGM2)} \implies \textbf{(GGM1)}$$

General results

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Analogous result for K-mixing

If every global observable is more or less a sum of local observables with pairwise disjoint supports, then uniform global-local mixing implies the "strongest" form of global-global mixing:

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Proposition

Suppose that every $G \in \mathcal{G}$ can be written μ-a.e. as

$$G(x) = \sum_{j \in \mathbb{N}} g_j(x), \qquad \text{with } g_j \in \mathcal{L},$$

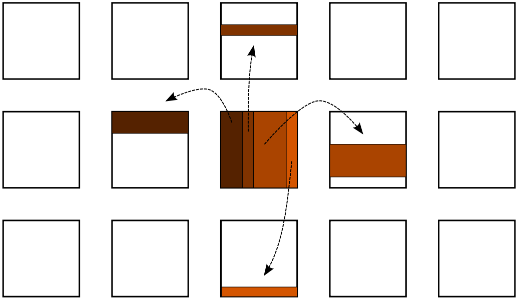
and, $\forall V \in \mathscr{V}$, $\exists$ finite $\mathbb{J}_V \subset \mathbb{N}$, such that

$$\mu\left(\left| G 1_V - \sum_{j \in \mathbb{J}_V} g_j \right| \right) = o(\mu(V));$$

$$\sum_{j \in \mathbb{J}_V} \| g_j \|_{L^1} = O(\mu(V)).$$

Then **(GLM3)** $\Longrightarrow$ **(GGM2)**

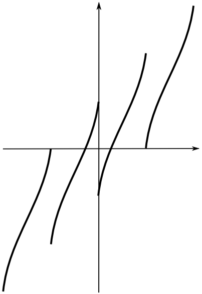
Random walk *(L 2010)*



For a *strongly aperiodic* (homogeneous) random walk on $\mathbb{Z}^d$ with sufficiently fast-decaying transition probabilities, **(GGMi), (GLMj)** $\forall i, j$ hold for suitable choices of $\mathscr{V}, \mathcal{G}, \mathcal{L}$.

Applications to prototypical examples

Uniformly expanding Markov maps of $\mathbb{R}$ *(L, 2014)*

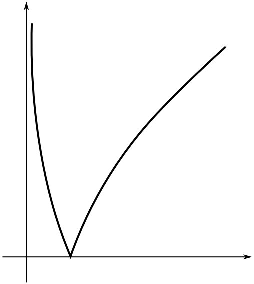


$\exists$ large class of maps for which exactness and **(GLM1)** hold.

Quasi-lifts verify **(GGMi),** $(i = 1, 2)$ **(GLMj)** $(j = 1, 2)$ (for suitable choices of $\mathscr{V}, \mathcal{G}, \mathcal{L}$).

Weaker results for finite modifications of quasi-lifts

Interval maps with indifferent fixed point *(Bonanno, Giulietti, L, in progress)*



Large class of such maps (including Pomeau-Manneville, Farey, Boole) verifies **(GLMj)** ($j = 1, 2$) (with "best" choice of $\mathscr{V}, \mathcal{G}, \mathcal{L}$).
Does not verify **(GLM3)**.

Does not verify **(GGMi)**, $\forall i$