Dynamics of close encounters of NEOs

Öpik theory and MOID regularization

Giacomo Tommei

Dipartimento di Matematica, Università di Pisa

Seminario di Fisica Matematica, 5 Marzo 2008

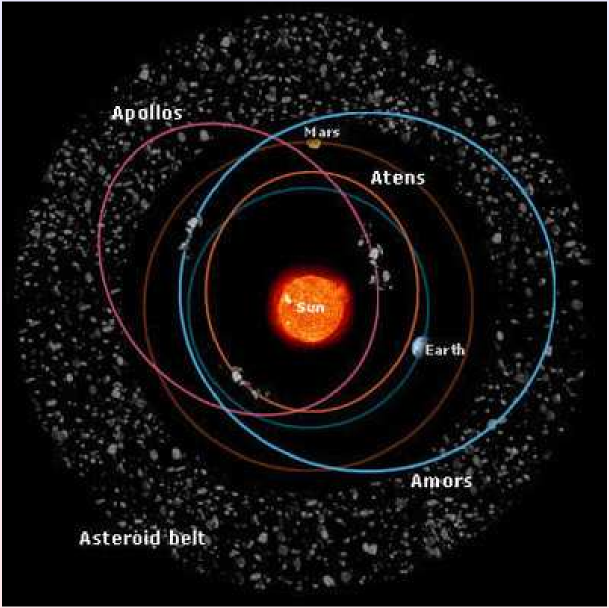
Contents

Classification of NEOs

CLASS	DESCRIPTION	DEFINITION
NECs	Near-Earth Comets	$q < 1.3\ AU$ $P < 200\ y$
NEAs	Near-Earth Asteroids	$q < 1.3\ AU$
Atens	Earth-crossing asteroids	$a < 1.0\ AU$ $Q > 0.983\ AU$
Apollo	Earth-crossing asteroids	$a > 1.0\ AU$ $q < 1.017\ AU$
Amors	NEAs with external-Earth orbit	$a > 1.0\ AU$ $1.017 < q < 1.3\ AU$
IEO	NEAs with internal-Earth orbit	$a < 1.0\ AU$ $Q < 0.983\ AU$

q: perihelion distance Q : aphelion distance P : period

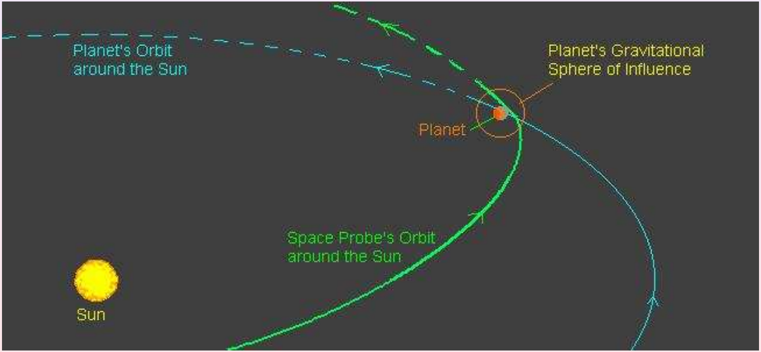
$1.017 AU$ is the Earth aphelion distance, $0.983 AU$ is the Earth perihelion distance


Apollos
Mars
Atens
Sun
Earth
Amors
Asteroid belt

Close Encounters of NEOs

Definition

A close encounter of a NEO is defined as a passage of the small body near the Earth: with the word "near" we usually mean inside the sphere of influence of our planet.



Contents

Description of the theory

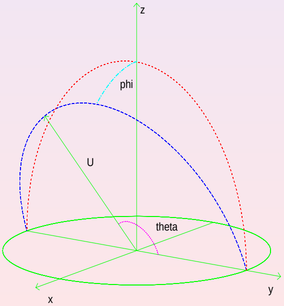
Öpik's theory of close encounters (1976)

- The motion of a small body approaching a planet is modelled as a planetocentric two-body scattering:
  1. heliocentric orbit until the time of the encounter with; the planet
  2. planetocentric hyperbolic orbit during the close approach.
- Direction and speed of the incoming asymptote of the planetocentric hyperbolic orbit defined by the relative velocity of the small body with respect to the planet are simple functions of the semimajor axis, eccentricity and inclination (a, e, i) of the heliocentric orbit of the small body (note that we assume the position of the small body coinciding with that of the planet).
- The effect of the encounter is an instantaneous deflection of the velocity vector in the direction of the outgoing asymptote of the planetocentric hyperbolic orbit, ignoring the perturbation due to the Sun and the time it actually takes for the small body to travel along the curved path that 'joins' the two asymptotes.
- The errors involved in such an approach are smaller for closer approaches, and the theory is exact only in the limit for the minimum approach distance (MOID) going to zero.

Basic Geometric Setup

PLANETOCENTRIC REFERENCE FRAME

- Planet: at the origin, moving in the direction of Y
- Sun: at unit distance on the negative X-axis
- $\vec{U}$: planetocentric velocity vector of the small body
- (θ, ϕ) : polar coordinates specifying the direction of $\vec{U}$



The components of the planetocentric velocity vector are

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} \pm\sqrt{2 - 1/a - a(1-e^2)} \\ \sqrt{a(1-e^2)}\cos i - 1 \\ \pm\sqrt{a(1-e^2)}\sin i \end{bmatrix}$$

and its length is

$$U = \sqrt{3 - \frac{1}{a} - 2\sqrt{a(1-e^2)}\cos i}.$$

This can be rewritten as

$$U = \sqrt{3 - T}$$

where T is the Tisserand parameter with respect to the planet

$$T = \frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i.$$

The direction of the incoming asymptote is defined by the angles, θ and ϕ, such that

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} U\sin\theta\sin\phi \\ U\cos\theta \\ U\sin\theta\cos\phi \end{bmatrix}$$

and, conversely

$$\begin{bmatrix} \cos\theta \\ \tan\phi \end{bmatrix} = \begin{bmatrix} U_y/U \\ U_x/U_z \end{bmatrix}.$$

Contents

Complete set of variables

ORIGINAL FORMULATION: (U, θ, ϕ) depending on (a, e, i)
Valsecchi et al. (2003) introduced corrections to first order in miss distance to extend the formulation to close encounters and they use a **non-canonical** set of elements for their analysis:

$$(U, \theta, \phi, \xi, \zeta, t_0)\ .$$

(ξ, ζ): coordinates on the Target Plane (TP)
t_0: time of the crossing of the ecliptic plane.

Definition

The **Target Plane** (TP), or b-plane, is the plane containing the center of the Earth and orthogonal to the velocity vector of the small body, that is orthogonal to the *incoming asymptote of the geocentric hyperbola* on which the small body travels when it is closest to the planet

The ξ-axis is perpendicular to the heliocentric velocity of the planet, and the ζ-axis is in the direction opposite to the projection on the b-plane of the heliocentric velocity of the planet.

Contents

PROBLEMS

1. to find canonical elements describing planetary encounters and non-singular at collision
2. to find canonical elements containing information about the position of the small body on the TP

RESULTS

1. derived a set of canonical hyperbolic collision elements
2. proved that is not possible to solve point 2

Canonical Hyperbolic Collision Elements

We look for canonical elements for **hyperbolic collision orbits:**

$$e \longrightarrow 1^+, \quad a \ \text{fixed} \ .$$

Tremaine (2001) found a set of canonical elements for elliptic collision orbits:

$$e \longrightarrow 1^-, \quad a \ \text{fixed} \ .$$

STARTING POINT: hyperbolic Delaunay elements

$$\mathcal{D}_{\mathbf{hyp}} = (L, G, H, l, g, h)$$

$$\begin{aligned} L &= -(\mu\, a)^{1/2} & \quad l &= e \sinh F - F = n\, t + \text{const.} \\ G &= [\mu\, a\, (e^2 - 1)]^{1/2} & \quad g &= \omega \\ H &= G \cos i & \quad h &= \Omega \end{aligned}$$

FINAL RESULT:

$$\mathcal{C}_{\mathbf{hyp}} = (L, \Theta, H, l, \theta_C, \phi_C)$$

is a set of canonical elements well defined at collision.

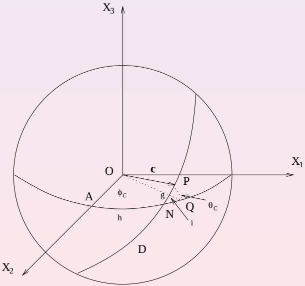
Canonical Hyperbolic Collision Elements

(θ_C, ϕ_C) are the polar coordinates defining the direction from the planet to the center of the hyperbola. This direction (given by the versor **c** in the figure below) coincides with that of pericenter of the orbit when it is defined.

Momentum Θ is the projection of the angular momentum $\mathbf{G}$ along a line defined by the versor
$\hat{\mathbf{t}} = (\hat{X}_3 \times \hat{\mathbf{c}})/\cos\theta_c = (-\sin\phi_C, \cos\phi_c, 0)\,.$
The angles (θ_C, ϕ_C) satisfy the following relations:

$$\sin\theta_C = \sin g \sin i$$
$$\cos\theta_C \sin(\phi_C - h) = \sin g \cos i$$
$$\cos\theta_C \cos(\phi_C - h) = \cos g$$

Planet is at the origin O and the orbital plane of the small body intersects the sphere along a great circle D.



Canonical Hyperbolic Collision Elements

The transformation from $\mathcal{D}_{\mathbf{hyp}}$ to $\mathcal{C}_{\mathbf{hyp}}$ is governed by a suitable *generating function* depending on the old (Delaunay) momenta and the new (angle-like) coordinates:

$$S(L,G,H,\mathbf{w}) = -w_1\,L + (\frac{\pi}{2} - w_3)\,H - \frac{\pi}{2}\,G$$
$$\mp G\ \arccos\left[\frac{G\ \sin w_2}{(G^2-H^2)^{1/2}}\right] \pm H\ \arccos\left[\frac{H\ \tan w_2}{(G^2-H^2)^{1/2}}\right]$$

where $0 \le w_2 \le \pi$ and $0 \le w_3 \le 2\pi$.

$$I_1 = -\frac{\partial S}{\partial w_1} = L \qquad I_2 = -\frac{\partial S}{\partial w_2} = \mp\left(G^2 - \frac{H^2}{\cos^2 w_2}\right)^{1/2}$$

$$I_3 = -\frac{\partial S}{\partial w_3} = H$$

$$l = -\frac{\partial S}{\partial L} = w_1 \quad g = -\frac{\partial S}{\partial G} = \frac{\pi}{2} \pm \arccos\left(\frac{\sin w_2}{\sin i}\right)$$

$$h = \frac{\partial S}{\partial H} = w_3 - \frac{\pi}{2} \mp \arccos\left(\frac{\tan w_2}{\tan i}\right)$$

Canonical Hyperbolic Collision Elements

Using previous relations and from simple computations we obtain

$$\begin{aligned} \sin w_2 &= \sin g \ \sin i \\ \cos w_2 \ \sin(w_3 - h) &= \sin g \ \cos i \\ \cos w_2 \cos(w_3 - h) &= \cos g \end{aligned}$$

Comparing these equations with the equation defining θ_C and ϕ_c, we deduce

$$w_2 = \theta_C \qquad w_3 = \phi_C \ .$$

CONCLUSION:

$$\mathcal{C}_{\mathbf{hyp}} = (L, \Theta, H, l, \theta_C, \phi_C)$$

is a set of canonical elements well defined at collision.

Replacing (L, l) with (U, η)

We want to construct a new canonical set $\mathcal{C}_{\mathbf{opik}}$, applicable within the framework of Öpik's Theory, by replacing the pair of canonically conjugate variables (L, l) of $\mathcal{C}_{\mathbf{hyp}}$ with the couple (U, η) where

$$U = |\mathbf{U}| = \left(\frac{\mu}{a}\right)^{1/2}$$

is the norm of the planetocentric unperturbed velocity vector of the small body and

$$\eta = U\,(t - t_0)$$

is the distance covered by the small body along the asymptote. In terms of Delaunay hyperbolic elements we have

$$U = U(L) = -\frac{\mu}{L} \qquad \eta = \eta(L, l) = \frac{L^2\, l}{\mu}$$

Replacing (L, l) with (U, η)

The transformation from $\mathcal{C}_{\mathbf{hyp}}$ to $\mathcal{C}_{\mathbf{opik}}$ is canonical (completely canonical) iff the Jacobian matrix is symplectic, iff

$$\frac{\partial U}{\partial L} \frac{\partial \eta}{\partial l} = 1 \; .$$

This condition is indeed satisfied, since

$$\frac{\partial U}{\partial L} = \frac{\mu}{L^2} \qquad \text{and} \qquad \frac{\partial \eta}{\partial l} = \frac{L^2}{\mu} \; .$$

The standard Keplerian Hamiltonian becomes

$$\mathcal{K}_{\mathbf{C_{opik}}} = \mathcal{K}_{\mathbf{C_{opik}}}(U) = \frac{1}{2} \, U^2 \, ,$$

and the canonical equation of motion for the coordinate η gives its conjugate momentum U

$$\dot{\eta} = \frac{\partial \mathcal{K}}{\partial U} = U \, .$$

The Encounter

The introduction of this set of elements gives prominence to the local behavior of the small body: around the time of crossing the TP, the small body travels with constant velocity U on a straight line having the direction of the asymptote.
Pre-encounter state vector $(U, \Theta, H, \eta, \theta_C, \phi_C) \rightarrow$ Post-encounter state vector $(U', \Theta', H', \eta', \theta'_C, \phi'_C)$:

$$
\begin{aligned}
U' &= U & \qquad \eta' &= \eta + U(t_2 - t_1) \\
\Theta' &= \Theta & \qquad \theta'_C &= \theta_C \\
H' &= H & \qquad \phi'_C &= \phi_C
\end{aligned}
$$

t_1 is the time of crossing the pre-encounter TP, while t_2 is the time of crossing the post-encounter TP.
The 2-body propagation, like in ordinary treatment of Keplerian motion, is described by five constants and a time-dependent variable. This peculiarity make this set less interesting to study the dynamics of the future close approaches, in particular the structure of resonance and **keyholes**.

- Keyholes: small regions on the TP such that, if the small body passes through one of them, an impact with the planet will occur at the next encounter

Canonical Elements Containing TP Coordinates

We look for three functions acting as the new *coordinate-type canonical variables*

$$\begin{aligned} \xi &: \mathcal{D}^{\mathbf{5}}_{\mathbf{hyp}} \longrightarrow \mathbf{R} \\ \zeta &: \mathcal{D}^{\mathbf{5}}_{\mathbf{hyp}} \longrightarrow \mathbf{R} \\ \eta &: \mathcal{D}_{\mathbf{hyp}} \longrightarrow \mathbf{R} \end{aligned}$$

such that

$$\{\xi,\zeta\} = 0 \qquad \{\xi,\eta\} = 0 \qquad \{\zeta,\eta\} = 0 \ ,$$

and

$$\xi^2 + \zeta^2 = \mathcal{R}^2(b) \ ,$$

where $\mathcal{R}(b)$ is a rescaling function of b. The impact parameter can be expressed as function of the $\mathcal{D}_{\mathbf{hyp}}$ elements, using the angular momentum computed when the small body intersects the TP

$$b = \frac{G}{U} = -\frac{L\,G}{\mu} \ .$$

Results

Proposition

There exist two functions

$$\xi : \mathcal{D}^5_{\mathbf{hyp}} \to \mathbf{R}$$
$$\zeta : \mathcal{D}^5_{\mathbf{hyp}} \to \mathbf{R}$$

which characterize the position of the small body on the TP in some reference system such that

$$\{\xi, \zeta\} = 0 \ .$$

REMARK. If ξ and ζ are functions $\mathcal{D}^5_{\mathbf{hyp}} \to \mathbf{R}$ representing the position of the small body on the TP, then

$$\frac{\partial \xi}{\partial L} \neq 0 \qquad \frac{\partial \zeta}{\partial L} \neq 0 \ ,$$

that is they depend on L. This dependence follows from the definition of TP.

Results

Proposition

Let ξ *and* ζ *be two functions as in Proposition 1. Let us suppose that*

$$\eta(\mathcal{D}_{\mathbf{hyp}}) = l^N\,\bar{\eta}(\mathcal{D}^{\mathbf{5}}_{\mathbf{hyp}}) + \tilde{\eta}(\mathcal{D}^{\mathbf{5}}_{\mathbf{hyp}}),\ \ N \in \mathbf{Z},\ \ \bar{\eta} \neq 0\ ,$$

where l *is the hyperbolic mean anomaly.*
Then (ξ, ζ, η) *are not canonical coordinates.*

Corollary

Let ξ *and* ζ *be two functions as in Proposition 1. If* η *is the distance covered by the small body along the asymptote (the coordinate conjugate to the momentum* U*), then* (ξ, ζ, η) *are not canonical coordinates.*

Main result

Theorem

If ξ *and* ζ *are two functions as in Proposition 1, then it is* **NOT possible** *to find a function*

$$\eta : \mathcal{D}_{\mathbf{hyp}} \longrightarrow \mathbf{R}$$

such that

$$\{\xi, \eta\} = 0 \quad \textit{and} \quad \{\zeta, \eta\} = 0 \ .$$

Sketch of the Proof

1. Let us suppose that there exists a function

$$\eta : \mathcal{D}_{\mathbf{hyp}} \longrightarrow \mathbf{R}$$

such that

$$\{\xi, \eta\} = 0 \quad \text{and} \quad \{\zeta, \eta\} = 0 \ ,$$

where ξ and ζ are as in Proposition 1.

2. After some computations we arrive to prove that

$$\begin{aligned} & \exists\ \eta(\mathcal{D}_{hyp}) \ : \ \{\xi, \eta\} = 0 \text{ and } \{\zeta, \eta\} = 0 \\ \Rightarrow \quad & \eta(\mathcal{D}_{hyp}) \in \mathcal{S} \ , \end{aligned}$$

where $\mathcal{S}$ is the family of solutions of the linear homogeneous partial differential equation

$$G \frac{\partial \eta}{\partial l} + L \frac{\partial \eta}{\partial g} = 0 \ .$$

Then also the following implication is true

$$\eta(\mathcal{D}_{\mathbf{hyp}}) \notin \mathcal{S}$$

$$\Rightarrow \forall\ \eta(\mathcal{D}_{\mathbf{hyp}}) \ \ \{\xi, \eta\} \neq 0 \text{ or } \{\zeta, \eta\} \neq 0 \ ,$$

Sketch of the Proof

3. To conclude the proof we show that if η belongs to $\mathcal{S}$, then (ξ, ζ, η) cannot be canonical coordinates. If η belongs to $\mathcal{S}$ then ξ and ζ must satisfy the following PDEs:

$$L\,\frac{\partial \xi}{\partial L} = (G-g)\,\frac{\partial \xi}{\partial G} \qquad L\,\frac{\partial \zeta}{\partial L} = (G-g)\,\frac{\partial \zeta}{\partial G}\,.$$

Solutions:

$$\xi = \xi(L\,(G-g), H, g, h) \qquad \zeta = \zeta(L\,(G-g), H, g, h)$$

But functions of this form do not satisfy the relation on the TP coordinates required by the hyphoteses of the theorem:

$$\xi^2 + \zeta^2 = \mathcal{R}^2(b),$$

$$b = b(L, G) = -\frac{L\,G}{\mu}$$

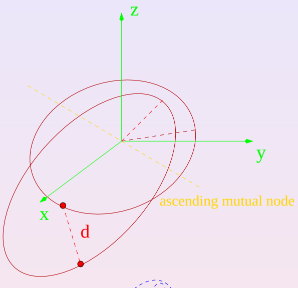
This contradiction concludes the proof.

Contents

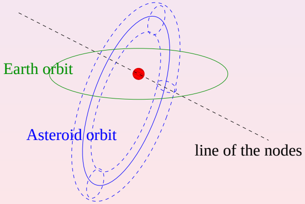
Minimal Orbit Intersection Distance

MOID (Minimal Orbit Intersection Distance): minimal distance between two confocal Keplerian orbits

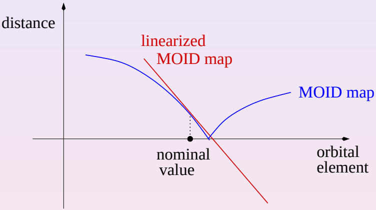
Potentially Hazardous Asteroid (PHA): asteroid having MOID $\leq 0.05\ AU$ and absolute magnitude $H \leq 22$.

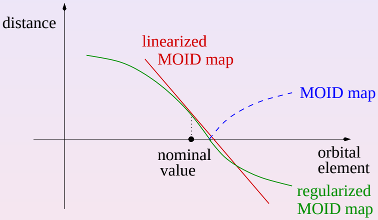


PROBLEM: even if an asteroid is not a PHA taking into account its nominal orbit, considering the uncertainty of its orbit it could have a significant probability to be a PHA.



Given a nominal orbit $\bar{\mathcal{E}}$, with its covariance matrix $\Gamma_{\bar{\mathcal{E}}}$, the propagation of the covariance of a function of the orbit consists in a linearization of the function in a neighbourhood of $\bar{\mathcal{E}}$.





Note that $d_{min}(\mathcal{E})$ is not smooth where it vanishes, thus the linearization is not a good approximation (Figure on the left).

PROBLEM: is it possible to give a sign to the minimal distance in such a way that the linearization makes sense? (Figure on the right)

Contents

The Keplerian distance function and its critical points

$\mathcal{E} = (E_1, E_2)$: set of 10 elements that defines the geometric configuration of the 2 orbits
$V = (v_1, v_2)$: parameters along the orbits
$\mathcal{X}_1 = \mathcal{X}_1(E_1, v_1)$, $\mathcal{X}_2 = \mathcal{X}_2(E_2, v_2) \in \mathbb{R}^3$: Cartesian coordinates of two bodies on the two orbits

$\mathcal{X}_r$ is an analytic function of the elements (E_r, v_r) for $r = 1, 2$.

Definition

For each choice of the orbit parameters $\mathcal{E}$ we define the *Keplerian distance function* d as the map

$$\mathcal{V} \ni V \mapsto d(\mathcal{E}, V) \overset{def}{=} \sqrt{\langle \mathcal{X}_1 - \mathcal{X}_2, \mathcal{X}_1 - \mathcal{X}_2 \rangle} \in \mathbb{R}^+ \ ,$$

where $\mathcal{V} = \mathbb{T}^2 = S^1 \times S^1$ (a two–dimensional torus) if both orbits are bounded, $\mathcal{V} = S^1 \times \mathbb{R}$ (an infinite cylinder) if only one is bounded, and $\mathcal{V} = \mathbb{R} \times \mathbb{R}$ if they are both unbounded.

The Keplerian distance function and its critical points

Let

$$V_j(\mathcal{E}) = (v_1^{(j)}(\mathcal{E}), v_2^{(j)}(\mathcal{E}))$$

be the values of the j-th critical point of $d^2(\mathcal{E}, \cdot)$, solution of

$$\nabla_V d^2(\mathcal{E}, V) = 0\,, \qquad (1)$$

with

$$\nabla_V d^2 = \left(\frac{\partial d^2}{\partial v_1}, \frac{\partial d^2}{\partial v_2}\right)^t,$$

and let

$$\mathcal{X}_1^{(j)}(\mathcal{E}) = \mathcal{X}_1(E_1, v_1^{(j)}(\mathcal{E}))\,; \qquad \mathcal{X}_2^{(j)}(\mathcal{E}) = \mathcal{X}_2(E_2, v_2^{(j)}(\mathcal{E}))$$

be the corresponding Cartesian coordinates.

The Keplerian distance function and its critical points

The number of critical points of d^2 is generically finite; Gronchi (2002) has proved that they can be infinitely many only in the case of two coplanar (concentric) circles or two overlapping conics. Except for these two very peculiar cases, we can define the the **Keplerian distance at the j-th critical point of d^2** is

$$\begin{aligned} d_j(\mathcal{E}) &\stackrel{def}{=} d(\mathcal{E}, V_j(\mathcal{E})) = \\ &= \sqrt{\langle \mathcal{X}_1^{(j)}(\mathcal{E}) - \mathcal{X}_2^{(j)}(\mathcal{E}), \mathcal{X}_1^{(j)}(\mathcal{E}) - \mathcal{X}_2^{(j)}(\mathcal{E}) \rangle} \,. \end{aligned}$$

Definition

Calling $\mathfrak{E}$ the *two–orbit configuration space*, locally homeomorphic to $\mathbb{R}^{10}$, we define the maps

$$\mathfrak{E} \ni \mathcal{E} \mapsto V_j(\mathcal{E}) \in \mathcal{V}\,; \qquad \mathfrak{E} \ni \mathcal{E} \mapsto d_j(\mathcal{E}) \in \mathbb{R}^+\,,$$

representing the j-th critical point of $d^2(\mathcal{E}, \cdot)$ and the corresponding value of the distance for a given configuration $\mathcal{E}$.

The Keplerian distance function and its critical points

Non-degeneracy condition

If

$$\det \mathcal{H}_V(d^2)(\bar{\mathcal{E}}, V_j(\bar{\mathcal{E}})) \neq 0 \qquad (2)$$

holds for a given configuration $\bar{\mathcal{E}}$ and for every index j of the critical points of $d^2(\bar{\mathcal{E}}, \cdot)$, then there exists an open neighborhood $\mathfrak{U} \subset \mathfrak{E}$ of $\bar{\mathcal{E}}$ such that the number of critical points of $d^2(\mathcal{E}, \cdot)$ is the same for each $\mathcal{E} \in \mathfrak{U}$. We can define the maps V_j and d_j in the neighborhood $\mathfrak{U}$ for every index j of such critical points. Moreover we can choose $\mathfrak{U}$ and the order of the critical points in a way that each map V_j is analytic.
The partial derivatives of V_j with respect to the element $\mathcal{E}_k$ at $\mathcal{E} \in \mathfrak{U}$ are given by

$$\frac{\partial V_j}{\partial \mathcal{E}_k}(\mathcal{E}) = -\left[\mathcal{H}_V(d^2)(\mathcal{E}, V_j(\mathcal{E}))\right]^{-1} \frac{\partial}{\partial \mathcal{E}_k} \nabla_V d^2(\mathcal{E}, V_j(\mathcal{E})), \qquad (3)$$

for $k = 1 \ldots 10$, where

$$\frac{\partial}{\partial \mathcal{E}_k} \nabla_V d^2 = \left(\frac{\partial^2 d^2}{\partial \mathcal{E}_k \partial v_1}, \frac{\partial^2 d^2}{\partial \mathcal{E}_k \partial v_2} \right)^t .$$

The Keplerian distance function and its critical points

We shall be particularly interested in the **local minimum points**, corresponding to the subset of indexes j_h:

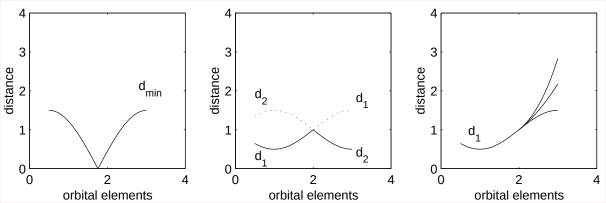
$$\mathcal{E} \mapsto d_{j_h}(\mathcal{E}) \stackrel{def}{=} d_h(\mathcal{E}) \qquad (\textit{locally minimal distance})\,. \tag{4}$$

When at least one orbit is bounded we define the **absolute minimum map**

$$\mathcal{E} \mapsto d_{min}(\mathcal{E}) \stackrel{def}{=} \min_h d_h(\mathcal{E}), \tag{5}$$

that for each two–orbit configuration returns the orbit distance.

Singularities of d_h and d_{min}



(I) d_h and d_{min} are not differentiable where they vanish;

(II) in a neighborhood of a two orbit configuration $\bar{\mathcal{E}}$, two local minima can exchange their role as absolute minimum: then d_{min} can lose its regularity even without vanishing;

(III) when a bifurcation occurs the definition of the maps d_h may become ambiguous after the bifurcation point. Note that this ambiguity does not occur for the d_{min} map. The bifurcation phenomena can occur only where the Hessian matrix of $d^2(E, V)$ is degenerate.

Regularization of the minimal distance maps

The goal is to prove that the maps d_h, defined in (4), are generically not regular functions of the orbital elements $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_{10})$ where they vanish, but it is possible to remove this singularity by performing a suitable cut–off of its definition domain and changing the sign of these maps on selected subsets of the smaller resulting domain. The same results are also valid for the map d_{min}, apart maybe the configurations with two intersection points.

Example:

$$f(x, y) = \sqrt{x^2 + y^2}\,;$$

its directional derivatives at $(x, y) = (0, 0)$ do not exist for every choice of the direction. We cut off the line $\{(x, y)\,|\,x = 0\}$ from the definition domain and change the sign of the function on the set $\{x > 0\}$: the result is the continuous function

$$\tilde{f}(x, y) = \begin{cases} -f(x, y) & \text{for } x > 0 \\ f(x, y) & \text{for } x < 0 \end{cases} .$$

We can extend $\tilde{f}$ by continuity to the origin by setting $\tilde{f}(0, 0) = 0$, thus we obtain a function having all the directional derivatives at $(x, y) = (0, 0)$.

Derivatives of the minimal distance maps

Minimal distance map $d_h : \mathfrak{U} \to \mathbb{R}^+$ and a two–orbit configuration $\bar{\mathcal{E}} \in \mathfrak{U}$ with $d_h(\bar{\mathcal{E}}) \neq 0$.

The derivative of d_h at $\bar{\mathcal{E}}$ with respect to the orbital element $\mathcal{E}_k$ is given by

$$\frac{\partial d_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = \frac{1}{2d_h(\bar{\mathcal{E}})} \frac{\partial d_h^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) \qquad \text{for } k = 1 \dots 10\,,$$

where, using the chain rule,

$$\frac{\partial d_h^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = \frac{\partial d^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})) + \frac{\partial d^2}{\partial V}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})) \frac{\partial V_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}})$$

with

$$\frac{\partial V_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = -\left[\mathcal{H}_V(d^2)(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\right]^{-1} \frac{\partial}{\partial \mathcal{E}_k} \nabla_V d^2(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\,.$$

Moreover we have

$$\frac{\partial d_h^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = \frac{\partial d^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\,, \qquad (6)$$

in fact

$$\frac{\partial d^2}{\partial V}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})) = 0$$

because $V_h(\bar{\mathcal{E}})$ is a critical point of $d^2(\bar{\mathcal{E}}, \cdot)$.

Derivatives of the minimal distance maps

Using (6) and the differences

$$\Delta = \mathcal{X}_1 - \mathcal{X}_2\,; \qquad \Delta_h = \mathcal{X}_1^{(h)} - \mathcal{X}_2^{(h)}$$

we can write

$$\frac{\partial d_h^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = 2\left\langle \Delta_h(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})), \frac{\partial \Delta}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\right\rangle\ ,$$

so that, if $d_h(\bar{\mathcal{E}}) \neq 0$, we have

$$\frac{\partial d_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = \left\langle \hat{\Delta}_h(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})), \frac{\partial \Delta}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\right\rangle \tag{7}$$

where

$$\hat{\Delta}_h = \frac{\Delta_h}{d_h} \tag{8}$$

is the unit vector map having the direction of the line joining the points on the two orbits that correspond to the local minimum point $V_h(\mathcal{E})$.
If $d_h(\bar{\mathcal{E}}) = 0$, then (8) becomes singular and the limit of $\hat{\Delta}_h(\mathcal{E})$ for $\mathcal{E} \to \bar{\mathcal{E}}$ does not exist.
Generically the direction (but *not* the orientation) of the unit vector $\hat{\Delta}_h$ is unique also in the limit $\mathcal{E} \to \bar{\mathcal{E}}$ with $d_h(\bar{\mathcal{E}}) = 0$.

Intuitively this is due to a geometric characterization of the critical points of the squared distance function: the line joining two points on the curves that correspond to a critical point must be orthogonal to both tangent vectors to the curves at those points.

Sketch of regularization

We can remove the singularity appearing in (7) for the configurations $\bar{\mathcal{E}} \in \mathfrak{U}$ such that $d_h(\bar{\mathcal{E}}) = 0$ by performing the following operations:

1. we choose a subset of the domain $\mathfrak{U}$ to cut–off, that properly contains the set
$$\{d_h = 0\} \overset{def}{=} \{\mathcal{E} \in \mathfrak{U} : d_h(\mathcal{E}) = 0\}\,;$$
2. we change the sign of $\hat{\Delta}_h$ in different subsets of the smaller resulting domain $\mathfrak{U}_h$, depending on the selected minimum point index h;
3. $\forall\, \mathcal{E} \in \mathfrak{U}_h$, we give $d_h(\mathcal{E})$ the same sign as the one selected for $\hat{\Delta}_h(\mathcal{E})$ in the previous step;
4. we show that the resulting function, called $\tilde{d}_h$, is continuous and continuously extendable to a wider domain $\tilde{\mathfrak{U}}_h$, that includes all the orbit crossings in $\mathfrak{U}$ but the tangent ones.

Results

$\tau_1(\mathcal{E})$, $\tau_2(\mathcal{E})$: tangent vectors to the two orbits at the points $\mathcal{X}_1^{(h)}(\mathcal{E})$, $\mathcal{X}_2^{(h)}(\mathcal{E})$, corresponding to $V_h(\mathcal{E})$.

$$\mathcal{T} = \begin{pmatrix} \tau_{1,x} & \tau_{1,y} & \tau_{1,z} \\ \tau_{2,x} & \tau_{2,y} & \tau_{2,z} \end{pmatrix}$$

$$\mathcal{T}_1 = \begin{pmatrix} \tau_{1,y} & \tau_{1,z} \\ \tau_{2,y} & \tau_{2,z} \end{pmatrix}; \quad \mathcal{T}_2 = \begin{pmatrix} \tau_{1,z} & \tau_{1,x} \\ \tau_{2,z} & \tau_{2,x} \end{pmatrix}; \quad \mathcal{T}_3 = \begin{pmatrix} \tau_{1,x} & \tau_{1,y} \\ \tau_{2,x} & \tau_{2,y} \end{pmatrix}.$$

The matrix $\mathcal{T}(\mathcal{E})$ has rank < 2 if and only if the two tangent vectors $\tau_1(\mathcal{E}), \tau_2(\mathcal{E})$ are parallel. In case of orbit crossing the matrix $\mathcal{T}(\mathcal{E})$ has rank < 2 if and only if $\mathcal{E}$ is a tangent crossing configuration. We introduce the maps

$$S_1 = \Delta_x^{(h)} \det(\mathcal{T}_1); \quad S_2 = \Delta_y^{(h)} \det(\mathcal{T}_2); \quad S_3 = \Delta_z^{(h)} \det(\mathcal{T}_3);$$

$$\tau_3 = \tau_1 \times \tau_2 = (\det(\mathcal{T}_1), \det(\mathcal{T}_2), \det(\mathcal{T}_3))$$

Results

We define the regularized function $\tilde{d}_h : \mathfrak{U}_h \rightarrow \mathbb{R}$ by giving a sign to d_h, restricted to $\mathfrak{U}_h$, according to the following rules:

Definition

$$\tilde{d}_h := \begin{cases} \mathrm{sign}(S_1)\, d_h & \text{where } S_1 \neq 0 \\ \mathrm{sign}(S_2)\, d_h & \text{where } S_2 \neq 0 \\ \mathrm{sign}(S_3)\, d_h & \text{where } S_3 \neq 0 \end{cases} . \tag{9}$$

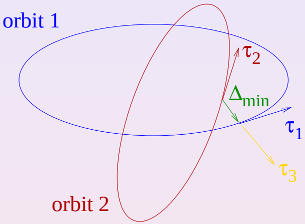
Proposition

The continuous map $\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ is analytic in $\tilde{\mathfrak{U}}_h$ and relation

$$\frac{\partial \tilde{d}_h}{\partial \mathcal{E}_k}(\mathcal{E}) = \left\langle \hat{\tau}_3(\mathcal{E}), \frac{\partial \Delta}{\partial \mathcal{E}_k}(\mathcal{E}, V_h(\mathcal{E})) \right\rangle \qquad k = 1 \ldots 10 \, . \tag{10}$$

gives a formula to compute its partial derivatives.

Geometric Characterization



τ_1, τ_2: tangent vectors to the orbits at the minimum point.

$$\tau_3 = \tau_1 \times \tau_2$$

Regularized map $\tilde{d}_{min}$: $|\tilde{d}_{min}| = d_{min}$ and we choose the sign $+$ for $\tilde{d}_{min}$ if Δ_{min} and τ_3 have the same orientation, the sign $-$ otherwise. This sign is well defined, with the only exception of the cases in which τ_1 and τ_2 are parallel.

Computation of the uncertainty of d_h and d_{min}

Computing the uncertainty of the values of $\tilde{d}_h(\bar{\mathcal{E}})$ we make the following assumptions:

i) we can approximate the target function with the quadratic function defined by the normal matrix, as explained in the previous section;

ii) we can approximate the map $\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ with its linearization around the nominal configuration $\bar{\mathcal{E}}$;

iii) the determination of the two orbits are independent .

$$\Gamma_{\bar{\mathcal{E}}} = \begin{bmatrix} \Gamma_{\bar{E}_1} & 0 \\ 0 & \Gamma_{\bar{E}_2} \end{bmatrix} .$$

We compute the covariance of $\tilde{d}_h(\bar{\mathcal{E}})$ by performing a linear propagation of the matrix $\Gamma_{\bar{\mathcal{E}}}$ (i.e. using assumption *ii*)):

$$\Gamma_{\tilde{d}_h(\bar{\mathcal{E}})} = \left[\frac{\partial \tilde{d}_h}{\partial \mathcal{E}}(\bar{\mathcal{E}})\right] \Gamma_{\bar{\mathcal{E}}} \left[\frac{\partial \tilde{d}_h}{\partial \mathcal{E}}(\bar{\mathcal{E}})\right]^t . \tag{11}$$

The *standard deviation*, defined as

$$\sigma_h(\bar{\mathcal{E}}) = \sqrt{\Gamma_{\tilde{d}_h(\bar{\mathcal{E}})}},$$

gives us a way to define a range of uncertainty for $\tilde{d}_h(\bar{\mathcal{E}})$: if we assume that the minimal distance $\tilde{d}_h(\bar{\mathcal{E}})$ is a Gaussian random variable, there is a high probability ($\sim 99.7\%$) that its value is within the interval

$$I_h(\bar{\mathcal{E}}) = [\tilde{d}_h(\bar{\mathcal{E}}) - 3\sigma_h(\bar{\mathcal{E}}), \tilde{d}_h(\bar{\mathcal{E}}) + 3\sigma_h(\bar{\mathcal{E}})] . \tag{12}$$

Virtual PHAs

VPHA	dist	RMS	H	prob
1994XG	0.063	0.030	18.58	33%
2006FW$_{33}$	0.066	0.111	20.12	30%
2000VZ$_{44}$	-0.052	0.003	21.03	25%
2006FW$_{33}$	0.108	0.115	20.12	22%
2006KT$_{67}$	0.111	0.145	19.59	20%
2006CD	-0.142	0.155	20.46	17%
1999UZ$_{5}$	0.055	0.004	21.87	12%
1984QY$_{1}$	0.179	0.084	14.16	6%
2006OV$_{5}$	0.192	0.090	19.02	6%
2000RK$_{12}$	0.056	0.004	21.27	5%

Table: VPHAs (down to probability $> 5\%$) in the "official" list of NEAs, that are not PHAs according to their nominal orbit.

References

- G. F. Gronchi **On the stationary points of the squared distance between two ellipses with a common focus**, SIAM Journ, Sci. Comp., Vol. 24, pp. 61-80, 2002
- G.B. Valsecchi, A. Milani, G.F. Gronchi and S.R. Chesley **Resonant returns to close approaches: Analytical theory**, Astronomy and Astrophysics, Vol. 408, p.1179-1196, 2003
- G. T. **Canonical elements for Öpik theory**, Celestial Mechanics & Dynamical Astronomy, Vol. 94, pp. 173-195, 2006
- G.F. Gronchi and G. T. **On the uncertainty of the minimal distances between two confocal Keplerian orbits**, Discrete and Continuous Dynamical Systems Series B, Volume 7, Number 4, pp. 755 - 778, 2007
- G.F. Gronchi, G. T. and A. Milani **Mutual geometry of confocal Keplerian orbits: uncertainty of the MOID and search for Virtual PHAs**, Proceedings of IAU Symposium 236, Cambridge University Press, pp.3-14, 2007