

Esercizio (6 giugno 2017)

Si consideri la funzione hamiltoniana

$$H_\varepsilon(I, \varphi) = h(I) + \varepsilon f(\varphi)$$

con

$$h(I) = I_1 + I_2$$

$$f(\varphi) = \cos^2(\varphi_1 + 2\varphi_2) \sin^2(\varphi_1 - 2\varphi_2)$$

$$I = (I_1, I_2) \in \mathbb{R}^2,$$

$$\varphi = (\varphi_1, \varphi_2) \in \mathbb{T}^2, \quad \varepsilon \ll 1$$

Determinare una funzione generatrice di una trasformazione canonica vicina all'identità

$$(I, \varphi) \xrightarrow{\Psi_\varepsilon} (\tilde{I}, \tilde{\varphi})$$

tale che

$$K_\varepsilon = H_\varepsilon \circ \Psi_\varepsilon^{-1}$$

non dipenda da $\tilde{\varphi}$ al primo ordine in ε

Sol.

Funzione generatrice

$$S(\varphi, \tilde{I}; \varepsilon) = \varphi \cdot \tilde{I} + \varepsilon W(\varphi, \tilde{I})$$

$$\begin{cases} \tilde{\varphi} = \frac{\partial S}{\partial \tilde{I}} = \varphi + \varepsilon \frac{\partial W}{\partial \tilde{I}} \\ I = \frac{\partial S}{\partial \varphi} = \tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi} \end{cases}$$

Abbiamo

$$\begin{aligned}K_\varepsilon &= H_\varepsilon \circ \psi_\varepsilon^{-1} = h \left(\tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi} \right) + \\&\quad \varepsilon \phi \left(\tilde{\varphi} - \varepsilon \frac{\partial W}{\partial \tilde{I}} \right) \\&= h(\tilde{I}) + \varepsilon \frac{\partial h}{\partial I}(\tilde{I}) \cdot \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) + \\&\quad \varepsilon \phi(\tilde{\varphi}) + O(\varepsilon^2)\end{aligned}$$

scriviamo l'equazione omologica

$$\frac{\partial h}{\partial I}(\tilde{I}) \cdot \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) + \phi(\tilde{\varphi}) = \langle \phi \rangle_\varphi$$

le frequenze sono date da

$$\omega = \frac{\partial h}{\partial I} = (1, 1)^T$$

$$\omega \cdot \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) + \phi(\tilde{\varphi}) = \langle \phi \rangle_\varphi$$

$$\phi(\varphi) = \cos^2(\varphi_1 + 2\varphi_2) \sin^2(\varphi_1 - 2\varphi_2)$$

$$\cos^2(\varphi_1 + 2\varphi_2) = \left(\frac{e^{i(\varphi_1 + 2\varphi_2)} + e^{-i(\varphi_1 + 2\varphi_2)}}{2} \right)^2$$

$$= \frac{1}{4} \left(e^{i(2\varphi_1 + 4\varphi_2)} + e^{-i(2\varphi_1 + 4\varphi_2)} + 2 \right)$$

$$\sin^2(\varphi_1 - 2\varphi_2) = \left(\frac{e^{i(\varphi_1 - 2\varphi_2)} - e^{-i(\varphi_1 - 2\varphi_2)}}{2i} \right)^2$$

$$= -\frac{1}{4} \left(e^{i(2\varphi_1 - 4\varphi_2)} + e^{-i(2\varphi_1 - 4\varphi_2)} - 2 \right)$$

$$\cos^2(\varphi_1 + 2\varphi_2) \sin^2(\varphi_1 - 2\varphi_2) =$$

$$-\frac{1}{16} \left(e^{i(4\varphi_1)} + e^{i(8\varphi_2)} - 2e^{i(2\varphi_1 + 4\varphi_2)} \right. \\ \left. + e^{-i(8\varphi_2)} + e^{-i(4\varphi_1)} - 2e^{-i(2\varphi_1 + 4\varphi_2)} \right. \\ \left. + 2e^{i(2\varphi_1 - 4\varphi_2)} + 2e^{-i(2\varphi_1 - 4\varphi_2)} - 4 \right)$$

scriviamo lo sviluppo in serie di Fourier di ϕ

$$\phi(\varphi) = \sum_{k \in K} \hat{\phi}_k e^{ik \cdot \varphi}$$

dove l'insieme K è dato da

$$K = \{(0,0), \pm(4,0), (0, \pm 8), \pm(2,4), \\ \pm(2,-4)\}$$

notiamo che la condizione di non risonanza

$$\omega \cdot k \neq 0 \quad (\omega = (1,1)^T)$$

è soddisfatta per ogni $k \in K$

la media sugli angoli di ϕ è

$$\langle \phi \rangle_{\varphi} = \frac{1}{4} = \hat{\phi}_{(0,0)}$$

e gli altri termini sono

$$\hat{\phi}_{(4,0)} = -\frac{1}{16}, \quad \hat{\phi}_{-(4,0)} = -\frac{1}{16} \quad k = \pm(4,0)$$

$$\hat{\phi}_{(0,8)} = -\frac{1}{16}, \quad \hat{\phi}_{-(0,8)} = -\frac{1}{16} \quad k = \pm(0,8)$$

$$\hat{\phi}_{(2,4)} = \frac{1}{8}, \quad \hat{\phi}_{-(2,4)} = \frac{1}{8} \quad k = \pm(2,4)$$

$$\hat{\phi}_{(2,-4)} = -\frac{1}{8}, \quad \hat{\phi}_{-(2,-4)} = -\frac{1}{8} \quad k = \pm(2,-4)$$

Possiamo ora determinare $W(\varphi, \tilde{I})$

Notiamo che W non dipenderà da $\tilde{I}$; scriviamo

$$W(\varphi) = \sum_{k \in K \setminus (0,0)} \hat{W}_k e^{i k \cdot \varphi}$$

dove i termini $\hat{W}_k$ si trovano imponendo l'equazione omologica, si ha

$$\hat{W}_k = -\frac{\hat{\phi}_k}{i \omega \cdot k}$$

procediamo

$$\hat{W}_{(4,0)} = -\frac{\hat{\phi}_{(4,0)}}{4i} = -\left(-\frac{1}{16}\right) \frac{1}{4i} = \frac{1}{64i}$$

$$k = (4,0), \quad \omega = (1,1) \quad k \cdot \omega = 4$$

$$\hat{W}_{-(4,0)} = - \frac{\hat{\mathcal{L}}_{-(4,0)}}{-4i} = \left(-\frac{1}{16}\right) \frac{1}{4i} = -\frac{1}{64i}$$

$\kappa = -(4,0), \omega = (1,1) \quad \kappa \cdot \omega = -4$

$$\hat{W}_{(0,8)} = - \frac{\hat{\mathcal{L}}_{(0,8)}}{8i} = - \left(-\frac{1}{16}\right) \frac{1}{8i} = \frac{1}{128i}$$

$\kappa = (0,8), \omega = (1,1) \quad \kappa \cdot \omega = 8$

$$\hat{W}_{-(8,0)} = - \frac{\hat{\mathcal{L}}_{-(8,0)}}{-8i} = \left(-\frac{1}{16}\right) \frac{1}{8i} = -\frac{1}{128i}$$

$\kappa = -(0,8), \omega = (1,1) \quad \kappa \cdot \omega = -8$

$$\hat{W}_{(2,4)} = - \frac{\hat{\mathcal{L}}_{(2,4)}}{6i} = - \left(\frac{1}{8}\right) \frac{1}{6i} = -\frac{1}{48i}$$

$\kappa = (2,4), \omega = (1,1) \quad \kappa \cdot \omega = 6$

$$\hat{W}_{-(2,4)} = - \frac{\hat{\mathcal{L}}_{-(2,4)}}{-6i} = \left(\frac{1}{8}\right) \frac{1}{6i} = \frac{1}{48i}$$

$\kappa = -(2,4), \omega = (1,1) \quad \kappa \cdot \omega = -6$

$$\hat{W}_{(2,-4)} = - \frac{\hat{\mathcal{L}}_{(2,-4)}}{-2i} = \left(-\frac{1}{8}\right) \frac{1}{2i} = -\frac{1}{16i}$$

$\kappa = (2,-4), \omega = (1,1) \quad \kappa \cdot \omega = -2$

$$\hat{W}_{-(2,-4)} = - \frac{\hat{\mathcal{F}}_{-(2,-4)}}{\underset{\substack{\uparrow \\ k = -(2,-4), \omega = (1,1) \quad k \cdot \omega = 2}}{2i}} = - \left(-\frac{1}{8} \right) \frac{1}{2i} = \frac{1}{16i}$$

Si è ottenuto

$$W(\varphi) = \frac{1}{64i} \left(e^{i4\varphi_1} - e^{-i4\varphi_1} \right) +$$

$$\frac{1}{128i} \left(e^{i8\varphi_2} - e^{-i8\varphi_2} \right) - \frac{1}{48i} \left(e^{i(2\varphi_1+4\varphi_2)} \right.$$

$$\left. - e^{-i(2\varphi_1+4\varphi_2)} \right) - \frac{1}{16i} \left(e^{i(2\varphi_1-4\varphi_2)} - e^{-i(2\varphi_1-4\varphi_2)} \right)$$

cioè

$$W(\varphi) = \frac{1}{32} \sin 4\varphi_1 + \frac{1}{64} \sin 8\varphi_2 - \frac{1}{24} \sin (2\varphi_1 + 4\varphi_2)$$

$$- \frac{1}{8} \sin (2\varphi_1 - 4\varphi_2)$$

e la funzione generatrice creata è

$$S(\varphi, \tilde{I}; \varepsilon) = \varphi \cdot \tilde{I} + \varepsilon \left(\frac{1}{32} \sin 4\varphi_1 + \frac{1}{64} \sin 8\varphi_2 \right.$$

$$\left. - \frac{1}{24} \sin (2\varphi_1 + 4\varphi_2) - \frac{1}{8} \sin (2\varphi_1 - 4\varphi_2) \right)$$

La forma normale non risonante è

$$K_\varepsilon(\tilde{I}, \tilde{\varphi}) = h(\tilde{I}) + \varepsilon \hat{\mathcal{F}}_{(0,0)} + O(\varepsilon^2)$$

$$K_\varepsilon(\tilde{I}, \tilde{q}) = \tilde{I}_1 + \tilde{I}_2 + \frac{\varepsilon}{4} + O(\varepsilon^2)$$

Esercizio (18 dicembre 2014)

Consideriamo la funzione hamiltoniana

$$H_\varepsilon(I, \varphi) = h(I) + \varepsilon \phi(I, \varphi)$$

$I \in \mathbb{R}^n$, $\varphi \in \mathbb{T}^n$, $\varepsilon \ll 1$, con

$$h(I) = \frac{1}{2} |I|^2$$

$$\phi(I, \varphi) = \sum_{j=1}^n I_j \cos(j \varphi_j) + |I|^2 \cos^2 \left(\sum_{j=1}^n \varphi_j \right)$$

Determinare una funzione generatrice di una trasformazione canonica vicina all'identità

$$(I, \varphi) \xrightarrow{\psi_\varepsilon} (\tilde{I}, \tilde{q})$$

tale che

$$K_\varepsilon = H_\varepsilon \circ \psi_\varepsilon^{-1}$$

non dipenda da $\tilde{q}$ al primo ordine in ε

Sol.

Ricordiamo l'equazione omologica

$$\omega \cdot \frac{\partial W}{\partial \varphi} = g - \phi$$

$$\omega(I) = \frac{\partial h}{\partial I} = I, \quad \omega_j = I_j, \quad j = 1, \dots, n$$

$$g = \langle f \rangle_\psi$$

calcoliamo la media $\langle f \rangle_\psi$

$$\langle f \rangle_\psi = \frac{1}{(2\pi)^n} \int_{\Pi^n} f \, d\psi$$

$$= \frac{1}{(2\pi)^n} \int_{\Pi^n} \left[\sum_{j=1}^n I_j \cos(j\psi_j) + |I|^2 \cos^2 \left(\sum_{j=1}^n \psi_j \right) \right] d\psi$$

$$= \frac{1}{(2\pi)^n} \int_{\Pi^n} |I|^2 \cos^2 \left(\sum_{j=1}^n \psi_j \right) d\psi$$

$$= \frac{1}{(2\pi)^n} \int_{\Pi^{n-1}} |I|^2 \left[\frac{1}{2} \psi_1 + \frac{1}{4} \sin \left(2 \sum_{j=1}^n \psi_j \right) \right] \Big|_0^{2\pi} d\psi_2 \dots d\psi_n$$

$$\cos^2 \alpha = \frac{1}{2} (1 + \cos 2\alpha)$$

$$= \frac{1}{(2\pi)^n} \int_{\Pi^{n-1}} \frac{|I|^2}{2} (2\pi) d\psi_2 \dots d\psi_n$$

quindi si ottiene

$$\langle f \rangle_\psi(I) = \frac{1}{2} |I|^2$$

$$\phi(I, \varphi) = \sum_{k \in K} \hat{\phi}_k(I) e^{i k \cdot \varphi}$$

$$\begin{aligned} \phi(I, \varphi) &= \sum_{j=1}^m I_j \left(\frac{e^{i(j\varphi_j)} + e^{-i(j\varphi_j)}}{2} \right) + \\ &\quad \frac{|I|^2}{4} \left(e^{i \sum_{j=1}^m \varphi_j} + e^{-i \sum_{j=1}^m \varphi_j} \right)^2 \\ &= \sum_{j=1}^m I_j \left(\frac{e^{i(j\varphi_j)} + e^{-i(j\varphi_j)}}{2} \right) + \\ &\quad \frac{|I|^2}{4} \left(e^{2i \sum_{j=1}^m \varphi_j} + e^{-2i \sum_{j=1}^m \varphi_j} + 2 \right) \end{aligned}$$

determiniamo ora l'insieme K

$$K = \{0, \pm e_1, \dots, \pm e_j, \dots, \pm e_m, \pm 2e\}$$

con

e_j , $j = 1, \dots, m$, vettori della base canonica di $\mathbb{R}^m$

$$e_1 = (1, 0, 0, \dots, 0)$$

$\vdots$

$$e_j = (0, 0, \dots, 1, \dots, 0)$$

$\vdots$

$$e_m = (0, 0, \dots, 0, 1)$$

inoltre

$$e = \sum_{j=1}^m e_j$$

possiamo trovare i termini $\hat{\phi}_k(I)$:

$$\hat{\phi}_0(I) = \langle \phi \rangle_e(I) = \frac{|I|^2}{2}$$

$$\hat{\phi}_{\pm j e_j}(I) = \frac{I_j}{2}, \quad j = 1, \dots, m$$

$$\hat{\phi}_{\pm 2e}(I) = \frac{|I|^2}{4}$$

e da questi, si determina

$$W(e, I) = \sum_{k \in K \setminus \{0\}} \hat{W}_k(I) e^{ik \cdot e}$$

con

$$\hat{W}_{j e_j}(I) = - \frac{\hat{\phi}_{j e_j}(I)}{j i \omega \cdot e_j} = - \frac{1}{i j I_j} \frac{I_j}{2}$$

$$\omega = I \quad \omega \cdot e_j = \omega_j = I_j$$

$$\hat{W}_{j e_j}(I) = - \frac{1}{2 i j}$$

$$\hat{W}_{-j e_j}(I) = - \frac{\hat{\phi}_{-j e_j}(I)}{-j i \omega \cdot e_j} = \frac{1}{2 i j}$$

$$\hat{W}_{2e}(I) = - \frac{\hat{\phi}_{2e}(I)}{i \omega \cdot e} = - \frac{1}{i \sum_{j=1}^m I_j} \frac{|I|^2}{4}$$

$$\omega \cdot e = \sum_{j=1}^m I_j$$

$$\hat{W}_{-2e}(I) = - \frac{\hat{\phi}_{-2e}(I)}{-i \sum_{j=1}^m I_j} = \frac{1}{i \sum_{j=1}^m I_j} \frac{|I|^2}{4}$$

in time

$$W(\varphi, \tilde{I}) = \sum_{j=1}^m -\frac{1}{2ij} (e^{i(j\varphi_j)} - e^{-i(j\varphi_j)})$$

$$-\frac{|\tilde{I}|^2}{4} \frac{1}{i \sum_{j=1}^m \tilde{I}_j} \left(e^{i\left(2 \sum_{j=1}^m \varphi_j\right)} - e^{-i\left(2 \sum_{j=1}^m \varphi_j\right)} \right)$$

in space

$$W(\varphi, \tilde{I}) = \sum_{j=1}^m -\frac{1}{j} \sin(j\varphi_j) \\ - \frac{|\tilde{I}|^2}{2} \frac{1}{\sum_{j=1}^m \tilde{I}_j} \sin\left(2 \sum_{j=1}^m \varphi_j\right)$$

la funzione generatrice creata è

$$S(\varphi, \tilde{I}; \varepsilon) = \varphi \cdot \tilde{I} + \varepsilon W(\varphi, \tilde{I})$$

con $W(\varphi, \tilde{I})$ scritta sopra

Esercizio (9 gennaio 2015)

Si consideri la funzione hamiltoniana

$$H_\varepsilon(p, q) = H_0(p, q) + \varepsilon H_1(p, q)$$

$$p, q \in \mathbb{R}, \quad \varepsilon \ll 1$$

con

$$H_0 = \frac{1}{2}(p^2 + \omega^2 q^2), \quad H_1 = pq \quad (\omega \neq 0)$$

- i) Scrivere il sistema hamiltoniano e la corrispondente funzione hamiltoniana $K_\varepsilon(I, \varphi)$ ottenuta da H_ε introducendo le variabili azione - angolo dell'oscillatore armonico con hamiltoniana H_0

Suggerimento: scrivere l'equazione di Hamilton - Jacobi associata a $H_0(p, q)$, scegliendo I in modo che $H_0 = \omega I$

Sol.

L'equazione di Hamilton - Jacobi per la funzione caratteristica $W(q, I)$ è

$$\frac{1}{2} \left(\frac{\partial W}{\partial q} \right)^2 + \frac{1}{2} \omega^2 q^2 = e(I)$$

poniamo, secondo il suggerimento

$$e(I) = \omega I$$

$$\frac{1}{2} \left(\frac{\partial W}{\partial q} \right)^2 + \frac{1}{2} \omega^2 q^2 = \omega I$$

$$\left(\frac{\partial W}{\partial q} \right)^2 = 2\omega I - \omega^2 q^2, \quad \frac{\partial W}{\partial q} = \sqrt{2\omega I - \omega^2 q^2}$$

$$W(q, I) = \int \sqrt{2\omega I - \omega^2 q^2} \, dq$$

usando poi la relazione

$$\varphi = \frac{\partial W}{\partial I} \quad \text{si ha} \quad \varphi = \int \frac{\partial}{\partial I} (\sqrt{2\omega I - \omega^2 q^2}) \, dq =$$

$$= \left\{ \frac{\omega}{\sqrt{2\omega I}} \int \frac{1}{\sqrt{1 - \left(\frac{\omega}{\sqrt{2I}} \right)^2 q^2}} \, dq \right\} = \arcsen \sqrt{\frac{\omega}{2I}} q$$

$$\varphi = \arcsen \sqrt{\frac{\omega}{2I}} q, \quad q = \sqrt{\frac{2I}{\omega}} \sin \varphi$$

$$\frac{1}{2} (p^2 + \omega^2 q^2) = I \omega$$

$$\frac{1}{2} p^2 + \frac{1}{2} \omega^2 \frac{2I}{\omega} \sin^2 \varphi = I \omega$$

$$p^2 = 2 (I \omega - I \omega \sin^2 \varphi) = 2 I \omega \cos^2 \varphi$$

$$p = \sqrt{2 I \omega} \cos \varphi, \quad q = \sqrt{\frac{2 I}{\omega}} \sin \varphi$$

Dopo aver introdotto le variabili azioni - angolo
riprendiamo l'espressione di H_ε

$$H_\varepsilon(p, q) = \frac{1}{2} (p^2 + \omega^2 q^2) + \varepsilon p q$$

e otteniamo

$$K_\varepsilon(I, \varphi) = H_\varepsilon(p(I, \varphi), q(I, \varphi))$$

$$\begin{aligned} K_\varepsilon(I, \varphi) &= \frac{1}{2} \left(2 I \omega \cos^2 \varphi + \omega^2 \frac{2 I}{\omega} \sin^2 \varphi \right) \\ &\quad + \varepsilon \sqrt{2 I \omega} \cos \varphi \sqrt{\frac{2 I}{\omega}} \sin \varphi \\ &= I \omega + \varepsilon 2 I \cos \varphi \sin \varphi \end{aligned}$$

$$K_\varepsilon(I, \varphi) = I \omega + \varepsilon I \sin 2 \varphi$$

$$\begin{cases} \dot{I} = - \frac{\partial K_\varepsilon}{\partial \varphi} = - 2 \varepsilon I \cos 2 \varphi \\ \dot{\varphi} = \frac{\partial K_\varepsilon}{\partial I} = \omega + \varepsilon \sin 2 \varphi \end{cases}$$

ii) Determinare una funzione generatrice di una trasformazione canonica vicina all'identità

$$(I, \varphi) \xrightarrow{\Psi_\varepsilon} (\tilde{I}, \tilde{\varphi})$$

tale che

$$\tilde{K}_\varepsilon = K_\varepsilon \circ \Psi_\varepsilon^{-1}$$

non dipenda da $\tilde{\varphi}$ al primo ordine in ε

Sol.

$$-\frac{\partial h}{\partial I} \frac{\partial W}{\partial \varphi} = \mathcal{F} - g$$

$$h(I) = I\omega, \quad \mathcal{F}(I, \varphi) = I \sin 2\varphi$$

$$\langle \mathcal{F} \rangle_\varphi = 0, \quad g = 0$$

$$\frac{\partial h}{\partial I} = \omega$$

e l'equazione omologica si riduce a

$$-\omega \frac{\partial W}{\partial \varphi} = \mathcal{F}$$

Consideriamo ora $\mathcal{F}(I, \omega)$ e scriviamo

$$\mathcal{F}(I, \varphi) = \sum_{k \in \mathbb{K}} \hat{\mathcal{F}}_k(I) e^{i k \cdot \varphi}$$

$$\mathcal{F}(I, \varphi) = \frac{I}{2i} (e^{i2\varphi} - e^{-i2\varphi})$$

$$\hat{\phi}_2(I) = \frac{I}{2i}, \quad \hat{\phi}_{-2}(I) = -\frac{I}{2i}$$

$$K = \{2, -2\}$$

Per la funzione $W(\varphi, I)$ si ha

$$W(\varphi, I) = \sum_{k \in K} \hat{W}_k(I) e^{i k \cdot \varphi}$$

$$\hat{W}_2(I) = -\frac{\hat{\phi}_2(I)}{i2\omega} = -\frac{1}{2i\omega} \left(\frac{I}{2i} \right) = \frac{I}{4\omega}$$

$$\hat{W}_{-2}(I) = -\frac{\hat{\phi}_{-2}(I)}{-i2\omega} = \frac{1}{2i\omega} \left(-\frac{I}{2i} \right) = \frac{I}{4\omega}$$

$$\begin{aligned} W(\varphi, I) &= \frac{I}{4\omega} \left(e^{i(2\varphi)} + e^{-i(2\varphi)} \right) \\ &= \frac{I}{2\omega} \cos 2\varphi \end{aligned}$$

quindi

$$S(\varphi, \tilde{I}; \varepsilon) = \varphi \tilde{I} + \varepsilon \frac{\tilde{I}}{2\omega} \cos 2\varphi$$

e la forma normale non risonante è

$$\begin{aligned} \tilde{K}_\varepsilon &= K_\varepsilon \circ \psi_\varepsilon^{-1} = \omega \left(\tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) \right) \\ &+ \varepsilon \tilde{I} \sin 2\tilde{\varphi} + O(\varepsilon^2) = \omega \tilde{I} + O(\varepsilon^2) \end{aligned}$$

Osservazione L'espressione di $W(\varphi, \tilde{I})$

potrebbe essere ottenuta direttamente dall'equazione omologica per integrazione

$$\omega \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) + \tilde{I} \sin 2\tilde{\varphi} = 0$$

$$\omega \frac{\partial W}{\partial \varphi} = -\tilde{I} \sin 2\tilde{\varphi}$$

$$W(\tilde{\varphi}, \tilde{I}) = \frac{1}{\omega} \int -\tilde{I} \sin 2\tilde{\varphi} d\tilde{\varphi}$$

$$W(\tilde{\varphi}, \tilde{I}) = \frac{\tilde{I}}{2\omega} \cos 2\tilde{\varphi} \rightarrow W(\varphi, \tilde{I}) = \frac{\tilde{I}}{2\omega} \cos 2\varphi$$

iii) Si scriva il sistema hamiltoniano nelle variabili $(\tilde{I}, \tilde{\varphi})$ corrispondenti allo sviluppo di Taylor di K_ε fino al secondo ordine in ε

Sol.

$$K_\varepsilon(I, \varphi) = I\omega + \varepsilon I \sin 2\varphi$$

$$I = \frac{\partial S}{\partial \varphi} = \tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}(\varphi, \tilde{I})$$

$$\tilde{\varphi} = \frac{\partial S}{\partial \tilde{I}} = \varphi + \varepsilon \frac{\partial W}{\partial \tilde{I}}(\varphi, \tilde{I})$$

$$\varphi = \tilde{\varphi} - \varepsilon \frac{\partial W}{\partial \tilde{I}}(\varphi, \tilde{I})$$

abbiamo

$$\tilde{K}_\varepsilon(\tilde{I}, \tilde{\varphi}) = \left(\tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}(\varphi, \tilde{I}) \right) \omega + \varepsilon \left(\tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}(\varphi, \tilde{I}) \right) \sin 2\varphi$$

$$\text{con } \varphi = \varphi(\tilde{\varphi}, \tilde{I})$$

$$\begin{aligned} \tilde{K}_\varepsilon(\tilde{I}, \tilde{\varphi}) &= \tilde{I} \omega + \varepsilon \left(\omega \frac{\partial W}{\partial \varphi}(\varphi, \tilde{I}) + \tilde{I} \sin 2\varphi \right) \\ &\quad + \varepsilon^2 \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) \sin 2\tilde{\varphi} + O(\varepsilon^3) \end{aligned}$$

$\downarrow = 0$

$$\left(W(\varphi, \tilde{I}) = \frac{\tilde{I}}{2\omega} \cos 2\varphi \right)$$

$$\tilde{K}(\tilde{I}, \tilde{\varphi}) = \tilde{I} \omega + \varepsilon^2 \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) \sin 2\tilde{\varphi} + O(\varepsilon^2)$$

$$\frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) = -\frac{\tilde{I}}{\omega} \sin 2\tilde{\varphi}$$

la forma normale non risonante fino al secondo ordine è

$$\begin{aligned} \tilde{K}^*(\tilde{I}, \tilde{\varphi}) &= \tilde{I} \omega + \varepsilon^2 \left(-\frac{\tilde{I}}{\omega} \sin 2\tilde{\varphi} \right) \sin 2\tilde{\varphi} \\ &= \tilde{I} \omega - \varepsilon^2 \frac{\tilde{I}}{\omega} (\sin^2 2\tilde{\varphi}) \end{aligned}$$

e il sistema hamiltoniano ad essa associato è

$$\left\{ \begin{aligned} \dot{\tilde{I}} &= -\frac{\partial \tilde{K}^*}{\partial \tilde{\varphi}} = \frac{2\varepsilon^2}{\omega} \tilde{I} (\sin 2\tilde{\varphi}) 2 \cos 2\tilde{\varphi} \\ &= \frac{2\varepsilon^2 \tilde{I}}{\omega} \sin 4\tilde{\varphi} \\ \dot{\tilde{\varphi}} &= \frac{\partial \tilde{K}^*}{\partial \tilde{I}} = \omega - \frac{\varepsilon^2}{\omega} (\sin^2 2\tilde{\varphi}) \end{aligned} \right.$$

Riotteniamo ora lo stesso risultato partendo dagli sviluppi in serie di I , $\sin 2\varphi$

$$\begin{aligned} I &= \tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi} \left(\underbrace{\tilde{\varphi} - \varepsilon \frac{\partial W}{\partial \tilde{I}}(\varphi, \tilde{I})}_{=\varphi}, \tilde{I} \right) \\ &= \tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) + \varepsilon \frac{\partial^2 W}{\partial \varphi^2}(\tilde{\varphi}, \tilde{I}) \left(-\varepsilon \frac{\partial W}{\partial \tilde{I}}(\tilde{\varphi}, \tilde{I}) \right) \\ &\quad + O(\varepsilon^3) \end{aligned}$$

$$\begin{aligned} I &= \tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) - \varepsilon^2 \frac{\partial^2 W}{\partial \varphi^2}(\tilde{\varphi}, \tilde{I}) \frac{\partial W}{\partial \tilde{I}}(\tilde{\varphi}, \tilde{I}) \\ &\quad + O(\varepsilon^3) \end{aligned}$$

$$\sin 2\varphi = \sin \left(2 \left(\tilde{\varphi} - \varepsilon \frac{\partial W}{\partial \tilde{I}}(\varphi, \tilde{I}) \right) \right)$$

$$= \sin 2\tilde{\varphi} + \cos 2\tilde{\varphi} \left(-2\varepsilon \frac{\partial W}{\partial \tilde{I}}(\tilde{\varphi}, \tilde{I}) \right) + O(\varepsilon^2)$$

$$\sin 2\varphi = \sin 2\tilde{\varphi} - 2\varepsilon \cos 2\tilde{\varphi} \frac{\partial W}{\partial \tilde{I}}(\tilde{\varphi}, \tilde{I}) + O(\varepsilon^2)$$

sostituiamo ora in $K_\varepsilon(I, \varphi)$

$$\begin{aligned}\tilde{K}_\varepsilon(\tilde{I}, \tilde{\varphi}) &= \tilde{I} \omega + \varepsilon \frac{\partial W}{\partial \varphi} \omega - \varepsilon^2 \frac{\partial^2 W}{\partial \varphi^2} \frac{\partial W}{\partial \tilde{I}} \omega \\ &+ \varepsilon \left(\tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi} \right) \left(\sin 2\tilde{\varphi} - 2\varepsilon \cos 2\tilde{\varphi} \frac{\partial W}{\partial \tilde{I}} \right) \\ &+ O(\varepsilon^3)\end{aligned}$$

$$\begin{aligned}\tilde{K}_\varepsilon(\tilde{I}, \tilde{\varphi}) &= \tilde{I} \omega + \varepsilon \left(\frac{\partial W}{\partial \varphi} \omega + \tilde{I} \sin 2\tilde{\varphi} \right) \\ &+ \varepsilon^2 \left(-\frac{\partial^2 W}{\partial \varphi^2} \frac{\partial W}{\partial \tilde{I}} \omega - 2\tilde{I} \cos 2\tilde{\varphi} \frac{\partial W}{\partial \tilde{I}} + \frac{\partial W}{\partial \varphi} \sin 2\tilde{\varphi} \right) \\ &+ O(\varepsilon^3)\end{aligned}$$

$$\omega \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) + \tilde{I} \sin 2\tilde{\varphi} = 0$$

(equazione omologica)

ricordando che $W(\varphi, \tilde{I}) = \frac{\tilde{I}}{2\omega} \cos 2\varphi$

si ha

$$\frac{\partial W}{\partial \tilde{I}}(\tilde{\varphi}, \tilde{I}) = \frac{\cos 2\tilde{\varphi}}{2\omega}, \quad \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) = -\frac{\tilde{I}}{\omega} \sin 2\tilde{\varphi}$$

$$\frac{\partial^2 W}{\partial \varphi^2}(\tilde{\varphi}, \tilde{I}) = -\frac{2\tilde{I}}{\omega} \cos 2\tilde{\varphi}$$

$$-\frac{\partial^2 W}{\partial \varphi^2} \frac{\partial W}{\partial \tilde{I}} \omega - 2\tilde{I} \cos 2\tilde{\varphi} \frac{\partial W}{\partial \tilde{I}} + \frac{\partial W}{\partial \varphi} \sin 2\tilde{\varphi} =$$

$$-\cancel{\omega} \left(-\cancel{\frac{2\tilde{I}}{\omega}} \cos 2\tilde{\varphi} \right) \frac{\cos 2\tilde{\varphi}}{\cancel{2\omega}} - \cancel{2\tilde{I}} \cos 2\tilde{\varphi} \frac{\cos 2\tilde{\varphi}}{\cancel{2\omega}}$$

$$+ \sin 2\tilde{\varphi} \left(-\frac{\tilde{I}}{\omega} \sin 2\tilde{\varphi} \right) =$$

$$\frac{\tilde{I}}{\omega} \cancel{\cos^2 2\tilde{\varphi}} - \frac{\tilde{I}}{\omega} \cancel{\cos^2 2\tilde{\varphi}} - \frac{\tilde{I}}{\omega} \sin^2 2\tilde{\varphi}$$

$$\tilde{K}_\varepsilon(\tilde{I}, \tilde{\varphi}) = \tilde{I}\omega - \varepsilon^2 \frac{\tilde{I}}{\omega} (\sin^2 2\tilde{\varphi}) + O(\varepsilon^3)$$

che è la stessa espressione trovata prima