The total time derivatives of the eccentric and mean anomalies are then

$$\frac{dE}{dt} = \frac{na}{r} + \frac{\partial E}{\partial \mathbf{v}}\mathbf{a}_d$$
$$\frac{dM}{dt} = n + \frac{\partial M}{\partial \mathbf{v}}\mathbf{a}_d \tag{10.89}$$

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10.6 Applications of the Variational Method

In this section we consider several interesting and important applications of the concepts thus far developed in this chapter. The first example utilizes the Lagrange planetary equations to study the average effect of the J_2 term in the earth's gravitational potential on the motion of an earth orbiting satellite. The second example is an application of Gauss' form of the variational equations to analyze the effect of atmospheric drag on the orbital elements of a satellite in earth orbit.

Effect of J_2 on Satellite Orbits

The disturbing function associated with the J_2 term in the earth's gravitational field

$$R = -\frac{Gm}{r} J_2 \left(\frac{r_{eq}}{r}\right)^2 P_2(\cos\phi) \tag{10.90}$$

is obtained from Eq. (8.92). The colatitude angle ϕ is related to the orbital elements and calculated from

$$\cos\phi = \mathbf{i}_r \cdot \mathbf{i}_z = \sin(\omega + f)\sin i$$

using the results of Prob. 3–21. Hence, the Legendre polynomial $P_2(\cos\phi)$ is expressed as

$$P_2(\cos\phi) = \tfrac{1}{2}[3\sin^2(\omega + f)\sin^2 i - 1]$$

so that the disturbing function assumes the form

$$R = -\frac{GmJ_2 r_{eq}^2}{2p^3}(1 + e\cos f)^3[3\sin^2(\omega + f)\sin^2 i - 1] \tag{10.91}$$

where r has been replaced by the equation of orbit.

The disturbing function can be expanded as a Fourier series in the mean anomaly M using the technique of Sect. 5.3. The constant term in the series is simply the average value of R over one orbit, i.e.,

$$\overline{R} = \frac{1}{2\pi}\int_0^{2\pi} R\, dM$$

Since $dM = n\,dt$ and $r^2 df = h\,dt$, then clearly,

$$\overline{R} = \frac{1}{2\pi}\int_0^{2\pi} \frac{n}{h} R r^2\,df$$

Substituting from Eq. (10.91) and performing the integration yields

$$\overline{R} = \frac{n^2 J_2 r_{eq}^2}{4(1-e^2)^{\frac{3}{2}}}(2 - 3\sin^2 i) \tag{10.92}$$

Thus, the average value of the disturbing function depends only on the three orbital elements a, e, and i.

When $\overline{R}$ is used for R in Lagrange's planetary equations (10.31), we have, immediately, expressions for the average rates of change of the satellite orbital elements during a single revolution. For example, since $\overline{R}$ is not a function of Ω, ω, or λ, we see that

$$\overline{\frac{da}{dt}} = 0 \qquad \overline{\frac{de}{dt}} = 0 \tag{10.93}$$

On the other hand, we obtain for the longitude of the node

$$\overline{\frac{d\Omega}{dt}} = -\frac{3}{2}J_2\left(\frac{r_{eq}}{p}\right)^2 n\cos i \tag{10.94}$$

Thus, the plane of the orbit rotates about the earth's polar axis in a direction opposite to that of the motion of the satellite with a mean rate of rotation given by Eq. (10.94). This phenomenon is referred to as the *regression of the node.*

In a similar manner, we obtain for the mean rate of rotation of the line of apsides

$$\overline{\frac{d\omega}{dt}} = \frac{3}{4}J_2\left(\frac{r_{eq}}{p}\right)^2 n(5\cos^2 i - 1) \tag{10.95}$$

It is apparent that there exists a *critical inclination angle*

$$i_{crit} = 63^\circ\, 26'.1$$

such that, if i exceeds i_{crit}, the line of apsides will regress while, if i is smaller than i_{crit}, the apsidal line will advance.

◇ Problem 10–12

For an earth orbiting satellite, show that

$$\overline{\frac{d\Omega}{dt}} = -9.96\left(\frac{r_{eq}}{a}\right)^{3.5}(1-e^2)^{-2}\cos i \quad \text{degrees/day}$$

$$\overline{\frac{d\omega}{dt}} = 5.0\left(\frac{r_{eq}}{a}\right)^{3.5}(1-e^2)^{-2}(5\cos^2 i - 1) \quad \text{degrees/day}$$

using appropriate values for the physical data of the earth.