Esercizio (6 giugno 2017)

Si consideri la funzione hamiltoniana

$$H_\varepsilon(I,\varphi) = h(I) + \varepsilon f(\varphi)$$

con

$$h(I) = I_1 + I_2$$

$$f(\varphi) = \cos^2(\varphi_1 + 2\varphi_2)\,\text{sen}^2(\varphi_1 - 2\varphi_2)$$

$$I = (I_1, I_2) \in \mathbb{R}^2,$$

$$\varphi = (\varphi_1, \varphi_2) \in \mathbb{T}^2, \qquad \varepsilon \ll 1$$

Determinare una funzione generatrice di una trasforma=zione canonica vicina all'identità

$$(I,\varphi) \xrightarrow{\Psi_\varepsilon} (\tilde{I}, \tilde{\varphi})$$

tale che

$$K_\varepsilon = H_\varepsilon \circ \Psi_\varepsilon^{-1}$$

non dipenda da $\tilde{\varphi}$ al primo ordine in ε

Sol.

Funzione generatrice

$$S(\varphi, \tilde{I}; \varepsilon) = \varphi \cdot \tilde{I} + \varepsilon W(\varphi, \tilde{I})$$

$$\begin{cases} \tilde{\varphi} = \dfrac{\partial S}{\partial \tilde{I}} = \varphi + \varepsilon \dfrac{\partial W}{\partial \tilde{I}} \\[2ex] I = \dfrac{\partial S}{\partial \varphi} = \tilde{I} + \varepsilon \dfrac{\partial W}{\partial \varphi} \end{cases}$$

Abbiamo

$$K_\varepsilon = H_\varepsilon \circ \psi_\varepsilon^{-1} = h\left(\tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}\right) +$$

$$\varepsilon f\left(\tilde{\varphi} - \varepsilon \frac{\partial W}{\partial \tilde{I}}\right)$$

$$= h(\tilde{I}) + \varepsilon \frac{\partial h}{\partial I}(\tilde{I}) \cdot \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) +$$

$$\varepsilon f(\tilde{\varphi}) + O(\varepsilon^2)$$

scriviamo l'equazione omologica

$$\frac{\partial h}{\partial I}(\tilde{I}) \cdot \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) + f(\tilde{\varphi}) = \langle f \rangle_\varphi$$

le frequenze sono date da

$$\omega = \frac{\partial h}{\partial I} = (1,1)^T$$

$$\omega \cdot \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) + f(\tilde{\varphi}) = \langle f \rangle_\varphi$$

$$f(\varphi) = \cos^2(\varphi_1 + 2\varphi_2)\, \operatorname{sen}^2(\varphi_1 - 2\varphi_2)$$

$$\cos^2(\varphi_1 + 2\varphi_2) = \left(\frac{e^{i(\varphi_1 + 2\varphi_2)} + e^{-i(\varphi_1 + 2\varphi_2)}}{2}\right)^2$$

$$= \frac{1}{4}\left(e^{i(2\varphi_1 + 4\varphi_2)} + e^{-i(2\varphi_1 + 4\varphi_2)} + 2\right)$$

$$\operatorname{sen}^2(\varphi_1 - 2\varphi_2) = \left(\frac{e^{i(\varphi_1 - 2\varphi_2)} - e^{-i(\varphi_1 - 2\varphi_2)}}{2i}\right)^2$$

$$= \frac{1}{4}\left(e^{i(2\varphi_1 - 4\varphi_2)} + e^{-i(2\varphi_1 - 4\varphi_2)} - 2\right)$$

$$\cos^2(\varphi_1 + 2\varphi_2)\,\mathrm{sen}^2(\varphi_1 - 2\varphi_2) =$$

$$-\frac{1}{16}\Big(e^{i(4\varphi_1)} + e^{i(8\varphi_2)} - 2e^{i(2\varphi_1 + 4\varphi_2)}$$

$$+ e^{-i(8\varphi_2)} + e^{-i(4\varphi_1)} - 2e^{-i(2\varphi_1 + 4\varphi_2)}$$

$$+ 2e^{i(2\varphi_1 - 4\varphi_2)} + 2e^{-i(2\varphi_1 - 4\varphi_2)} - 4\Big)$$

scriviamo lo sviluppo in serie di Fourier di f

$$f(\varphi) = \sum_{k \in K} \hat{f}_k \, e^{ik\cdot\varphi}$$

dove l'insieme K è dato da

$$K = \{(0,0), \pm(4,0), (0,\pm 8), \pm(2,4), \pm(2,-4)\}$$

notiamo che la condizione di non risonanza

$$\omega \cdot k \neq 0 \qquad (\omega = (1,1)^T)$$

è soddisfatta per ogni $k \in K$

la media sugli angoli di f è

$$\langle f \rangle_\varphi = \frac{1}{4} = \hat{f}(0,0)$$

e gli altri termini sono

$$\hat{f}_{(4,0)} = -\frac{1}{16}, \quad \hat{f}_{-(4,0)} = -\frac{1}{16} \qquad k = \pm(4,0)$$

$$\hat{f}_{(0,8)} = -\frac{1}{16}, \quad \hat{f}_{-(0,8)} = -\frac{1}{16} \qquad k = \pm(0,8)$$

$$\hat{f}_{(2,4)} = \frac{1}{8}, \quad \hat{f}_{-(2,4)} = \frac{1}{8} \qquad k = \pm(2,4)$$

$$\hat{f}_{(2,-4)} = -\frac{1}{8}, \quad \hat{f}_{-(2,-4)} = -\frac{1}{8} \qquad k = \pm(2,-4)$$

Possiamo ora determinare $W(\varphi, \tilde{I})$

Notiamo che W non dipenderà da $\tilde{I}$; scriviamo

$$W(\varphi) = \sum_{k \in K \setminus (0,0)} \hat{W}_k \, e^{i k \cdot \varphi}$$

dove i termini $\hat{W}_k$ si trovano imponendo l'equazione omologica, si ha

$$\hat{W}_k = -\frac{\hat{f}_k}{i \omega \cdot k}$$

procediamo

$$\hat{W}_{(4,0)} = -\frac{\hat{f}_{(4,0)}}{4i} = -\left(-\frac{1}{16}\right)\frac{1}{4i} = \frac{1}{64i}$$

$k = (4,0), \ \omega = (1,1) \qquad k \cdot \omega = 4$

$$\hat{W}_{-(4,0)} = -\frac{\hat{f}_{-(4,0)}}{-4i} = \left(-\frac{1}{16}\right)\frac{1}{4i} = -\frac{1}{64i}$$

$$k = -(4,0),\ w = (1,1) \qquad k \cdot w = -4$$

$$\hat{W}_{(0,8)} = -\frac{\hat{f}_{(0,8)}}{8i} = -\left(-\frac{1}{16}\right)\frac{1}{8i} = \frac{1}{128i}$$

$$k = (0,8),\ w = (1,1) \qquad k \cdot w = 8$$

$$\hat{W}_{-(8,0)} = -\frac{\hat{f}_{-(8,0)}}{-8i} = \left(-\frac{1}{16}\right)\frac{1}{8i} = -\frac{1}{128i}$$

$$k = -(0,8),\ w = (1,1) \qquad k \cdot w = -8$$

$$\hat{W}_{(2,4)} = -\frac{\hat{f}_{(2,4)}}{6i} = -\left(\frac{1}{8}\right)\frac{1}{6i} = -\frac{1}{48i}$$

$$k = (2,4),\ w = (1,1) \qquad k \cdot w = 6$$

$$\hat{W}_{-(2,4)} = -\frac{\hat{f}_{-(2,4)}}{-6i} = \left(\frac{1}{8}\right)\frac{1}{6i} = \frac{1}{48i}$$

$$k = -(2,4),\ w = (1,1) \qquad k \cdot w = -6$$

$$\hat{W}_{(2,-4)} = -\frac{\hat{f}_{(2,-4)}}{-2i} = \left(-\frac{1}{8}\right)\frac{1}{2i} = -\frac{1}{16i}$$

$$k = (2,-4),\ w = (1,1) \qquad k \cdot w = -2$$

$$\hat{W}_{-(2,-4)} = -\frac{\hat{f}_{-(2,-4)}}{2i} = -\left(-\frac{1}{8}\right)\frac{1}{2i} = \frac{1}{16i}$$

$k = -(2,-4)$, $\omega = (1,1)$ $\quad k\cdot\omega = 2$

Si è ottenuto

$$W(\varphi) = \frac{1}{64i}\left(e^{i4\varphi_1} - e^{-i4\varphi_1}\right) +$$

$$\frac{1}{128i}\left(e^{i8\varphi_2} - e^{-i8\varphi_2}\right) - \frac{1}{48i}\left(e^{i(2\varphi_1+4\varphi_2)}\right.$$

$$\left. - e^{-i(2\varphi_1+4\varphi_2)}\right) - \frac{1}{16i}\left(e^{i(2\varphi_1-4\varphi_2)} - e^{-i(2\varphi_1-4\varphi_2)}\right)$$

cioè

$$W(\varphi) = \frac{1}{32}\,\mathrm{sen}\,4\varphi_1 + \frac{1}{64}\,\mathrm{sen}\,8\varphi_2 - \frac{1}{24}\,\mathrm{sen}\,(2\varphi_1+4\varphi_2)$$

$$-\frac{1}{8}\,\mathrm{sen}\,(2\varphi_1-4\varphi_2)$$

e la funzione generatrice cercata è

$$S(\varphi,\tilde{I};\varepsilon) = \varphi\cdot\tilde{I} + \varepsilon\left(\frac{1}{32}\,\mathrm{sen}\,4\varphi_1 + \frac{1}{64}\,\mathrm{sen}\,8\varphi_2\right.$$

$$\left. - \frac{1}{24}\,\mathrm{sen}\,(2\varphi_1+4\varphi_2) - \frac{1}{8}\,\mathrm{sen}\,(2\varphi_1-4\varphi_2)\right)$$

La forma normale non risonante è

$$K_\varepsilon(\tilde{I},\tilde{\varphi}) = h(\tilde{I}) + \varepsilon\hat{f}_{(0,0)} + O(\varepsilon^2)$$

$$K_\varepsilon(\tilde{I},\tilde{\varphi}) = \tilde{I}_1 + \tilde{I}_2 + \frac{\varepsilon}{4} + O(\varepsilon^2)$$

Esercizio (18 dicembre 2014)

Consideriamo la funzione hamiltoniana

$$H_\varepsilon(I,\varphi) = h(I) + \varepsilon f(I,\varphi)$$

$I \in \mathbb{R}^n$, $\varphi \in \mathbb{T}^n$, $\varepsilon \ll 1$, con

$$h(I) = \frac{1}{2}|I|^2$$

$$f(I,\varphi) = \sum_{j=1}^{n} I_j \cos(j\varphi_j) + |I|^2 \cos^2\left(\sum_{j=1}^{n} \varphi_j\right)$$

Determinare una funzione generatrice di una trasforma= zione canonica vicina all'identità

$$(I,\varphi) \xrightarrow{\Psi_\varepsilon} (\tilde{I},\tilde{\varphi})$$

tale che

$$K_\varepsilon = H_\varepsilon \circ \Psi_\varepsilon^{-1}$$

non dipenda da $\tilde{\varphi}$ al primo ordine in ε

Sol.

Ricordiamo l'equazione omologica

$$\omega \cdot \frac{\partial W}{\partial \varphi} = g - f$$

$$\omega(I) = \frac{\partial h}{\partial I} = I \ , \quad \omega_j = I_j \ , \quad j = 1, \dots, n$$

$$g = \langle f \rangle_\varphi$$

calcoliamo la media $\langle f \rangle_\varphi$

$$\langle f \rangle_\varphi = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f \, d\varphi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \left[\sum_{j=1}^n I_j \cos(j\varphi_j) + |I|^2 \cos^2\left(\sum_{j=1}^n \varphi_j \right) \right] d\varphi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |I|^2 \cos^2\left(\sum_{j=1}^n \varphi_j \right) d\varphi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^{n-1}} |I|^2 \left[\frac{1}{2}\varphi_1 + \frac{1}{4} \operatorname{sen}\left(2 \sum_{j=1}^n \varphi_j \right) \right] \Bigg|_0^{2\pi} d\varphi_2 \dots d\varphi_n$$

$$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^{n-1}} \frac{|I|^2}{2} (2\pi) \, d\varphi_2 \dots d\varphi_n$$

quindi si ottiene

$$\langle f \rangle_\varphi (I) = \frac{1}{2} |I|^2$$

$$f(I,\varphi) = \sum_{k \in K} \hat{f}_k(I)\, e^{ik\cdot\varphi}$$

$$f(I,\varphi) = \sum_{j=1}^{n} I_j \left(\frac{e^{i(j\varphi_j)} + e^{-i(j\varphi_j)}}{2}\right) +$$

$$\frac{|I|^2}{4}\left(e^{i\sum_{j=1}^{n}\varphi_j} + e^{-i\sum_{j=1}^{n}\varphi_j}\right)^2$$

$$= \sum_{j=1}^{n} I_j \left(\frac{e^{i(j\varphi_j)} + e^{-i(j\varphi_j)}}{2}\right) +$$

$$\frac{|I|^2}{4}\left(e^{2i\sum_{j=1}^{n}\varphi_j} + e^{-2i\sum_{j=1}^{n}\varphi_j} + 2\right)$$

determiniamo ora l'insieme K

$$K = \{0, \pm e_1, \dots, \pm e_j, \dots, \pm e_n, \pm 2e\}$$

con

e_j , $j = 1, \dots, n$, vettori della base canonica di $\mathbb{R}^n$

$$e_1 = (1, 0, 0, \dots, 0)$$

$$\vdots$$

$$e_j = (0, 0, \dots, 1, \dots 0)$$

$$\vdots$$

$$e_n = (0, 0, \dots, 0, 1)$$

inoltre

$$e = \sum_{j=1}^{n} e_j$$

possiamo trovare i termini $\hat{f}_k(I)$:

$$\hat{f}_0(I) = \langle f \rangle_\varphi (I) = \frac{|I|^2}{2}$$

$$\hat{f}_{\pm j e_j}(I) = \frac{I_j}{2}, \quad j = 1, \dots, n$$

$$\hat{f}_{\pm 2e}(I) = \frac{|I|^2}{4}$$

e da questi, si determina

$$W(\varphi, I) = \sum_{k \in K \setminus (0)} \hat{W}_k(I)\, e^{i k \cdot \varphi}$$

con

$$\hat{W}_{j e_j}(I) = -\frac{\hat{f}_{j e_j}(I)}{j i \omega \cdot e_j} = -\frac{1}{i j I_j} \frac{I_j}{2}$$

$$\omega = I \qquad \omega \cdot e_j = \omega_j = I_j$$

$$\hat{W}_{j e_j}(I) = -\frac{1}{2ij}$$

$$\hat{W}_{-j e_j}(I) = -\frac{\hat{f}_{-j e_j}(I)}{-j i \omega \cdot e_j} = \frac{1}{2ij}$$

$$\hat{W}_{2e}(I) = -\frac{\hat{f}_{2e}(I)}{i\omega\cdot e} = -\frac{1}{i\sum_{j=1}^{n} I_j}\frac{|I|^2}{4}$$

$$\omega\cdot e = \sum_{j=1}^{n} I_j$$

$$\hat{W}_{-2e}(I) = -\frac{\hat{f}_{-2e}(I)}{-i\sum_{j=1}^{n} I_j} = \frac{1}{i\sum_{j=1}^{n} I_j}\frac{|I|^2}{4}$$

si ottiene

$$W(\varphi,\tilde{I}) = \sum_{j=1}^{n} -\frac{1}{2ij}\left(e^{i(j\varphi_j)} - e^{-i(j\varphi_j)}\right)$$

$$-\frac{|\tilde{I}|^2}{4}\frac{1}{i\sum_{j=1}^{n}\tilde{I}_j}\left(e^{i\left(2\sum_{j=1}^{n}\varphi_j\right)} - e^{-i\left(2\sum_{j=1}^{n}\varphi_j\right)}\right)$$

e infine

$$W(\varphi,\tilde{I}) = \sum_{j=1}^{n} -\frac{1}{j}\,\mathrm{sen}(j\varphi_j)$$

$$-\frac{|\tilde{I}|^2}{2}\frac{1}{\sum_{j=1}^{n}\tilde{I}_j}\,\mathrm{sen}\left(2\sum_{j=1}^{n}\varphi_j\right)$$

la funzione generatrice cercata è

$$S(\varphi, \tilde{I}; \varepsilon) = \varphi \cdot \tilde{I} + \varepsilon W(\varphi, \tilde{I})$$

con $W(\varphi, \tilde{I})$ scritta sopra

Esercizio (9 gennaio 2015)

Si consideri la funzione hamiltoniana

$$H_\varepsilon(p,q) = H_0(p,q) + \varepsilon H_1(p,q)$$

$p, q \in \mathbb{R}$, $\varepsilon \ll 1$

con

$$H_0 = \frac{1}{2}(p^2 + \omega^2 q^2), \quad H_1 = pq \qquad (\omega \neq 0)$$

i) Scrivere il sistema hamiltoniano e la corrispondente funzione hamiltoniana $K_\varepsilon(I, \varphi)$ ottenuta da H_ε introducendo le variabili azione-angolo dell'oscillatore armonico con hamiltoniana H_0

Suggerimento: scrivere l'equazione di Hamilton-Jacobi associata a $H_0(p,q)$, scegliendo I in modo che $H_0 = \omega I$

Sol.

L'equazione di Hamilton - Jacobi per la funzione caratteristica $W(q, I)$ è

$$\frac{1}{2}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}\omega^2 q^2 = e(I)$$

poniamo, seguendo il suggerimento

$$e(I) = \omega I$$

$$\frac{1}{2}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}\omega^2 q^2 = \omega I$$

$$\left(\frac{\partial W}{\partial q}\right)^2 = 2\omega I - \omega^2 q^2, \quad \frac{\partial W}{\partial q} = \sqrt{2\omega I - \omega^2 q^2}$$

$$W(q, I) = \int \sqrt{2\omega I - \omega^2 q^2}\, dq$$

usando poi la relazione

$$\varphi = \frac{\partial W}{\partial I} \quad \text{si ha} \quad \varphi = \int \frac{\partial}{\partial I}\left(\sqrt{2\omega I - \omega^2 q^2}\right) dq =$$

$$= \left\{ \frac{\omega}{\sqrt{2\omega I}} \int \frac{1}{\sqrt{1 - \left(\frac{\omega}{2I}\right) q^2}}\, dq = \operatorname{arcsen} \sqrt{\frac{\omega}{2I}}\, q\right.$$

$$\sqrt{\frac{\omega}{2I}}$$

$$\varphi = \operatorname{arcsen} \sqrt{\frac{\omega}{2I}}\, q, \quad q = \sqrt{\frac{2I}{\omega}} \operatorname{sen} \varphi$$

$$\frac{1}{2}(p^2 + w^2 q^2) = Iw$$

$$\frac{1}{2}p^2 + \frac{1}{\cancel{2}} w^2 \frac{\cancel{2}I}{\cancel{w}} \operatorname{sen}^2\varphi = Iw$$

$$p^2 = 2(Iw - Iw \operatorname{sen}^2\varphi) = 2Iw\cos^2\varphi$$

$$p = \sqrt{2Iw}\cos\varphi, \quad q = \sqrt{\frac{2I}{w}}\operatorname{sen}\varphi$$

Dopo aver introdotto le variabili azioni-angolo riprendiamo l'espressione di H_ε

$$H_\varepsilon(p,q) = \frac{1}{2}(p^2 + w^2 q^2) + \varepsilon pq$$

e otteniamo

$$K_\varepsilon(I,\varphi) = H_\varepsilon(p(I,\varphi), q(I,\varphi))$$

$$\begin{aligned} K_\varepsilon(I,\varphi) &= \frac{1}{2}\left(2Iw\cos^2\varphi + w^2\frac{2I}{\cancel{w}}\operatorname{sen}^2\varphi\right) \\ &\quad + \varepsilon\sqrt{2Iw}\cos\varphi\sqrt{\frac{2I}{w}}\operatorname{sen}\varphi \\ &= Iw + \varepsilon 2I\cos\varphi\operatorname{sen}\varphi \end{aligned}$$

$$K_\varepsilon(I,\varphi) = Iw + \varepsilon I\operatorname{sen}2\varphi$$

$$\begin{cases} \dot{I} = -\dfrac{\partial K_\varepsilon}{\partial\varphi} = -2\varepsilon I\cos 2\varphi \\[2ex] \dot{\varphi} = \dfrac{\partial K_\varepsilon}{\partial I} = w + \varepsilon\operatorname{sen}2\varphi \end{cases}$$

ii) Determinare una funzione generatrice di una trasforma= zione canonica vicina all'identità

$$(I, \varphi) \xrightarrow{\psi_\varepsilon} (\tilde{I}, \tilde{\varphi})$$

tale che

$$\tilde{K}_\varepsilon = K_\varepsilon \circ \psi_\varepsilon^{-1}$$

non dipenda da $\tilde{\varphi}$ al primo ordine in ε

Sol.

$$-\frac{\partial h}{\partial I} \frac{\partial W}{\partial \varphi} = f - g$$

$$h(I) = I\omega, \quad f(I, \varphi) = I \operatorname{sen} 2\varphi$$

$$\langle f \rangle_\varphi = 0, \quad g = 0$$

$$\frac{\partial h}{\partial I} = \omega$$

e l'equazione omologica si riduce a

$$-\omega \frac{\partial W}{\partial \varphi} = f$$

Consideriamo ora $f(I, \omega)$ e scriviamo

$$f(I, \varphi) = \sum_{k \in K}' \hat{f}_k(I) e^{ik \cdot \varphi}$$

$$f(I, \varphi) = \frac{I}{2i} \left(e^{i2\varphi} - e^{-i2\varphi} \right)$$

$$\hat{f}_2(I) = \frac{I}{2i} \quad , \quad \hat{f}_{-2}(I) = -\frac{I}{2i}$$

$$K = \{2, -2\}$$

Per la funzione $W(\varphi, I)$ si ha

$$W(\varphi, I) = \sum_{k \in K} \hat{W}_k(I)\, e^{ik\cdot\varphi}$$

$$\hat{W}_2(I) = -\frac{\hat{f}_2(I)}{i2\omega} = -\frac{1}{2i\omega}\left(\frac{I}{2i}\right) = \frac{I}{4\omega}$$

$$\hat{W}_{-2}(I) = -\frac{\hat{f}_{-2}(I)}{-i2\omega} = \frac{1}{2i\omega}\left(-\frac{I}{2i}\right) = \frac{I}{4\omega}$$

$$W(\varphi, I) = \frac{I}{4\omega}\left(e^{i(2\varphi)} + e^{-i(2\varphi)}\right)$$

$$= \frac{I}{2\omega}\cos 2\varphi$$

quindi

$$S(\varphi, \tilde{I}; \varepsilon) = \varphi\, \tilde{I} + \varepsilon \frac{\tilde{I}}{2\omega}\cos 2\varphi$$

e la forma normale non risonante è

$$\tilde{K}_\varepsilon = K_\varepsilon \circ \psi_\varepsilon^{-1} = \omega\left(\tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I})\right)$$

$$+ \varepsilon \tilde{I} \operatorname{sen} 2\tilde{\varphi} + O(\varepsilon^2) = \omega \tilde{I} + O(\varepsilon^2)$$

<u>Osservazione</u> L'espressione di $W(\varphi, \tilde{I})$

poteva essere ottenuta direttamente dall'equazione omologica per integrazione

$$\omega \frac{\partial W}{\partial \varphi}(\tilde{\varphi}, \tilde{I}) + \tilde{I} \operatorname{sen} 2\tilde{\varphi} = 0$$

$$\omega \frac{\partial W}{\partial \varphi} = -\tilde{I} \operatorname{sen} 2\tilde{\varphi}$$

$$W(\tilde{\varphi}, \tilde{I}) = \frac{1}{\omega} \int -\tilde{I} \operatorname{sen} 2\tilde{\varphi} \, d\tilde{\varphi}$$

$$W(\tilde{\varphi}, \tilde{I}) = \frac{\tilde{I}}{2\omega} \cos 2\tilde{\varphi} \longrightarrow W(\varphi, \tilde{I}) = \frac{\tilde{I}}{2\omega} \cos 2\varphi$$

iii) Si scriva il sistema hamiltoniano nelle variabili $(\tilde{I}, \tilde{\varphi})$ corrispondente allo sviluppo di Taylor di $\tilde{K}_\varepsilon$ fino al secondo ordine in ε

Sol.

$$K_\varepsilon(I, \varphi) = I\omega + \varepsilon I \operatorname{sen} 2\varphi$$

$$I = \frac{\partial S}{\partial \varphi} = \tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}(\varphi, \tilde{I})$$

$$\tilde{\varphi} = \frac{\partial S}{\partial \tilde{I}} = \varphi + \varepsilon \frac{\partial W}{\partial \tilde{I}}(\varphi, \tilde{I})$$

$$\varphi = \tilde{\varphi} - \varepsilon \frac{\partial W}{\partial \tilde{I}}(\varphi, \tilde{I})$$

abbiamo

$$\tilde{K}_\varepsilon(\tilde{I},\tilde{\varphi}) = \left(\tilde{I} + \varepsilon\frac{\partial W}{\partial \varphi}(\varphi,\tilde{I})\right)\omega +$$

$$\varepsilon\left(\tilde{I} + \varepsilon\frac{\partial W}{\partial \varphi}(\varphi,\tilde{I})\right)\operatorname{sen} 2\varphi$$

con $\varphi = \varphi(\tilde{\varphi},\tilde{I})$

$$\tilde{K}_\varepsilon(\tilde{I},\tilde{\varphi}) = \tilde{I}\omega + \varepsilon\underbrace{\left(\omega\frac{\partial W}{\partial \varphi}(\varphi,\tilde{I}) + \tilde{I}\operatorname{sen} 2\varphi\right)}_{=0}$$

$$+ \varepsilon^2\frac{\partial W}{\partial \varphi}(\tilde{\varphi},\tilde{I})\operatorname{sen} 2\tilde{\varphi} + O(\varepsilon^3)$$

$$\left(W(\varphi,\tilde{I}) = \frac{\tilde{I}}{2\omega}\cos 2\varphi\right)$$

$$\tilde{K}(\tilde{I},\tilde{\varphi}) = \tilde{I}\omega + \varepsilon^2\frac{\partial W}{\partial \varphi}(\tilde{\varphi},\tilde{I})\operatorname{sen} 2\tilde{\varphi} + O(\varepsilon^2)$$

$$\frac{\partial W}{\partial \varphi}(\tilde{\varphi},\tilde{I}) = -\frac{\tilde{I}}{\omega}\operatorname{sen} 2\tilde{\varphi}$$

la forma normale non risonante fino al secondo ordine è

$$\tilde{K}^*(\tilde{I},\tilde{\varphi}) = \tilde{I}\omega + \varepsilon^2\left(-\frac{\tilde{I}}{\omega}\operatorname{sen} 2\tilde{\varphi}\right)\operatorname{sen} 2\tilde{\varphi}$$

$$= \tilde{I}\omega - \varepsilon^2\frac{\tilde{I}}{\omega}(\operatorname{sen}^2 2\tilde{\varphi})$$

e il sistema hamiltoniano ad essa associato è

$$\begin{cases} \dot{\tilde{I}} = -\dfrac{\partial \tilde{K}^*}{\partial \tilde{\varphi}} = \dfrac{2\varepsilon^2}{\omega} \tilde{I} (\operatorname{sen} 2\tilde{\varphi}) 2 \cos 2\tilde{\varphi} \\ \qquad\qquad\qquad = \dfrac{2\varepsilon^2 \tilde{I}}{\omega} \operatorname{sen} 4\tilde{\varphi} \\ \dot{\tilde{\varphi}} = \dfrac{\partial \tilde{K}^*}{\partial \tilde{I}} = \omega - \dfrac{\varepsilon^2}{\omega} (\operatorname{sen}^2 2\tilde{\varphi}) \end{cases}$$

Riotteniamo ora lo stesso risultato partendo dagli sviluppi in serie di I, $\operatorname{sen} 2\varphi$

$$I = \tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi} \Big(\underbrace{\tilde{\varphi} - \varepsilon \frac{\partial W}{\partial \tilde{I}} (\varphi, \tilde{I})}_{= \varphi}, \tilde{I} \Big)$$

$$= \tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi} (\tilde{\varphi}, \tilde{I}) + \varepsilon \frac{\partial^2 W}{\partial \varphi^2} (\tilde{\varphi}, \tilde{I}) \left(-\varepsilon \frac{\partial W}{\partial \tilde{I}} (\tilde{\varphi}, \tilde{I}) \right)$$
$$+ O(\varepsilon^3)$$

$$I = \tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi} (\tilde{\varphi}, \tilde{I}) - \varepsilon^2 \frac{\partial^2 W}{\partial \varphi^2} (\tilde{\varphi}, \tilde{I}) \frac{\partial W}{\partial \tilde{I}} (\tilde{\varphi}, \tilde{I})$$
$$+ O(\varepsilon^3)$$

$$\operatorname{sen} 2\varphi = \operatorname{sen} \left(2 \left(\tilde{\varphi} - \varepsilon \frac{\partial W}{\partial \tilde{I}} (\varphi, \tilde{I}) \right) \right)$$

$$= \operatorname{sen} 2\tilde{\varphi} + \cos 2\tilde{\varphi} \left(-2\varepsilon \frac{\partial W}{\partial \tilde{I}} (\tilde{\varphi}, \tilde{I}) \right) + O(\varepsilon^3)$$

$$\operatorname{sen} 2\varphi = \operatorname{sen} 2\tilde{\varphi} - 2\varepsilon \cos 2\tilde{\varphi} \frac{\partial W}{\partial \tilde{I}} (\tilde{\varphi}, \tilde{I}) + O(\varepsilon^2)$$

sostituiamo ora in $K_\varepsilon(I,\varphi)$

$$\tilde{K}_\varepsilon(\tilde{I},\tilde{\varphi}) = \tilde{I}\omega + \varepsilon \frac{\partial W}{\partial \varphi}\omega - \varepsilon^2 \frac{\partial^2 W}{\partial \varphi^2}\frac{\partial W}{\partial \tilde{I}}\omega$$

$$+ \varepsilon\left(\tilde{I} + \varepsilon \frac{\partial W}{\partial \varphi}\right)\left(\operatorname{sen} 2\tilde{\varphi} - 2\varepsilon \cos 2\tilde{\varphi}\frac{\partial W}{\partial \tilde{I}}\right)$$

$$+ O(\varepsilon^3)$$

$$\tilde{K}_\varepsilon(\tilde{I},\tilde{\varphi}) = \tilde{I}\omega + \varepsilon\left(\frac{\partial W}{\partial \varphi}\omega + \tilde{I}\operatorname{sen} 2\tilde{\varphi}\right)$$

$$+ \varepsilon^2\left(-\frac{\partial^2 W}{\partial \varphi^2}\frac{\partial W}{\partial \tilde{I}}\omega - 2\tilde{I}\cos 2\tilde{\varphi}\frac{\partial W}{\partial \tilde{I}} + \frac{\partial W}{\partial \varphi}\operatorname{sen} 2\tilde{\varphi}\right)$$

$$+ O(\varepsilon^3)$$

$$\omega \frac{\partial W}{\partial \varphi}(\tilde{\varphi},\tilde{I}) + \tilde{I}\operatorname{sen} 2\tilde{\varphi} = 0$$

(equazione omologica)

ricordando che $W(\varphi,\tilde{I}) = \dfrac{\tilde{I}}{2\omega}\cos 2\varphi$

si ha

$$\frac{\partial W}{\partial \tilde{I}}(\tilde{\varphi},\tilde{I}) = \frac{\cos 2\tilde{\varphi}}{2\omega}, \quad \frac{\partial W}{\partial \varphi}(\tilde{\varphi},\tilde{I}) = -\frac{\tilde{I}}{\omega}\operatorname{sen} 2\tilde{\varphi}$$

$$\frac{\partial^2 W}{\partial \varphi^2}(\tilde{\varphi},\tilde{I}) = -\frac{2\tilde{I}}{\omega}\cos 2\tilde{\varphi}$$

$$-\frac{\partial^2 W}{\partial \varphi^2}\frac{\partial W}{\partial \tilde{I}}\omega - 2\tilde{I}\cos 2\tilde{\varphi}\frac{\partial W}{\partial \tilde{I}} + \frac{\partial W}{\partial \varphi}\operatorname{sen} 2\tilde{\varphi} =$$

$$-\cancel{\omega}\left(-\frac{\cancel{2}\tilde{I}}{\cancel{\omega}}\cos 2\tilde{\varphi}\right)\frac{\cos 2\tilde{\varphi}}{\cancel{2}\omega}-\cancel{2}\tilde{I}\cos 2\tilde{\varphi}\,\frac{\cos 2\tilde{\varphi}}{\cancel{2}\omega}$$

$$+\operatorname{sen} 2\tilde{\varphi}\left(-\frac{\tilde{I}}{\omega}\operatorname{sen} 2\tilde{\varphi}\right)=$$

$$\frac{\tilde{I}}{\omega}\cancel{\cos^2 2\tilde{\varphi}}-\frac{\tilde{I}}{\omega}\cancel{\cos^2 2\tilde{\varphi}}-\frac{\tilde{I}}{\omega}\operatorname{sen}^2 2\tilde{\varphi}$$

$$\tilde{K}_\varepsilon(\tilde{I},\tilde{\varphi})=\tilde{I}\omega-\varepsilon^2\frac{\tilde{I}}{\omega}\left(\operatorname{sen}^2 2\tilde{\varphi}\right)+O(\varepsilon^3)$$

che è la stessa espressione trovata prima