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« Theory of orbits »

Chapter 3

Regularization

3.1 Introduction

A significant difference between the motion of natural (celestial) bodies and that of artificial bodies is that close approaches are common occurrences in the latter case while in the former they happen but seldom.

The consequences of this fact can be understood when the property of the Newtonian gravitational force field is recalled, according to which the forces acting between particles approach infinity when the distance between the bodies approaches zero. Therefore at collision (when r_1 or r_2 is zero) the equations show singularities.

Space probes often require orbits which connect two celestial bodies. The actual orbits, of course, do not go through the points of singularity since before this happens the trajectory ends at the point of impact between the probe and the surface of the celestial body. In the framework of the problem of three bodies this impact can take place only at the singularity since the participating bodies are point masses. Therefore, in the physical sense singularities are never reached by means of collision in celestial mechanics. In numerical (computational) and conceptual respects the singularity aspects of the problem of three bodies are of utmost importance.

Both the force acting on the third body and its velocity increase as the body approaches the vicinity of one of the primaries. The step size of a numerical integration must be decreased significantly in this region in order to obtain reliable results. The physical aspects of space

trajectories also require an increased accuracy in these critical regions near the singularities.

The conceptual aspects of the singularities in the field are connected with the existence of the solution of differential equations.

Since the singularities occurring at collisions are not of essential character, they can be eliminated by the proper choice of the independent variable. Once this has been done the following have been effected:

(i) existence of solutions has been established for an arbitrary selection of the initial conditions;
(ii) solutions going through singularities can be traced analytically;
(iii) solutions may be established numerically before, at, and after collision;
(iv) close approaches may be treated with analytical and numerical precision.

In this chapter we proceed from discussing the simplest straightline collision-orbit without regularization, to the complete regularization of the restricted problem. Accordingly, the first subject is the problem of two bodies. The corresponding equations of motion are regularized in two steps. First we treat collision orbits (Section 3.2), then the general problem is regularized (Section 3.3). This is followed by the regularization of the equations of motion of the restricted problem. Here first we solve the problem "locally," by which in this context we mean that we regularize the equations of motion only at one of the two singularities (Section 3.4). Following Birkhoff's terminology, we then affect "global" regularizations; we eliminate both singularities simultaneously. Both the "global" and the "local" regularizations may be, according to mathematical usage, considered local operations; nevertheless, we accept Birkhoff's terminology at this point and defer additional remarks to Section 3.10. The global regularization may be performed with several transformations, and we describe three methods in Sections 3.5–3.7. Generalizations and comparisons of global regularization techniques are offered then in Sections 3.8 and 3.9. Section 3.10 contains the theorem of existence of solutions of the restricted problem for finite intervals of time. The chapter is concluded with two major areas of application: space dynamics and stellar dynamics,

The following two additional remarks are in order at this point. The *regularization of the differential equations* of motion is the principal problem and the main subject of this chapter. The *regularization of the solution* at collision can always be accomplished by introducing the eccentric anomaly since the collision of two bodies in any problem can be regularized in this way.

The second remark is that the inclusion of a chapter on regularization in a book on celestial mechanics might serve two purposes. It helps to establish the existence of solutions from the point of view of analysis and it extends the applicability of celestial mechanics to collision orbits. Books on classical celestial mechanics often omit entirely the subject of regularization since the analyticity of solutions is not of central interest to astronomers and because collisions do not occur in classical celestial mechanics.

For our purposes inclusion of the chapter on regularization is a necessity in order to satisfy our interest in applications to space dynamics as well as to establish the existence of the solutions of the restricted problem. To these solutions, after all, our whole volume is dedicated.

The questions of existence are treated at the end of the chapter, only after a relatively leisurely journey through the methods of regularization. The first three sections after the introduction (Sections 3.2, 3.3, and 3.4) the reader will find comfortable, mostly easy going, and at some places even repetitious. Teaching experience suggests such a treatment to lay a sound foundation for the more complex aspects. Since regularization is essential in space dynamics, the presentation of Sections 3.2, 3.3, and 3.4 is directed to workers in this field, leaving some of the analytical aspects to Section 3.10.

3.2 Regularization of collision orbits in the problem of two bodies

Part (A) of this section describes the dynamics of a simple collision orbit without the use of regularizing variables. In Part (B) regularizing variables are introduced in a general form, and in Part (C) a special transformation concerning only the independent variable is treated. Part (D) offers an explanation of the mathematical result according to which the particle is reflected after collision, shows the regularizing role of the eccentric anomaly, and proves that the distance to collision varies as the 2/3 power of the time. In Part (E) regularization is affected by transforming the independent as well as the dependent variables.

(A) Consider the equations of motion [Section 1.5, Eq. (52)] of the restricted problem and let $\mu_1 = 1$, $\mu_2 = 0$. The corresponding function Ω is

$$\Omega = \frac{1}{r} + \frac{1}{2} r^2 \tag{1}$$

since $r^2 = r_1^2 = x^2 + y^2$ [see Fig. 3.1(a)]. The equations of motion are

$$\ddot{x} - 2\dot{y} = x(1 - 1/r^3), \qquad \ddot{y} + 2\dot{x} = y(1 - 1/r^3), \tag{2}$$

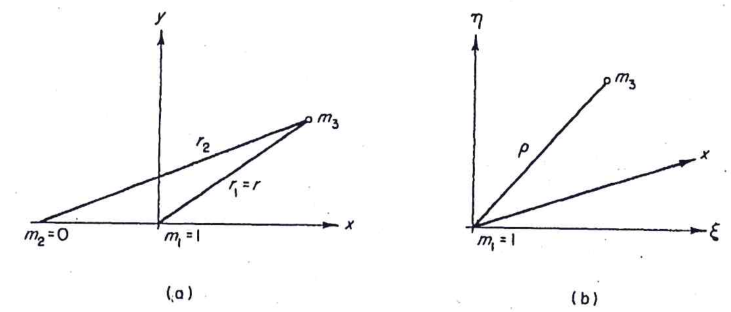

FIG. 3.1. Problem of two bodies.

and the Jacobian integral is

$$\dot{x}^2 + \dot{y}^2 = r^2 + 2/r - C. \tag{3}$$

Equations (2) describe the problem of two bodies; in fact they refer to a simplified restricted problem of three bodies in which the mass of one of the two primaries is zero. This description, however, is in a synodic coordinate system which renders the equations rather complicated. The equations of motion in a corresponding fixed system are

$$\ddot{\xi} = -\xi/\rho^3 \quad \text{and} \quad \ddot{\eta} = -\eta/\rho^3, \tag{4}$$

where $\rho^2 = r^2 = x^2 + y^2 = \xi^2 + \eta^2$ [see Fig. 3.1(b)]. The energy integral of Eqs. (4) is

$$\dot{\xi}^2 + \dot{\eta}^2 = 2/\rho - C. \tag{5}$$

In the first instance the simplest possible close approach, a collision, will be considered. Since a two-body collision orbit in a fixed system of coordinates is a straight line, we might specify the following initial conditions: at $t = 0$, $\xi = \xi_0$, $\dot{\xi} = \dot{\xi}_0$, and $\eta \equiv 0$, $\dot{\eta} \equiv 0$, for any t. Noting that $\rho = |\xi|$ we have

$$\ddot{\xi} = \mp 1/\xi^2 \tag{6}$$

for $\xi \gtrless 0$. The energy integral gives

$$\dot{\xi}^2 = 2/|\xi| - C = \pm 2/\xi - C, \tag{7}$$

again for $\xi \gtrless 0$, and in order to evaluate C we can substitute the initial conditions, obtaining

$$C = \pm 2/\xi_0 - \dot{\xi}_0^2. \tag{8}$$

In order to discuss a specific problem, let $\dot{\xi}_0 = 0$ and $\xi_0 > 0$. Thus $C = 2/\xi_0 > 0$, and Eq. (7) gives

$$\pm \int_{\xi_0}^{\xi} \left(\frac{\xi}{2 - C\xi}\right)^{1/2} d\xi = t. \tag{9}$$

Note that, since $2/|\xi| - 2/\xi_0 \geqslant 0$, $|\xi| \leqslant \xi_0$; the particle released at $t = 0$ from $\xi = \xi_0 > 0$ will never depart farther from the origin than its initial position. The velocity of the particle is directed toward the origin and is negative during the time interval $0 < t < t_c$, where t_c is the time of collision; that is,

$$\dot{\xi} = -\left(\frac{2}{|\xi|} - C\right)^{1/2}, \tag{10}$$

for $0 < \xi < \xi_0$. Equation (6) shows that this negative velocity increases in absolute value as the particle approaches the origin (point of collision), since $\ddot{\xi} < 0$ for $\xi > 0$. As $\xi \to 0$, $|\dot{\xi}| \to \infty$, and therefore Eq. (10) can be integrated from $t = 0$ to $t_c - \delta$, where the time $t_c - \delta$ corresponds to $\xi = \epsilon > 0$ and the time $t = 0$ to $\xi = \xi_0$. The limit is to be evaluated as $\epsilon \to 0$ and $\delta \to 0$. Now Eq. (9) may be evaluated with the negative sign, giving

$$t = \frac{1}{C}[\xi(2 - C\xi)]^{1/2} + \frac{2}{C^{3/2}} \arctan \left(\frac{2 - C\xi}{C\xi}\right)^{1/2}. \tag{11}$$

As $\xi \to 0$, from Eq. (11)

$$t \to t_c = \pi/C^{3/2} \tag{12}$$

which connects the initial conditions with the instant of collision since $C = 2/\xi_0$ and so $\xi_0 = 2(t_c/\pi)^{2/3}$.

The monotonically decreasing function $\xi = \xi(t)$ as obtained from Eq. (11) is shown in Fig. 3.2. From Eq. (10), $|\dot{\xi}| \to \infty$ as $\xi \to 0$ (i.e., as $t \to t_c$), so the curve intersects the t axis perpendicularly. The initial conditions show that the curve starts perpendicularly to the ξ axis; $t = 0$, $\xi = \xi_0$, $\dot{\xi} = \dot{\xi}_0 = 0$. Note that the second derivative $\ddot{\xi}$ is negative in the region $0 \leqslant t < t_c$ so the curve is concave.

What happens to the particle following the instant t_c for $t \geqslant t_c$ is, of course, not clear from the preceding results, which cease to be meaningful at $t = t_c$. The continuation of the orbit after collision is not feasible since the solution encounters the singularity present in the problem.

(B) To eliminate the singularity, new dependent and independent variables are introduced:

$$\xi = f(u) \tag{13}$$

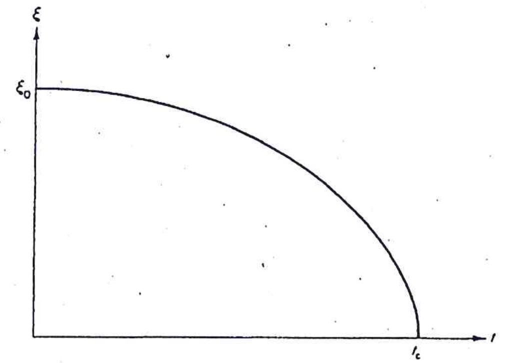

FIG. 3.2. One-dimensional collision orbit. Displacement ξ as a function of time t.

and

$$\tau = \int_{t_0}^{t} \frac{dt}{g(u)}. \tag{14}$$

The motion in the new system is described by the function $u = u(\tau)$; the new variable u depends on its own independent variable τ. When the functions $f(u)$ and $g(u)$ are known, Eq. (13) also gives u as a function of t since $\xi = \xi(t)$. Consequently Eq. (14) gives the relation between the old t and new τ time variables.

Another form of (14), which is frequently used in the literature, is

$$dt/d\tau = g(u). \tag{15}$$

This gives the ratio of the differentials of the old and new times as a function of the new, or, by Eq. (13), of the old dependent variable.

To understand what happens at and close to collision, the phenomenon must be slowed down by stretching the time scale so that the approach of the actual velocity to infinity can be handled.

The considerations leading to the selection of the functions f and g are presented in the following paragraphs.

The new velocity $u' = du/d\tau$ is related to the actual (physical) velocity $\dot{\xi} = d\xi/dt$ by

$$\frac{d\xi(t)}{dt} = \frac{df(u)}{du}\frac{du(\tau)}{d\tau}\frac{d\tau}{dt} \tag{16}$$

or

$$\dot{\xi} = u'f'/g, \tag{17}$$

if the notation $f' = df/du$ is introduced. Equation (16) follows from (13) using (15) and gives the new velocity:

$$u' = g\dot{\xi}/f'. \tag{18}$$

In order to have a finite value of this new velocity at collision, it is necessary that the ratio g/f' approach zero as $\dot{\xi} \to \infty$.

The energy integral (7) may be written as

$$\dot{\xi}^2 = 2/\xi - C = 2U, \tag{19}$$

where the $\pm$ sign is omitted; it should be remembered that $\xi > 0$. The energy integral in terms of the new variables becomes

$$(u')^2 = \frac{g^2}{(f')^2}\left(\frac{2}{f} - C\right) = \frac{g^2}{(f')^2}2U. \tag{20}$$

As $\xi \to 0$, $f(u) \to 0$ and $2U \to \infty$. In order to have a finite velocity u' at collision, we must have $[g^2/(f')^2]U$ finite as $\xi \to 0$.

Since $2U = (2/\xi) - C$, $U \to \infty$ as $\xi \to 0$, and close to collision $U = 1/\xi = 1/f$. So the requirement for finite velocity in the (u, τ) system is that

$$\frac{g^2}{(f')^2 f}$$

remain finite as $\xi \to 0$ or that

$$\frac{g}{f'}\frac{1}{f^{1/2}}$$

be finite as $f \to 0$.

If g/f' is represented by a power series in $f^{1/2}$, the lowest term in this series must be (const)$f^{1/2}$. This follows from writing $g/f' = (f^{1/2})^n$ which gives

$$\frac{g}{f'}\frac{1}{f^{1/2}} = (f^{1/2})^{n-1}.$$

Therefore the above limit requirement is satisfied if $n - 1 \geqslant 0$, and the lowest allowable power of $f^{1/2}$ in the series must be 1. The series is

$$\frac{g}{f'} = A_1 f^{1/2} + A_2 f + A_3 f^{3/2} + A_4 f^2 + \cdots. \tag{21}$$

Consequently

$$\frac{g}{f' f^{1/2}} = A_1 + A_2 f^{1/2} + \cdots, \tag{22}$$

and, as $f \to 0$, $(g/f' f^{1/2}) \to A_1$. Only if $A_1 = 0$ does the limit process lead to $g/(f' f^{1/2}) = 0$. Therefore, the velocity in the system u, τ is finite at the singularity, $u' = 2^{1/2}A_1$, provided g and f are selected satisfying Eq. (22).

For instance, if $\xi = f(u) = u^n$, we have

$$g = A_1 f' f^{1/2} = nA_1 u^{(3/2)n-1}. \tag{23}$$

The next step is to investigate the equation of motion regarding singularities. Similarly to Eq. (16), we have

$$\ddot{\xi} = f'u'\frac{d^2\tau}{dt^2} + (f'u'' + f''u'^2)\left(\frac{d\tau}{dt}\right)^2$$

or, since

$$\frac{d^2\tau}{dt^2} = \frac{d}{dt}\frac{1}{g(u)} = -\frac{g'u'}{g^3},$$

we have

$$\ddot{\xi} = -u'^2\frac{f'g'}{g^3} + \frac{f'u'' + f''u'^2}{g^2}. \tag{24}$$

The equation of motion in the system u, τ therefore becomes, from Eq. (6) using only the region $\xi > 0$,

$$u''\frac{f'}{g^2} + u'^2\left(\frac{f''}{g^2} - \frac{f'g'}{g^3}\right) = \frac{1}{f'}\frac{dU}{du} \tag{25}$$

or

$$u'' + u'^2\frac{gf'' - f'g'}{f'g} = \frac{g^2}{f'^2}\frac{dU}{du}. \tag{26}$$

In connection with the regularization of the velocity in the (u, τ) system, Eq. (20) leads to the requirement that $(g^2/f'^2)2U$ must be finite. In order to utilize this requirement we compute

$$\frac{d}{du}U\frac{g^2}{f'^2} = \frac{g^2}{f'^2}\frac{dU}{du} + \frac{u'^2}{f'g}(g'f' - gf''). \tag{27}$$

Solving this equation for the required term on the right side and substituting it into (26) we have

$$u'' = \frac{d}{du}\frac{g^2}{f'^2}U, \tag{28}$$

which also follows, of course, from (20) by differentiation since $du'/du = u''/u'$.

(C) Regarding the selection of the functions f and g we first recall Sundman's and Levi-Civita's idea according to which the essential part of the regularization is the time transformation, the selection of the function g. The basic idea follows from the previously mentioned process of slowing down the phenomenon, and the equations

$$dt/d\tau = g(\xi) \tag{29}$$

and

$$\dot{\xi} = u \tag{30}$$

take the place of Eqs. (13) and (14). The new velocity $u' = \xi' = d\xi/d\tau$ becomes, from Eq. (20),

$$\xi'^2 = g^2(2/\xi - C), \tag{31}$$

and in order to maintain a finite ξ' as $\xi \to 0$ we must have

$$g^2 = A_1\xi + B_1\xi^2 + C_1\xi^3 + \cdots. \tag{32}$$

This gives

$$\xi'^2 = 2A_1 + \xi(2B_1 - CA_1) + \xi^2(2C_1 - CB_1) + \cdots, \tag{33}$$

and so the new velocity is $(2A_1)^{1/2}$ at collision. If $A_1 = C_1 = \cdots = 0$, but $B_1 = 1$, we have $g = \xi$, $\xi'^2 = 2\xi - C\xi^2$, and from Eq. (29)

$$dt/d\tau = \xi, \tag{34}$$

or

$$d\tau = dt/\xi = \Omega(\xi)\, dt. \tag{35}$$

The new time variable τ is therefore directly related to the potential $\Omega(\xi) = 1/\xi$ of the dynamical problem. Since ξ in the present problem is the distance of the moving particle from the singularity, we have $\Omega(\xi) = 1/r$ or

$$\tau = \int_{t_0}^{t} \frac{dt}{r}, \tag{36}$$

which is a form popular in the literature.

The actual solution of the problem in the (ξ, τ) system follows from integrating Eq. (31). Instead of using the general form, Eq. (32), for $g(\xi)$, the special case $B_1 = 1$, $A_1 = C_1 = \cdots = 0$ is continued:

$$\xi'^2 = \xi(2 - C\xi). \tag{37}$$

The initial conditions are $t = 0$, $\tau = 0$, $\xi = 2/C$ and the solution is obtained from Eq. (37) in the form

$$\xi = \frac{1}{C}(1 + \cos C^{1/2}\tau) \tag{38}$$

with

$$t = \frac{1}{C}[\tau + (1/C^{1/2}) \sin C^{1/2}\tau]. \tag{39}$$

The functions $\xi(\tau)$ and $t(\tau)$ are shown in Figs. 3.3 and 3.4. Equations (38) may be obtained either from Eq. (37) or from

$$\xi'' + C\xi - 1 = 0, \tag{40}$$

which is a consequence of (26) or (28). Equation (39) follows from

$$= \int_0^\tau \xi\, d\tau. \tag{41}$$

At collision, $t = t_c = \pi/C^{3/2}$, $\tau = \tau_c = \pi/C^{1/2}$, and at the beginning of the motion $t = \tau = 0$.

The motion of the particle, according to Eqs. (38) and (39) and as shown in the corresponding Figs. 3.3 and 3.4, is oscillatory. At the beginning of the motion $t = \tau = 0$, $\xi = 2/C > 0$, $\dot{\xi} = 0$, and $\xi' = 0$. The particle moves toward the origin ($\xi = 0$) with $\dot{\xi} < 0$ and $\xi' < 0$ after the beginning of the motion. At $\tau = \tau_c/2$, $t = [(2 + \pi)/2\pi]\, t_c$, $\xi = 1/C$, $\dot{\xi} < 0$, $\xi' < 0$. As the particle approaches the origin, $|\dot{\xi}|$

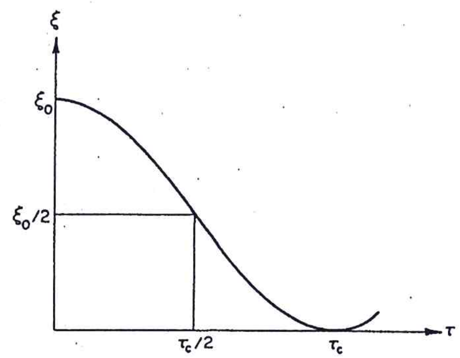

FIG. 3.3. Displacement ξ as a function of the regularized time τ for a one-dimensional collision orbit.

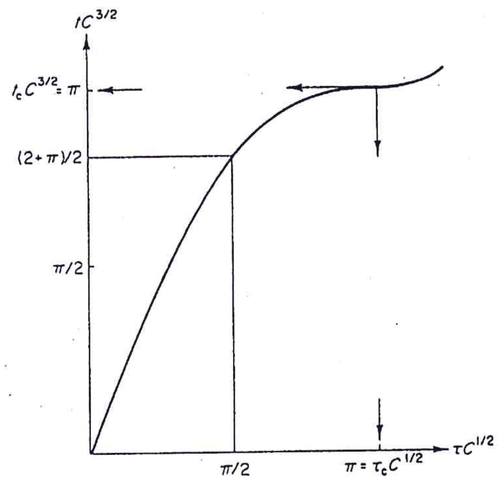

FIG. 3.4. Relation between physical t and pseudo time τ.

increases as $1/\xi^{1/2}$ and ξ' approaches zero. At collision $\xi = 0$, $t = t_c$, $\tau = \tau_c$, $\xi' = 0$, and $|\dot{\xi}| = +\infty$. Shortly after collision $t > t_c$, $\tau > \tau_c$, $\dot{\xi} > 0$, and $\xi' > 0$. At $t = 2t_c$, $\tau = 2\tau_c$, the particle is back at $\xi = 2/C$ with $\dot{\xi} = \xi' = 0$ and the cycle repeats itself. The function $\xi(\tau)$ is regular everywhere and so is the new velocity,

$$\xi'(\tau) = -\frac{1}{C^{1/2}} \sin C^{1/2} \tau.$$

The oscillation takes place along the positive ξ axis between $\xi_0 = 2/C$ and $\xi = 0$ with period $2t_c = 2\pi/C^{3/2}$. Therefore, for unit mean motion $n = 2\pi/2t_c = 1$, we have $t_c = \pi$, $C = 1$, and $\xi_0 = 2$.

(D) The regularized velocity ξ' changes sign at collision; $\xi' < 0$ before collision and $\xi' > 0$ after collision. Since the actual velocity $|\dot{\xi}| = \infty$ at collision, the incoming particle in this system arrives with a velocity $\dot{\xi} \to -\infty$ and leaves with a velocity $\dot{\xi} \to \infty$. Visualization of this is helped if the limiting degeneration of an elliptical two-dimensional orbit is considered.

As in Fig. 3.5, let $\dot{\eta}_0 > 0$ be a small vertical velocity at point $\xi = \xi_0$, $\eta = 0$ so that

$$2/\xi_0 > \dot{\eta}_0^2.$$

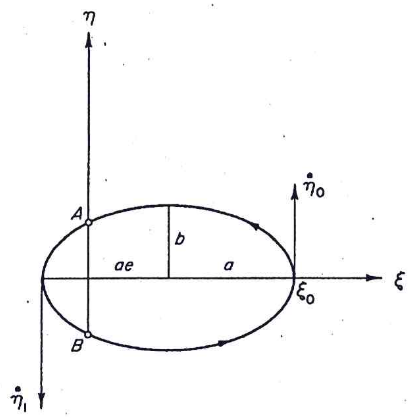

FIG. 3.5. Close approach as $\dot{\eta}_0 \to 0$.

The particle describes an ellipse with focus at the origin of the ξ, η coordinate system. The semimajor axis of this ellipse is related to the initial conditions by the energy integral $(2/r) - (1/a) = v^2$ or

$$\frac{1}{a} = \frac{2}{\xi_0} - \dot{\eta}_0^2, \tag{42}$$

and the eccentricity of the elliptic orbit is obtained from $\xi_0 = a(1 + e)$ or

$$e = \frac{\xi_0 - a}{a}. \tag{43}$$

When $\xi_0 = 2a$, then $e = 1$, $\dot{\eta}_0 = 0$, and with $v^2 = \dot{\xi}_0^2$ we have

$$\frac{2}{\xi_0} - \frac{1}{a} = \dot{\xi}_0^2 \tag{44}$$

from which $C = 1/a$.

The velocity at the pericenter is

$$\dot{\eta}_1 = \frac{1}{a^{1/2}} \left(\frac{1 + e}{1 - e} \right)^{1/2},$$

which goes to infinity as $e \to 1$. The velocity at the apocenter is

$$\dot{\eta}_0 = \frac{1}{a^{1/2}} \left(\frac{1 - e}{1 + e} \right)^{1/2}, \tag{45}$$

and when $e \to 1$, $\dot{\eta}_0 \to 0$.

The speed at point A with coordinates $\xi = 0$, $\eta = \eta_0$ is

$$v_A = \frac{1}{a^{1/2}}\left(\frac{1+e^2}{1-e^2}\right)^{1/2} \tag{46}$$

and the velocity components at points A and B become

$$\dot{\xi}_A = -\frac{1}{[a(1-e^2)]^{1/2}} = -\dot{\xi}_B,$$
$$\dot{\eta}_A = -\frac{e}{[a(1-e^2)]^{1/2}} = \dot{\eta}_B. \tag{47}$$

As $e \to 1$, the ellipse flattens out ($b = a(1-e^2)^{1/2} \to 0$) and the above components tend toward ∞. The ξ component of the velocity changes sign as the particle goes around the singularity, $\dot{\xi}_A = -\dot{\xi}_B$.

The period of the motion on the ellipse is $T = 2\pi/n$, with $n = a^{-3/2}$. When $\eta_0 \to 0$, $a^{-1} \to 2/\xi_0$ and so $T \to 2\pi(2/\xi_0)^{-3/2}$. Since $C = 2/\xi_0$ and $t_c = T/2$, the limiting condition of the elliptic motion furnishes the previously obtained time for collision, t_c.

It is not expected that the motion can be followed completely in the case $e = 1$; nevertheless, the preceding discussion suggests that the results obtained with the regularization process are reasonable not only from a mathematical but also from a physical point of view.

Prior to leaving this simplest case of regularization, attention is called to Eq. (38) connecting the distance $\xi = r$ with the new time variable τ. If the substitutions $a = 1/C$ and $na = aa^{-3/2} = C^{1/2}$ are made, we obtain

$$r = a(1 + \cos na\tau),$$

where $na\tau = u$ is the eccentric anomaly, and consequently Kepler's equation

$$nt = u + e \sin u$$

becomes Eq. (39) with $e = 1$. Note that in the last two equations the plus signs become minus signs if $u = \tau = 0$ corresponds to perihelion instead of to aphelion as in Fig. 3.5.

The conclusion is that the eccentric anomaly is a regularizing variable for the problem of two bodies. In fact, a comparison of Eq. (36),

$$d\tau = dt/r,$$

with

$$du = \frac{na}{r}\,dt$$

shows that our "new time" τ is essentially the eccentric anomaly.

If the time is measured from collision by $t^* = t - t_c$ or by $\tau^* = \tau - \tau_c$ with $Ct_c = \tau_c = \pi/C^{1/2}$ then Eqs. (38) and (39) become

$$\xi = \frac{1}{C}(1 - \cos C^{1/2}\tau^*)$$

and

$$t^* = \frac{1}{C}\left(\tau^* - \frac{1}{C^{1/2}}\sin C^{1/2}\tau^*\right).$$

Power series expansions for these functions may be written as

$$\xi = \frac{\tau^{*2}}{2!} - \frac{C\tau^{*4}}{4!} \pm \cdots = \tau^{*2}\,\Xi(\tau^*)$$

and

$$t^* = \frac{\tau^{*3}}{3!} - \frac{C\tau^{*5}}{5!} \pm \cdots = \tau^{*3}\,T(\tau^*),$$

where the series $\Xi(\tau^*)$ and $T(\tau^*)$ are convergent for any τ^* since they are essentially Taylor series for the functions sine and cosine. The constant terms in the series Ξ and T are $(2!)^{-1}$ and $(3!)^{-1}$, respectively. Consequently for sufficiently small t^* we have

$$\xi = (t^*)^{2/3}\,X(t^{*1/3}),$$ 2/3 VEDI NOTA

where once again the function $X(t^{*1/3})$ is a power series with the constant term $(9/2)^{1/3}$. The original solution therefore has at t^* a branch point of order 2 and the synodic path possesses a cusp at collision.

(E) Equations (13) and (14) propose the performance of the regularization by introducing two transformations,

$$\xi = f(u) \quad \text{and} \quad dt/d\tau = g(u).$$

The selection for $f(u)$ in the previous example was simply $f(u) = \xi = u$, and so the dependent variable was *not* transformed. The transformation of the independent variable followed the formula $dt/d\tau = g(u) = g(\xi) = \xi$. Regularization was performed, therefore, by transforming only the independent variable, as was expected since this was the essential step in the regularization process.

If the dependent variable is also transformed the situation is different because Eq. (23) furnishes a possible function $g(u)$, once $f(u)$ is selected. The selection of

$$\xi = f(u) = u^2 \tag{48}$$

is the next natural step. Note that, when n is even, Eq. (23) gives rational functions for g. In the present case

$$g(u) = Bu^2, \tag{49}$$

where $B =$ const. The new velocity, from Eq. (20), becomes

$$(u')^2 = \left(\frac{1}{2} - \frac{Cu^2}{4}\right) B^2$$

and the selection of $B = 4$ gives

$$u' = \pm\, 2(2 - Cu^2)^{1/2}. \tag{50}$$

The selection of the constant B is quite arbitrary. If $B = 4$ we have $g = 4u^2 = (f')^2$, which will be of interest later.

Regarding the sign in Eq. (50), note that, since $u = \pm\xi^{1/2}$, from Eq. (48), and

$$\dot{\xi} = \frac{u'f'}{g} = \frac{u'}{2u}, \tag{51}$$

from Eq. (17), we have the result that, when $\dot{\xi} > 0$, either both $u > 0$, $u' > 0$ or both $u < 0$, $u' < 0$. When $\dot{\xi} < 0$, the signs of u and u' must be opposite. It is to be realized that there are two distinct values of u corresponding to any value of ξ except to $\xi = 0$.

The equation of motion is Eq. (50). The second-order equation can be established either by differentiating Eq. (50) or by using Eq. (28). The result is

$$u'' + 4Cu = 0. \tag{52}$$

The solution of Eq. (50) or (52) is

$$u = (2/C)^{1/2} \cos 2C^{1/2}\tau, \tag{53}$$

where the initial conditions $\tau = 0$, $u = u_0 = \xi_0^{1/2} = (2/C)^{1/2}$, and $u_0' = 0$ are used. The last condition follows from Eq. (51), since

$$u_0' = 2u_0\dot{\xi}_0 = 0.$$

Collision corresponds to $\xi = u = 0$, i.e., to $\tau_c = \pi/4C^{1/2}$. The new velocity, from Eq. (53), becomes

$$u' = -\,2(2)^{1/2} \sin 2C^{1/2}\tau \tag{54}$$

and its value at collision is

$$u_c' = -2(2)^{1/2};$$

the same results can also be obtained from Eq. (50) with $u = u_c = 0$.

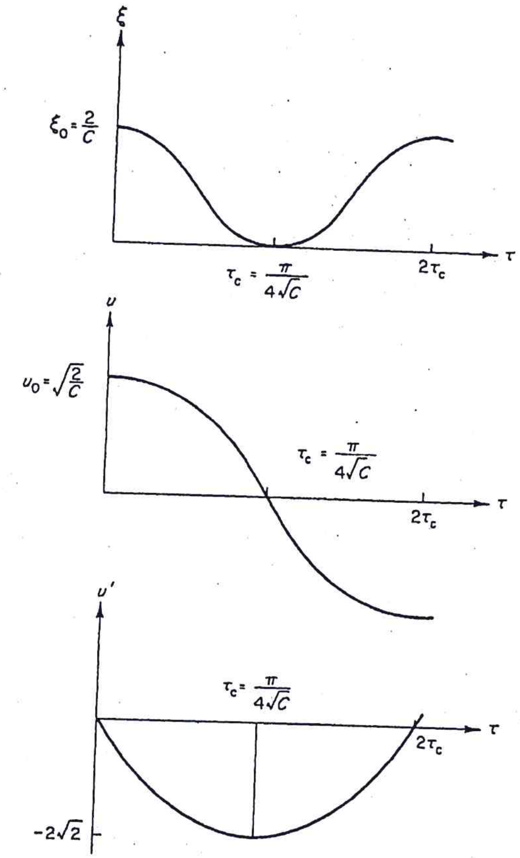

FIG. 3.6. Collision of two bodies with transformed time and coordinate.

In Fig. 3.6 are shown the relations $\xi(\tau)$, $u(\tau)$, and $u'(\tau)$. Only the positive value of the relation $u = \xi^{1/2}$ is shown. Between the initial position and collision, $0 \leqslant \tau \leqslant \tau_c$, $u_0 \geqslant u \geqslant 0$, $\xi_0 \geqslant \xi \geqslant 0$, and $0 \geqslant u' \geqslant -2(2)^{1/2}$. Observe that in this region $0 \geqslant \dot{\xi} \geqslant -\infty$, i.e., $\dot{\xi} < 0$ and sign $u = -$sign u'. After the collision in the time domain $\tau_c \leqslant \tau \leqslant 2\tau_c$ the particle returns from the origin to the position $\xi = \xi_0$, which it occupied at $\tau = 0$. During this time $0 \geqslant u \geqslant -u_0$, $0 \leqslant \xi \leqslant \xi_0$, and $-2(2)^{1/2} \leqslant u' \leqslant 0$. We have sign $u =$ sign u' in this region and $\dot{\xi} > 0$.

To find the relation between the new and old times τ and t we have $dt/d\tau = 4u^2$ from Eq. (15). Using (53) we obtain

$$t = \int_0^\tau \frac{8}{C} \cos^2 2C^{1/2}\tau \, d\tau,$$

or

$$t = \frac{4}{C}\left(\tau + \frac{\sin 4C^{1/2}\tau}{4C^{1/2}}\right), \tag{55}$$

the general shape of which is also shown in Fig. 3.4, since in this case $\tau_c = \pi/4C^{1/2}$.

There is no essential difference between the double transformation when both the time and the dependent variable are transformed, and the previously shown single time transformation. Both methods regularize the equation of motion and represent the solution. The first regularization results in zero velocity at collision, the second gives $2(2)^{1/2}$.

The transformation of the independent variable regularized the equation of motion which became

$$\xi'' + C\xi - 1 = 0$$

and the double transformation gave

$$u'' + 4Cu = 0.$$

Both these results are of the same mathematical simplicity.

The two-dimensional case to be discussed in the next section differs from the straight-line motion (from the point of view of regularization) in that the integral of energy does not furnish the equations of motion in the two-dimensional case. The introduction of the energy relation (20) into the equation of motion (26) eliminates the quadratic term in the velocity in Eq. (26), resulting in Eq. (28). It will be shown that in the two-dimensional case the introduction of a geometrical (coordinate) transformation, in addition to the time transformation, is necessary to eliminate the quadratic velocity term from the equations of motion.

Inasmuch as the regularization of the restricted problem is performed by double transformations, the above considerations [Eqs. (48) and (55)] might be looked upon as preparatory exercises.

3.3 Regularization of the general problem of two bodies

We now return to Eqs. (4) and study their general form. Introducing $\zeta = \xi + i\eta$ we have $\rho = |\zeta|$ and

$$\ddot{\zeta} = -\zeta/|\zeta|^3. \tag{56}$$

It is recognized that the singularity is at $\zeta = 0$, i.e., at collision; therefore, regularization from a mathematical point of view is of interest only when the conic-section orbit degenerates into a straight-line solution. From a practical point of view so-called close approach orbits also require regularization since, when an actual orbit is to be computed under these circumstances, problems of accuracy arise. For this reason we will regularize Eq. (56) by introducing a coordinate transformation,

$$\zeta = f(w), \tag{57}$$

and a time transformation,

$$dt/d\tau = g(w), \tag{58}$$

quite similar to Eqs. (13) and (14). Here $w = u + iv$, so while in the case of the straight-line motion the transformation established a relation between the points of the physical line ξ and the transformed line u, now the relation is between the physical plane ζ and the transformed plane w. The function $g(w)$ is a real function of the complex variable w, introducing in this way the new time τ as a real quantity.

The first transformation represents a conformal mapping; it contains the geometric information and it controls the accuracy of the shape of the orbit. The second transformation is the essential one, as was shown before, since it controls the kinematic aspects and it performs the regularization. The introduction of two transformations gives greater freedom and it will be shown in this section that significant simplifications of the transformed equations of motion are obtained by properly (and not independently) selecting the functions f and g.

The energy integral in the physical plane is

$$|\dot{\zeta}|^2 = 2/|\zeta| - C, \tag{59}$$

which is identical with the previously given Eq. (5).

Computation of $\dot{\zeta}$ follows the pattern established for the one-dimensional case:

$$\dot{\zeta} = \frac{d\zeta}{dw}\frac{dw}{d\tau}\frac{d\tau}{dt}, \tag{60}$$

or

$$|\dot{\zeta}|^2 = \frac{|f'(w)|^2 |w'|^2}{g^2}. \tag{61}$$

Combining this with Eq. (59) gives the energy integral in the new system of variables:

$$|w'|^2 = \left(\frac{2}{|f|} - C\right)\frac{g^2}{|f'|^2}, \tag{62}$$