MURRAY & DERMOTT,

« Solar System Dynamics »

6
The Disturbing Function

O polished perturbation!

William Shakespeare, *Henry IV (2), IV, v*

6.1 Introduction

In Chapter 3 we approached the three-body problem from the point of view of the location and stability of equilibrium points in the restricted problem. However, we made no attempt to tackle the more general problem of the motion of a third body under the gravitational effects of the two other bodies for arbitrary initial conditions. This problem is nonintegrable, but we can make some progress by analysing the accelerations experienced by the three bodies. If their motions are dominated by a central or primary body, then the orbits of the secondary bodies are conic sections with small deviations due to their mutual gravitational perturbations. In this chapter, we show how these deviations can be calculated by defining and analysing the *disturbing function*.

Consider a mass m_i orbiting a primary of mass m_c in an elliptical path. As we have seen in Chapter 2, this problem is integrable and the orbital elements a_i, e_i, I_i, ϖ_i, and Ω_i of the mass m_i are constant, provided the gravitational effect of the central body can be treated as arising from a point mass. If we now introduce a third mass, m_j, then the mutual gravitational force between the masses m_i and m_j results in accelerations in addition to the standard two-body accelerations due to m_c (see Fig. 6.1). These additional accelerations of the secondary masses *relative* to the primary can be obtained from the gradient of the perturbing potential, also called the *disturbing function*.

This chapter is concerned with a mathematical analysis of the properties of a Fourier series expansion of the disturbing function. We show how particular problems in solar system dynamics can be tackled by isolating the appropriate

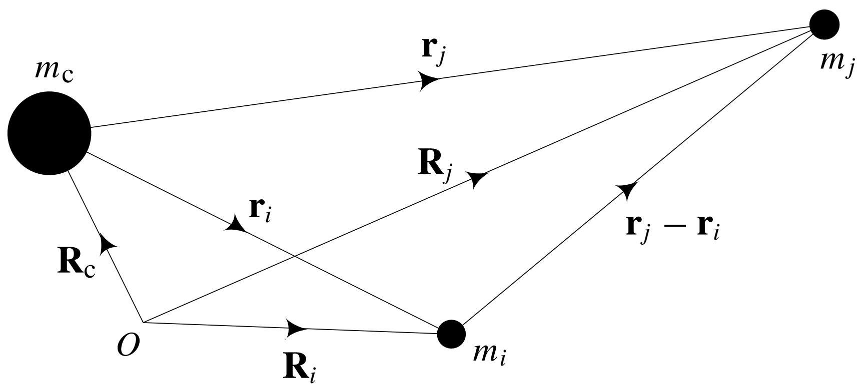

Fig. 6.1. The position vectors $\mathbf{r}_i$ and $\mathbf{r}_j$, of two masses m_i and m_j, with respect to the central mass m_c. The three masses have position vectors $\mathbf{R}$, $\mathbf{R}'$, and $\mathbf{R}_c$ with respect to an arbitrary, fixed origin O.

terms in the expansion of the disturbing function and by assuming that the time-averaged contributions to the equations of motion of all the other terms are negligible. An understanding of the properties of the disturbing function is the key to understanding the dynamics of resonance and other long-period motions in the solar system.

6.2 The Disturbing Function

Let the position vectors with respect to a fixed origin O of the three bodies of masses m_c, m_i, and m_j be $\mathbf{R}_c$, $\mathbf{R}_i$, and $\mathbf{R}_j$ respectively. Let $\mathbf{r}_i$ and $\mathbf{r}_j$ denote the position vectors of the secondary masses m_i and m_j relative to the primary, where

$$|\mathbf{r}_i| = r_i = \left(x_i^2 + y_i^2 + z_i^2\right)^{1/2}, \qquad |\mathbf{r}_j| = r_j = \left(x_j^2 + y_j^2 + z_j^2\right)^{1/2}, \tag{6.1}$$

and

$$|\mathbf{r}_j - \mathbf{r_i}| = \left[(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2\right]^{1/2} \tag{6.2}$$

and the primary is the origin of the coordinate system (see Fig. 6.1).

From Newton's laws of motion and the law of gravitation we obtain the equations of motion of the three masses in the inertial reference frame:

$$m_c \ddot{\mathbf{R}}_c = \mathcal{G} m_c m_i \frac{\mathbf{r}_i}{r_i^3} + \mathcal{G} m_c m_j \frac{\mathbf{r}_j}{r_j^3}, \tag{6.3}$$

$$m_i \ddot{\mathbf{R}}_i = \mathcal{G} m_i m_j \frac{(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} - \mathcal{G} m_i m_c \frac{\mathbf{r}_i}{r_i^3}, \tag{6.4}$$

$$m_j \ddot{\mathbf{R}}_j = \mathcal{G} m_j m_i \frac{(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} - \mathcal{G} m_j m_c \frac{\mathbf{r}_j}{r_j^3}. \tag{6.5}$$

The accelerations of the secondaries relative to the primary are given by

$$\ddot{\mathbf{r}}_i = \ddot{\mathbf{R}}_i - \ddot{\mathbf{R}}_{\mathrm{c}}, \tag{6.6}$$

$$\ddot{\mathbf{r}}_j = \ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}_{\mathrm{c}}. \tag{6.7}$$

Substituting the expressions for $\ddot{\mathbf{R}}_{\mathrm{c}}$, $\ddot{\mathbf{R}}_i$, and $\ddot{\mathbf{R}}_j$ from Eqs. (6.3)–(6.5) we get

$$\ddot{\mathbf{r}}_i + \mathcal{G}(m_{\mathrm{c}} + m_i)\frac{\mathbf{r}_i}{r_i^3} = \mathcal{G}m_j\left(\frac{\mathbf{r}_j - \mathbf{r}_i}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^3} - \frac{\mathbf{r}_j}{r_j^3}\right), \tag{6.8}$$

$$\ddot{\mathbf{r}}_j + \mathcal{G}(m_{\mathrm{c}} + m_j)\frac{\mathbf{r}_j}{r_j^3} = \mathcal{G}m_i\left(\frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3} - \frac{\mathbf{r}_i}{r_i^3}\right). \tag{6.9}$$

These relative accelerations can be written as gradients of scalar functions, that is, we can write

$$\ddot{\mathbf{r}}_i = \nabla_i(U_i + \mathcal{R}_i) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x_i} + \hat{\mathbf{j}}\frac{\partial}{\partial y_i} + \hat{\mathbf{k}}\frac{\partial}{\partial z_i}\right)(U_i + \mathcal{R}_i) \tag{6.10}$$

and

$$\ddot{\mathbf{r}}_j = \nabla_j(U_j + \mathcal{R}_j) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x_j} + \hat{\mathbf{j}}\frac{\partial}{\partial y_j} + \hat{\mathbf{k}}\frac{\partial}{\partial z_j}\right)(U_j + \mathcal{R}_j), \tag{6.11}$$

where

$$U_i = \mathcal{G}\frac{(m_{\mathrm{c}} + m_i)}{r_i} \quad \text{and} \quad U_j = \mathcal{G}\frac{(m_{\mathrm{c}} + m_j)}{r_j} \tag{6.12}$$

are the central, or two-body, parts of the total potential. The subscript i or j is included in the ∇ operator to emphasise that the gradient is with respect to the coordinates of the mass m_i or m_j. The $\mathcal{R}$ term in the potential is the *disturbing function*, which represents the potential that arises from the other secondary mass. Since $\mathbf{r}_i$ is not a function of x_j, y_j, and z_j, and $\mathbf{r}_j$ is not a function of x_i, y_i, and z_i, we can write

$$\mathcal{R}_i = \frac{\mathcal{G}m_j}{\left|\mathbf{r}_j - \mathbf{r}_i\right|} - \mathcal{G}m_j\frac{\mathbf{r}_i \cdot \mathbf{r}_j}{r_j^3}, \tag{6.13}$$

$$\mathcal{R}_j = \frac{\mathcal{G}m_i}{\left|\mathbf{r}_i - \mathbf{r}_j\right|} - \mathcal{G}m_i\frac{\mathbf{r}_i \cdot \mathbf{r}_j}{r_i^3}. \tag{6.14}$$

The leading terms in these expressions are called the *direct terms* while the other terms that arise from the choice of the origin of the coordinate system are called the *indirect terms*. If the origin of the coordinate system was at the centre of mass, then these indirect terms would not appear.

The above analysis can be extended to any number of bodies. In addition, the accelerations associated with the disturbing function can arise from any source and not just from point-mass gravitational forces. They could, for example, arise from a potential associated with the oblateness of the central mass (see Sect. 6.11). However, in what follows in this chapter we are mostly concerned with the particular case of two point-mass secondaries of masses m and m' and position vectors $\mathbf{r}$ and $\mathbf{r}'$ relative to the central mass, where $r < r'$ always. With this notation, the equation of motion of the inner secondary is

$$\ddot{\mathbf{r}} + \mathcal{G}\,(m_{\rm c} + m)\,\frac{\mathbf{r}}{r^3} = \mathcal{G}m'\left(\frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} - \frac{\mathbf{r}'}{r'^3}\right) \tag{6.15}$$

and its disturbing function can be written

$$\mathcal{R} = \frac{\mu'}{|\mathbf{r}' - \mathbf{r}|} - \mu'\frac{\mathbf{r}\cdot\mathbf{r}'}{r'^3}, \tag{6.16}$$

where $\mu' = \mathcal{G}m'$ and the associated reference orbit has osculating elements $n^2a^3 = \mathcal{G}\,(m_{\rm c} + m)$. Similar equations can be written for the outer secondary giving

$$\ddot{\mathbf{r}}' + \mathcal{G}\left(m_{\rm c} + m'\right)\frac{\mathbf{r}'}{r'^3} = \mathcal{G}m\left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} - \frac{\mathbf{r}}{r^3}\right). \tag{6.17}$$

The corresponding disturbing function for the outer secondary is then

$$\mathcal{R}' = \frac{\mu}{|\mathbf{r} - \mathbf{r}'|} - \mu\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}, \tag{6.18}$$

where $\mu = \mathcal{G}m$ and the associated reference orbit has osculating elements $n'^2a'^3 = \mathcal{G}\left(m_{\rm c} + m'\right)$.

Although this is the most straightforward way to derive expressions for $\mathcal{R}$ and $\mathcal{R}'$, it is worth pointing out that this procedure and the resulting expressions are not unique. For example, it is possible to add an additional term, $\mathcal{G}m\mathbf{r}'/r'^3$, to each side of the equation of motion for the mass m', Eq. (6.17), resulting in an additional term $-\mu/r'$ in the expression for $\mathcal{R}'$; however, this requires that the associated reference orbit for m' has osculating elements $n'^2a'^3 = \mathcal{G}\left(m_{\rm c} + m + m'\right)$.

6.3 Expansion Using Legendre Polynomials

Consider the configuration shown in Fig. 6.2 where $\mathbf{r}$ and $\mathbf{r}'$ denote the position vectors of the masses m and m' respectively. Let ψ denote the angle between the two position vectors. From the cosine rule we have

$$\left|\mathbf{r}' - \mathbf{r}\right|^2 = r^2 + r'^2 - 2rr'\cos\psi, \tag{6.19}$$

or, alternatively,

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{r'}\left[1 - 2\frac{r}{r'}\cos\psi + \left(\frac{r}{r'}\right)^2\right]^{-\frac{1}{2}}. \tag{6.20}$$

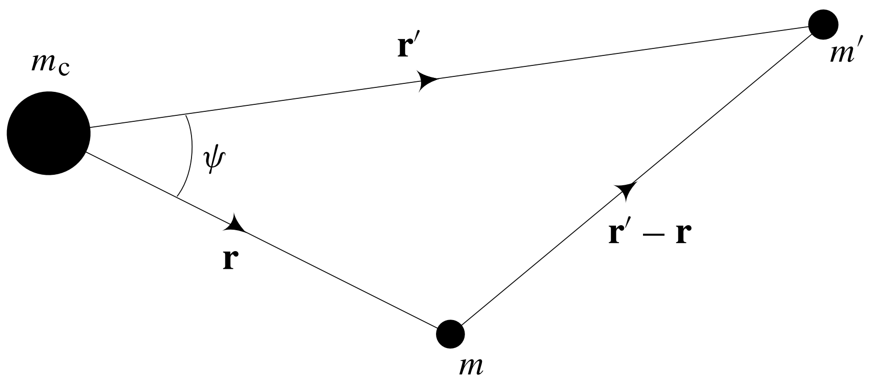

Fig. 6.2. The position vectors $\mathbf{r}$ and $\mathbf{r}'$ of two masses m and m', with respect to a central mass m_c. The angle between the position vectors is ψ.

This can be expanded in Legendre polynomials to give

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{r'} \sum_{l=0}^{\infty} \left(\frac{r}{r'}\right)^l P_l(\cos\psi), \tag{6.21}$$

where $P_0(\cos\psi) = 1$, $P_1(\cos\psi) = \cos\psi$, $P_2(\cos\psi) = \frac{1}{2}(3\cos^2\psi - 1)$, etc. (see Sect. 4.2).

Since $\mathbf{r} \cdot \mathbf{r}' = rr'\cos\psi = rr'P_1(\cos\psi)$, the disturbing function for the inner secondary can be written

$$\mathcal{R} = \frac{\mu'}{r'} \sum_{l=2}^{\infty} \left(\frac{r}{r'}\right)^l P_l(\cos\psi), \tag{6.22}$$

where the $P_0(\cos\psi)$ term has been omitted because it does not depend on r and, ultimately, we are only interested in the gradient of $\mathcal{R}$ with respect to the coordinates of the inner secondary. Similarly, the disturbing function for the outer secondary can be written

$$\mathcal{R}' = \frac{\mu}{r'} \sum_{l=2}^{\infty} \left(\frac{r}{r'}\right)^l P_l(\cos\psi) + \mu\frac{r}{r'^2}\cos\psi - \mu\frac{r'}{r^2}\cos\psi. \tag{6.23}$$

Thus, apart from two extra terms (that are actually unimportant for the applications discussed in the book), the expressions for $\mathcal{R}$ and $\mathcal{R}'$ are very similar.

This chapter is concerned with the series expansion of the disturbing functions $\mathcal{R}$ and $\mathcal{R}'$ in terms of the orbital elements (as opposed to the Cartesian coordinates) of m and m'. We use the standard orbital elements a, e, I, ϖ, Ω, and λ to denote the semi-major axis, eccentricity, inclination, longitude of pericentre, longitude of ascending node, and mean longitude, respectively, of the mass m, with similar primed quantities for the mass m'. We show that the expansion of $\mathcal{R}$ has the form

$$\mathcal{R} = \mu' \sum S(a, a', e, e', I, I')\cos\varphi. \tag{6.24}$$

Here φ is a permitted linear combination with general form

$$\varphi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega, \quad (6.25)$$

where the j_i $(i = 1, 2, \ldots, 6)$ are integers and

$$\sum_{i=1}^{6} j_i = 0. \quad (6.26)$$

This property stems from the azimuthal invariance of the primary's potential. By knowing the explicit form of the function S and the permissible combinations of the angles in φ, we can identify those terms that make the dominant contributions to the equations of motion and, conversely, those that can be neglected.

To illustrate the nature of this expansion let us consider the special case where the orbits of the two masses m and m' lie in the same plane and we can ignore any terms arising from the inclination. In this case we can write the angle ψ as the difference of the true longitudes,

$$\psi = (f' + \varpi') - (f + \varpi), \quad (6.27)$$

where f and f' denote the true anomalies of m and m'. Hence,

$$\begin{aligned}\cos\psi = {} & (\cos f'\cos\varpi' - \sin f'\sin\varpi')(\cos f\cos\varpi - \sin f\sin\varpi) \\ & + (\sin f'\cos\varpi' + \cos f'\sin\varpi')(\sin f\cos\varpi + \cos f\sin\varpi). \quad (6.28)\end{aligned}$$

We have already given series expansions for $\cos f$ and $\sin f$ in Sect. 2.5 and we can find similar series for $\cos f'$ and $\sin f'$ by substituting M' for M and e' for e. Taking these expansions to second degree in e and e' we find

$$\begin{aligned}\cos\psi = {} & \left(1 - e^2 - e'^2\right)\cos[M - M' + \varpi - \varpi'] \\ & - e\cos[M' - \varpi + \varpi'] - e'\cos[M + \varpi - \varpi'] \\ & + e\cos[2M - M' + \varpi - \varpi'] + e'\cos[M - 2M' + \varpi - \varpi'] \\ & - \frac{1}{8}e^2\cos[M + M' - \varpi + \varpi'] - \frac{1}{8}e'^2\cos[M + M' + \varpi - \varpi'] \\ & + \frac{9}{8}e^2\cos[3M - M' + \varpi - \varpi'] + \frac{9}{8}e'^2\cos[M - 3M' + \varpi - \varpi'] \\ & + ee'\cos[\varpi - \varpi'] + ee'\cos[2M - 2M' + \varpi - \varpi'] \\ & - ee'\cos[2M + \varpi - \varpi'] - ee'\cos[2M' - \varpi + \varpi']. \quad (6.29)\end{aligned}$$

Even at this stage some properties of the expression for $\cos\psi$ are evident. It is clear that the degree of the eccentricity term associated with each cosine argument is at least the modulus of the sum of the coefficients of the mean anomalies in the argument. Another property shows up if we express the angles in terms of the mean longitudes rather than the mean anomalies using the substitutions

$M = \lambda - \varpi$ and $M' = \lambda' - \varpi'$. This gives

$$\begin{aligned}\cos\psi = & \left(1 - e^2 - e'^2\right)\cos[\lambda - \lambda'] - e\cos[\lambda' - \varpi] - e'\cos[\lambda - \varpi'] \\ & + e\cos[2\lambda - \lambda' - \varpi] + e'\cos[\lambda - 2\lambda' + \varpi'] \\ & - \frac{1}{8}e^2\cos[\lambda + \lambda' - 2\varpi] - \frac{1}{8}e'^2\cos[\lambda + \lambda' - 2\varpi'] \\ & + \frac{9}{8}e^2\cos[3\lambda - \lambda' - 2\varpi] + \frac{9}{8}e'^2\cos[\lambda - 3\lambda' + 2\varpi'] \\ & + ee'\cos[\varpi - \varpi'] + ee'\cos[2\lambda - 2\lambda' - \varpi + \varpi'] \\ & - ee'\cos[2\lambda - \varpi - \varpi'] - ee'\cos[2\lambda' - \varpi - \varpi']. \end{aligned} \tag{6.30}$$

With this choice of angles it is clear that the sum of the integer coefficients of the longitudes in each argument is zero. This particular property is also true of the final expansion when the angles are expressed in terms of longitudes and it allows us to determine the permissible arguments.

If we now turn our attention to the radially dependent parts of the disturbing function Eq. (6.22), we can write

$$\mathcal{R} = \frac{\mu'}{a'}\sum_{l=2}^{\infty}\alpha^l\left(\frac{a'}{r'}\right)^{l+1}\left(\frac{r}{a}\right)^l P_l(\cos\psi), \tag{6.31}$$

where

$$\alpha = \frac{a}{a'} < 1 \tag{6.32}$$

is the ratio of the semi-major axes of the masses m and m'.

If we consider the terms with $l = 2$ then the series expansion for r/a given in Sect. 2.5 gives

$$\left(\frac{r}{a}\right)^2 \approx 1 - 2e\cos M + \left(\frac{1}{2}\right)e^2(3 - \cos 2M), \tag{6.33}$$

$$\left(\frac{a'}{r'}\right)^3 \approx 1 + 3e'\cos M' + \left(\frac{3}{2}\right)e'^2(1 + 3\cos 2M'), \tag{6.34}$$

with

$$\begin{aligned}\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3 \approx & 1 + \frac{3}{2}e^2 + \frac{3}{2}e'^2 - 2e\cos M + 3e'\cos M' \\ & - \frac{1}{2}e^2\cos 2M + \frac{9}{2}e'^2\cos 2M' \\ & - 3ee'\cos[M - M'] - 3ee'\cos[M + M']. \end{aligned} \tag{6.35}$$

Since $P_2(x) = (1/2)(3x^2 - 1)$, $P_3(x) = (1/2)(5x^3 - 3x)$, etc., considerable effort is required to calculate the $P_l(\cos\psi)$ given the complexity of our expression for $\cos\psi$. In fact, for $l = 2$ there are fourteen separate arguments, while for $l = 3$ there are thirty-six arguments. However, since the series for $(a'/r')^{l+1}(r/a)^l$ only involves sums and differences of the mean anomalies, this means that their product with the terms in the $P_l(\cos\psi)$ series will always preserve the property that the sum of the coefficients of the longitudes in any cosine argument is zero.

It is clear, even from this simple analysis, that the expansion of the disturbing function is a nontrivial task best undertaken with the assistance of computer algebra systems. The end result is a series in α involving a large number of different arguments. Before considering how best to deal with this series, it is essential to generalise the expansion to three dimensions and introduce the inclinations and nodes of the two orbits.

The disturbing function $\mathcal{R}$ can be expanded in terms of standard orbital elements using the method developed by Kaula (1961, 1962), in which the disturbing function for an inner secondary is expanded in an infinite series in the osculating (i.e., instantaneous) elliptic elements referred to the equator of the primary. The expression for $\mathcal{R}$ in Eq. (6.16) can be written

$$\begin{aligned}
\mathcal{R} = {} & \frac{\mu'}{a'} \sum_{l=2}^{\infty} \alpha^l \sum_{m=0}^{l} (-1)^{l-m} \kappa_m \frac{(l-m)!}{(l+m)!} \\
& \times \sum_{p,p'=0}^{l} F_{lmp}(I) F_{lmp'}(I') \sum_{q,q'=-\infty}^{\infty} X_{l-2p+q}^{l,l-2p}(e) X_{l-2p'+q'}^{-l-1,l-2p'}(e') \\
& \times \cos\big[(l-2p'+q')\lambda' - (l-2p+q)\lambda - q'\varpi' + q\varpi \\
& + (m-l+2p')\Omega' - (m-l+2p)\Omega\big],
\end{aligned} \tag{6.36}$$

where $\alpha = a/a'$, λ and λ' are mean longitudes, ϖ and ϖ' are the longitudes of pericentre, and $\kappa_0 = 1$ and $\kappa_m = 2$ for $m \neq 0$.

The $F_{lmp}(I)$ are the inclination functions defined as

$$\begin{aligned}
F_{lmp}(I) = {} & \frac{\mathrm{i}^{l-m}(l+m)!}{2^l p!(l-p)!} \\
& \times \sum_k (-1)^k \binom{2l-2p}{k} \binom{2p}{l-m-k} c^{3l-m-2p-2k} s^{m-l+2p+2k},
\end{aligned} \tag{6.37}$$

where $\mathrm{i} = \sqrt{-1}$, k is summed from $k = \max(0, l-m-2p)$ to $k = \min(l-m, 2l-2p)$, $s = \sin\frac{1}{2}I$, and $c = \cos\frac{1}{2}I$.

The quantities $X_c^{a,b}(e)$ are *Hansen coefficients*, which can be defined by

$$X_c^{a,b}(e) = e^{|c-b|} \sum_{\sigma=0}^{\infty} X_{\sigma+\alpha,\sigma+\beta}^{a,b} e^{2\sigma}. \tag{6.38}$$

In this context $\alpha = \max(0, c-b)$, $\beta = \max(0, b-c)$, and the $X_{c,d}^{a,b}$ are *Newcomb operators*, which can be defined recursively by

$$X_{0,0}^{a,b} = 1, \tag{6.39}$$

$$X_{1,0}^{a,b} = b - a/2, \tag{6.40}$$

and, for $d = 0$,

$$4cX_{c,0}^{a,b} = 2(2b-a)X_{c-1,0}^{a,b+1} + (b-a)X_{c-2,0}^{a,b+2}, \tag{6.41}$$

or, for $d \neq 0$,

$$\begin{aligned} 4dX_{c,d}^{a,b} = &-2(2b+a)X_{c,d-1}^{a,b-1} - (b+a)X_{c,d-2}^{a,b-2} \\ &- (c-5d+4+4b+a)X_{c-1,d-1}^{a,b} \\ &+ 2(c-d+b)\sum_{j\geq 2}(-1)^j \binom{3/2}{j} X_{c-j,d-j}^{a,b}. \end{aligned} \tag{6.42}$$

Also, $X_{c,d}^{a,b} = 0$ if $c < 0$ or $d < 0$. If $d > c$ then $X_{c,d}^{a,b} = X_{d,c}^{a,-b}$.

Additional information concerning Hansen coefficients and Newcomb operators can be found in Plummer (1918) and Hughes (1981). In particular, Hughes (1981) describes the properties of Hansen coefficients and their various recursive relations.

We must also consider the expansion of $\mathcal{R}'$. It is interesting to note that this expansion is curiously absent from the literature. It can only be assumed that since this form of expansion was developed for handling the perturbations on artificial satellites due to the exterior orbits of the Moon and Sun, the need for a similar expansion for $\mathcal{R}'$ never arose.

The expression for $\mathcal{R}'$ is

$$\begin{aligned} \mathcal{R}' = &\frac{\mu}{a'}\sum_{l=1}^{\infty}\alpha^l \sum_{m=0}^{l}\kappa_m \frac{(l-m)!}{(l+m)!} \\ &\times \sum_{p,p'=0}^{l} F_{lmp}(I)F_{lmp'}(I') \sum_{q,q'=-\infty}^{\infty} X_{l-2p+q}^{l,l-2p}(e) X_{l-2p'+q'}^{-(l+1),l-2p'}(e') \\ &\times \cos\big[(l-2p'+q')\lambda' - (l-2p+q)\lambda - q'\varpi' + q\varpi \\ &\qquad + (m-l+2p')\Omega' - (m-l+2p)\Omega\big] \\ &- \frac{\mu a'}{a^2}\sum_{m=0}^{1}\kappa_m \frac{(1-m)!}{(1+m)!} \\ &\times \sum_{p,p'=0}^{1} F_{1mp}(I)F_{1mp'}(I') \sum_{q,q'=-\infty}^{\infty} X_{1-2p+q}^{-2,1-2p}(e) X_{1-2p'+q'}^{1,1-2p'}(e') \\ &\times \cos\big[(1-2p'+q')\lambda' - (1-2p+q)\lambda - q'\varpi' + q\varpi \\ &+ (m-1+2p')\Omega' - (m-1+2p)\Omega\big]. \end{aligned} \tag{6.43}$$

6.4 Literal Expansion in Orbital Elements

Given the importance of the disturbing function in solar system dynamics, a number of authors have derived high-order expansions. Peirce (1849) derived an expansion to sixth order in the eccentricities and mutual inclination. One of the major expansions of the disturbing function, and one of the most commonly used, is due to Le Verrier (1855), who published a seventh-order expansion; Boquet (1889) extended Le Verrier's expansion to eighth order. Although Le Verrier's expansion contains a number of trivial errors, most of which were corrected in subsequent volumes of the *Annals of the Paris Observatory*, a single nontrivial mistake was found by Murray (1985). Other expansions include the symbolic development to sixth order by Newcomb (1895) and the low-order expansions by Brown & Shook (1933) and Brouwer & Clemence (1961). Although all these expansions were carried out in terms of the individual eccentricities and longitudes of pericentre of the two orbiting bodies, they all made use of a mutual inclination and a mutual ascending node. The reason for this was probably to reduce the amount of calculation required, but in the era of computer algebra such restrictions no longer apply. An expansion complete to second order in the individual eccentricities and inclinations is derived in Sect. 6.5, while Appendix B contains a literal expansion, which is complete to fourth order. Both expansions were derived using the method outlined below.

Given the complexity of the expansion, it is customary to distinguish between the direct and indirect parts of the disturbing function. Using the definitions in Eqs. (6.16) and (6.18), we can write

$$\mathcal{R} = \frac{\mu'}{a'}\mathcal{R}_{\mathrm{D}} + \frac{\mu'}{a'}\alpha\mathcal{R}_{\mathrm{E}} \tag{6.44}$$

and

$$\mathcal{R}' = \frac{\mu}{a'}\mathcal{R}_{\mathrm{D}} + \frac{\mu}{a'}\frac{1}{\alpha^2}\mathcal{R}_{\mathrm{I}}, \tag{6.45}$$

where

$$\mathcal{R}_{\mathrm{D}} = \frac{a'}{|\mathbf{r}' - \mathbf{r}|} \tag{6.46}$$

and

$$\mathcal{R}_{\mathrm{E}} = -\left(\frac{r}{a}\right)\left(\frac{a'}{r'}\right)^2 \cos\psi, \tag{6.47}$$

$$\mathcal{R}_{\mathrm{I}} = -\left(\frac{r'}{a'}\right)\left(\frac{a}{r}\right)^2 \cos\psi. \tag{6.48}$$

In these expressions $\mathcal{R}_{\mathrm{D}}$ is derived from the direct part of the disturbing function, $\mathcal{R}_{\mathrm{E}}$ comes from the indirect part due to an external perturber, and $\mathcal{R}_{\mathrm{I}}$ comes from the indirect part for an internal perturber. It is clear from Eqs. (6.44)–(6.46) that we can use an expansion of $\mathcal{R}_{\mathrm{D}}$ to obtain the direct part of either $\mathcal{R}$ or $\mathcal{R}'$.

To isolate the appropriate terms in the disturbing function for any particular problem in solar system dynamics, we need to obtain a series expansion of $\mathcal{R}$ or $\mathcal{R}'$ in terms of the individual orbital elements of the two orbiting bodies. This requires separate expansions of the direct part $\mathcal{R}_{\mathrm{D}}$ defined in Eq. (6.46) and the indirect parts $\mathcal{R}_E$ and $\mathcal{R}_I$ defined in Eqs. (6.47) and (6.48) respectively. The different cosine arguments in the expansion given in Appendix B are labelled D, E, or I according to which part of the disturbing function they are derived from.

Using Eq. (6.19) we can write

$$\frac{1}{\Delta} = \frac{\mathcal{R}_D}{a'} = \left[r^2 + r'^2 - 2rr'\cos\psi\right]^{-1/2}, \qquad (6.49)$$

where $\Delta = |\mathbf{r}' - \mathbf{r}|$ is the separation of the two masses and ψ is the angle between the two radius vectors (see Fig. 6.2). Since $\mathbf{r}\cdot\mathbf{r}' = rr'\cos\psi$ we can write

$$\cos\psi = \frac{xx' + yy' + zz'}{rr'}. \qquad (6.50)$$

From Eq. (2.122) we have

$$\frac{x}{r} = \cos\Omega\cos(\omega + f) - \sin\Omega\sin(\omega + f)\cos I, \qquad (6.51)$$

$$\frac{y}{r} = \sin\Omega\cos(\omega + f) + \cos\Omega\sin(\omega + f)\cos I, \qquad (6.52)$$

$$\frac{z}{r} = \sin(\omega + f)\sin I, \qquad (6.53)$$

with similar expressions for x'/r', y'/r', and z'/r'.

Each of the above equations can be expanded as a series in M and M' using the series expansions for $\cos f$ and $\sin f$ given in Sect. 2.5 and hence we can derive a series expansion for $\cos\psi$. If we define

$$\Psi = \cos\psi - \cos(\theta - \theta'), \qquad (6.54)$$

where $\theta = \varpi + f$ and $\theta' = \varpi' + f'$ are the true longitudes of the inner and outer bodies respectively, then, as we shall see later, the resulting series for Ψ is of second order in $\sin I$ and $\sin I'$ and the expression for Δ^{-1} can be expanded as a Taylor series in Ψ. We have

$$\begin{aligned}\frac{1}{\Delta} &= \left[r^2 + r'^2 - 2rr'\left(\cos(\theta - \theta') + \Psi\right)\right]^{-1/2} \\ &= \frac{1}{\Delta_0} + rr'\Psi\cdot\frac{1}{\Delta_0^3} + \frac{3}{2}\left(rr'\Psi\right)^2\cdot\frac{1}{\Delta_0^5} + \cdots \\ &= \sum_{i=0}^{\infty}\frac{(2i)!}{(i!)^2}\cdot\left(\frac{1}{2}rr'\Psi\right)^i\cdot\frac{1}{\Delta_0^{2i+1}}, \qquad (6.55)\end{aligned}$$

where

$$\frac{1}{\Delta_0} = \left[r^2 + r'^2 - 2rr'\cos(\theta - \theta')\right]^{-1/2}. \qquad (6.56)$$

Let

$$\rho_0 = \left[a^2 + a'^2 - 2aa'\cos(\theta - \theta')\right]^{1/2}. \qquad (6.57)$$

Using a Taylor series expansion in ρ_0, we can write

$$\frac{1}{\Delta_0^{2i+1}} = \frac{1}{\rho_0^{2i+1}} + (r-a)\frac{\partial}{\partial a}\left(\frac{1}{\rho_0^{2i+1}}\right) + (r'-a')\frac{\partial}{\partial a'}\left(\frac{1}{\rho_0^{2i+1}}\right) + \cdots. \tag{6.58}$$

Let $D_{m,n}$ denote the differential operator

$$D_{m,n} = a^m a'^n \frac{\partial^{m+n}}{\partial a^m \partial a'^n}, \tag{6.59}$$

and let

$$\varepsilon = \frac{r}{a} - 1, \qquad \varepsilon' = \frac{r'}{a'} - 1. \tag{6.60}$$

From the expansion of r/a given in Eq. (2.81), it is clear that ε is of $\mathcal{O}(e)$ and ε' is of $\mathcal{O}(e')$. Hence we have

$$\frac{1}{\Delta_0^{2i+1}} = \Big[1 + \varepsilon D_{1,0} + \varepsilon' D_{0,1} + \frac{1}{2!}\left(\varepsilon^2 D_{2,0} + 2\varepsilon\varepsilon' D_{1,1} + \varepsilon'^2 D_{0,2}\right) + \cdots\Big]\frac{1}{\rho_0^{2i+1}}. \tag{6.61}$$

However, from Eq. (6.57),

$$\begin{aligned}\frac{1}{\rho_0^{2i+1}} &= \left[a^2 + a'^2 - 2aa'\cos(\theta-\theta')\right]^{-\left(i+\frac{1}{2}\right)} \\ &= a'^{-(2i+1)}\left[1 + \alpha^2 - 2\alpha\cos(\theta-\theta')\right]^{-\left(i+\frac{1}{2}\right)} \\ &= a'^{-(2i+1)}\frac{1}{2}\sum_{j=-\infty}^{\infty} b_{i+\frac{1}{2}}^{(j)}(\alpha)\cos j(\theta-\theta'),\end{aligned} \tag{6.62}$$

where the $b_s^{(j)}(\alpha)$ are *Laplace coefficients*, each of which can be expressed as a uniformly convergent series in α for all $\alpha < 1$. Since the $D_{m,n}$ operators act only on the Laplace coefficients, we can define functions $A_{i,j,m,n}$ by

$$A_{i,j,m,n} = D_{m,n}\left(a'^{-(2i+1)} b_{i+\frac{1}{2}}^{(j)}(\alpha)\right) = a^m a'^n \frac{\partial^{m+n}}{\partial a^m \partial a'^n}\left(a'^{-(2i+1)} b_{i+\frac{1}{2}}^{(j)}(\alpha)\right), \tag{6.63}$$

and we can now write

$$\frac{1}{\Delta_0^{2i+1}} = \frac{1}{2}\sum_{j=-\infty}^{\infty}\left[A_{i,j,0,0} + \varepsilon A_{i,j,1,0} + \varepsilon' A_{i,j,0,1} + \cdots\right]\cos j(\theta-\theta'). \tag{6.64}$$

If we generalise this expression we obtain

$$\frac{1}{\Delta_0^{2i+1}} = \frac{1}{2}\sum_{j=-\infty}^{\infty}\left[\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{k=0}^{l}\binom{l}{k}\varepsilon^k \varepsilon'^{l-k} A_{i,j,k,l-k}\right]\cos j(\theta-\theta'). \tag{6.65}$$

Care must be taken in the calculation of the partial derivatives with respect to a and a' in the $A_{i,j,k,l-k}$ since a and a' are still contained implicitly within the Laplace coefficients $b^{(j)}_{i+\frac{1}{2}}(a/a')$.

Substituting Eq. (6.65) in Eq. (6.55) we obtain

$$\mathcal{R}_{\mathrm{D}} = \sum_{i=0}^{\infty} \frac{(2i)!}{(i!)^2} \left(\frac{1}{2}\frac{r}{a}\frac{r'}{a'}\Psi\right)^i \frac{a^i a'^{i+1}}{2} \times \sum_{j=-\infty}^{\infty} \left[\sum_{l=0}^{\infty} \frac{1}{l!} \sum_{k=0}^{l} \binom{l}{k} \varepsilon^k \varepsilon'^{l-k} A_{i,j,k,l-k}\right] \cos j(\theta - \theta'). \tag{6.66}$$

It is worthwhile noting that the inclinations, I and I', are only contained in Ψ and the eccentricities are only contained in the ε and ε' terms in Eq. (6.66).

The expansion of the indirect parts, $\mathcal{R}_{\mathrm{E}}$ and $\mathcal{R}_{\mathrm{I}}$, is more straightforward using the series obtained from expanding $\cos\psi$ in Eq. (6.50) and the series given in Sect. 2.5. Note that the expansion of these terms does not involve Laplace coefficients.

The literal expansion makes use of Laplace coefficients, which are explicit functions of α rather than the individual coefficients of powers of α that we encountered in Kaula's expansion. The Laplace coefficient $b^{(j)}_{i+\frac{1}{2}}(\alpha)$ in Eq. (6.62) is defined by

$$\frac{1}{2}b_s^{(j)}(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos j\psi \, \mathrm{d}\psi}{(1 - 2\alpha\cos\psi + \alpha^2)^s}, \tag{6.67}$$

where $s = i + 1/2$ is a half-integer (i.e., $s = 1/2, 3/2, 5/2, \ldots$) and $\alpha = a/a'$. Alternatively we can write this in series form as

$$\frac{1}{2}b_s^{(j)}(\alpha) = \frac{s(s+1)\ldots(s+j-1)}{1\cdot 2\cdot 3\ldots j}\alpha^j \times \left[1 + \frac{s(s+j)}{1(j+1)}\alpha^2 + \frac{s(s+1)(s+j)(s+j+1)}{1\cdot 2(j+1)(j+2)}\alpha^4 + \ldots\right]. \tag{6.68}$$

In the case where $j = 0$ the factor outside the brackets is equal to unity. It can be shown that the series definition of the Laplace coefficient is always convergent for $\alpha < 1$.

Useful relations between Laplace coefficients and their derivatives are given in Brouwer & Clemence (1961). These include

$$b_s^{(-j)} = b_s^{(j)}, \tag{6.69}$$

$$Db_s^{(j)} = s\left(b_{s+1}^{(j-1)} - 2\alpha b_{s+1}^{(j)} + b_{s+1}^{(j+1)}\right), \tag{6.70}$$

$$D^n b_s^{(j)} = s\Big(D^{n-1}b_{s+1}^{(j-1)} - 2\alpha D^{n-1}b_{s+1}^{(j)} + D^{n-1}b_{s+1}^{(j+1)} - 2(n-1)D^{n-2}b_{s+1}^{(j)}\Big), \tag{6.71}$$

and

$$\alpha^n \left(D^n b_s^{(j)} - D^n b_s^{(j-2)}\right) =$$
$$- (j+n-1)\alpha^{n-1} D^{n-1} b_s^{(j)} - (j-n-1)\alpha^{n-1} D^{n-1} b_s^{(j-2)}$$
$$+ 2(j-1)\left[\alpha^n D^{n-1} b_s^{(j-1)} + (n-1)\alpha^{n-1} D^{n-2} b_s^{(j-1)}\right], \qquad (6.72)$$

where $n \geq 2$ in the last two relations and $D \equiv \mathrm{d}/\mathrm{d}\alpha$ is a differential operator.

6.5 Literal Expansion to Second Order

As an illustration of the techniques outlined in Sect. 6.4, we will now derive an expansion of the disturbing function complete to second order in the eccentricities and inclinations.

To derive a series expansion for $\cos\psi$ we first need to make use of the expansions for $\sin f$ and $\cos f$ in terms of the mean anomaly M, given in Eqs. (2.84) and (2.85) respectively. To second order we have

$$\sin f = \sin M + e \sin 2M + e^2 \left(\frac{9}{8}\sin 3M - \frac{7}{8}\sin M\right), \qquad (6.73)$$

$$\cos f = \cos M + e\left(\cos 2M - 1\right) + e^2 \left(\frac{9}{8}\cos 3M - \frac{9}{8}\cos M\right). \qquad (6.74)$$

Hence

$$\cos[\omega + f] = \cos\omega\cos f - \sin\omega\sin f$$
$$\approx \cos[\omega + M] + e\left(\cos[\omega + 2M] - \cos\omega\right)$$
$$+ e^2 \left(-\cos[\omega + M] - \frac{1}{8}\cos[\omega - M] + \frac{9}{8}\cos[\omega + 3M]\right) \qquad (6.75)$$

and

$$\sin[\omega + f] = \sin\omega\cos f + \cos\omega\sin f$$
$$\approx \sin[\omega + M] + e\left(\sin[\omega + 2M] - \sin\omega\right)$$
$$+ e^2 \left(-\sin[\omega + M] + \frac{1}{8}\sin[\omega - M] + \frac{9}{8}\sin[\omega + 3M]\right). \qquad (6.76)$$

In keeping with a number of previous expansions (including that by Kaula discussed in Sect. 6.3) we wish to express the disturbing function in terms of powers of $\sin\frac{1}{2}I$ and $\sin\frac{1}{2}I'$ rather than sines and cosines of the inclinations. Therefore we make use of the relations

$$\cos I = 1 - 2\sin^2\frac{1}{2}I = 1 - 2s^2 \qquad (6.77)$$

and

$$\sin I = 2\sin\frac{1}{2}I\left(1 - \sin^2\frac{1}{2}I\right)^{\frac{1}{2}} = 2s + \mathcal{O}(s^3), \qquad (6.78)$$

where $s = \sin \frac{1}{2} I$. Substitution of these expressions and our expansions of $\cos[\omega + f]$ and $\sin[\omega + f]$ in Eqs. (6.51)–(6.53) gives

$$\begin{aligned}\frac{x}{r} \approx\ & \cos[\omega + \Omega + M] + e\left(\cos[\omega + \Omega + 2M] - \cos[\omega + \Omega]\right) \\ & + e^2\left(\frac{9}{8}\cos[\omega + \Omega + 3M] - \frac{1}{8}\cos[\omega + \Omega - M] - \cos[\omega + \Omega + M]\right) \\ & + s^2\left(\cos[\omega - \Omega + M] - \cos[\omega + \Omega + M]\right),\end{aligned} \tag{6.79}$$

$$\begin{aligned}\frac{y}{r} \approx\ & \sin[\omega + \Omega + M] + e\left(\sin[\omega + \Omega + 2M] - \sin[\omega + \Omega]\right) \\ & + e^2\left(\frac{9}{8}\sin[\omega + \Omega + 3M] - \frac{1}{8}\sin[\omega + \Omega - M] - \sin[\omega + \Omega + M]\right) \\ & - s^2\left(\sin[\omega - \Omega + M] + \sin[\omega + \Omega + M]\right),\end{aligned} \tag{6.80}$$

and

$$\frac{z}{r} \approx 2s\sin[\omega + M] + 2es\left(\sin[\omega + 2M] - \sin\omega\right). \tag{6.81}$$

Similar expressions can be obtained for x'/r', y'/r', and z'/r' by replacing unprimed quantities by primed quantities in the above equations. Hence we can derive an expression for $\cos\psi$ using Eq. (6.50). At the same time we can use the relations $M = \lambda - \varpi$ and $\omega = \varpi - \Omega$ to express the expansion in terms of longitudes. We get

$$\begin{aligned}\cos\psi \approx\ & \\ & (1 - e^2 - e'^2 - s^2 - s'^2)\cos[\lambda - \lambda'] + ee'\cos[2\lambda - 2\lambda' - \varpi + \varpi'] \\ & + ee'\cos[\varpi - \varpi'] + 2ss'\cos[\lambda - \lambda' - \Omega + \Omega'] \\ & + e\cos[2\lambda - \lambda' - \varpi] - e\cos[\lambda' - \varpi] \\ & + e'\cos[\lambda - 2\lambda' + \varpi'] - e'\cos[\lambda - \varpi'] \\ & + \frac{9}{8}e^2\cos[3\lambda - \lambda' - 2\varpi] - \frac{1}{8}e^2\cos[\lambda + \lambda' - 2\varpi] \\ & + \frac{9}{8}e'^2\cos[\lambda - 3\lambda' + 2\varpi'] - \frac{1}{8}e'^2\cos[\lambda + \lambda' - 2\varpi'] \\ & - ee'\cos[2\lambda - \varpi - \varpi'] - ee'\cos[2\lambda' - \varpi - \varpi'] \\ & + s^2\cos[\lambda + \lambda' - 2\Omega] + s'^2\cos[\lambda + \lambda' - 2\Omega'] \\ & - 2ss'\cos[\lambda + \lambda' - \Omega - \Omega'].\end{aligned} \tag{6.82}$$

Since $\theta = \omega + \Omega + f$ we have

$$\begin{aligned}\cos[\theta - \theta'] = & \left(\cos\Omega\cos[\omega + f] - \sin\Omega\sin[\omega + f]\right) \\ & \times \left(\cos\Omega'\cos[\omega' + f'] - \sin\Omega'\sin[\omega' + f']\right) \\ & + \left(\sin\Omega\cos[\omega + f] + \cos\Omega\sin[\omega + f]\right) \\ & \times \left(\sin\Omega'\cos[\omega' + f'] + \cos\Omega'\sin[\omega' + f']\right).\end{aligned} \tag{6.83}$$

By comparing this with Eqs. (6.51)–(6.53) we see that the expansion for $\cos[\theta - \theta']$ can be obtained from the expansion for $\cos\psi$ by setting $I = I' = 0$. Since $\Psi = \cos\psi - \cos[\theta - \theta']$, the expansion of $\cos\psi$ shows that Ψ is the inclination-dependent part of $\cos\psi$ and

$$\begin{aligned}\Psi = {} & s^2 \left(\cos[\lambda + \lambda' - 2\Omega] - \cos[\lambda - \lambda']\right) \\ & + 2ss' \left(\cos[\lambda - \lambda' - \Omega + \Omega'] - \cos[\lambda + \lambda' - \Omega - \Omega']\right) \\ & + s'^2 \left(\cos[\lambda + \lambda' - 2\Omega'] - \cos[\lambda - \lambda']\right). \end{aligned} \tag{6.84}$$

Note that Ψ is of second order in the inclinations.

Since $r/a = 1 + \mathcal{O}(e)$ and $r'/a' = 1 + \mathcal{O}(e')$, it is clear that, to second order in the eccentricities and inclinations, we can write

$$\begin{aligned}\left(\frac{1}{2}\frac{r}{a}\frac{r'}{a'}\Psi\right) = {} & \frac{1}{2}s^2 \left(\cos[\lambda + \lambda' - 2\Omega] - \cos[\lambda - \lambda']\right) \\ & + ss' \left(\cos[\lambda - \lambda' - \Omega + \Omega'] - \cos[\lambda + \lambda' - \Omega - \Omega']\right) \\ & + \frac{1}{2}s'^2 \left(\cos[\lambda + \lambda' - 2\Omega'] - \cos[\lambda - \lambda']\right), \end{aligned} \tag{6.85}$$

which is independent of e to this order. Since we are only interested in a second-order expansion, and since Ψ is already of second order, we can ignore second and higher powers of Ψ.

We have now obtained the first of the two major terms required for the series for $\mathcal{R}_{\mathrm{D}}$ (see Eq. (6.66)). We need to derive an expression for $\cos j[\theta - \theta']$, where j is an arbitrary integer. We start by noting that

$$\begin{aligned}\cos j[\theta - \theta'] = {} & \cos j[\omega + \Omega + f] \cos j[\omega' + \Omega' + f'] \\ & + \sin j[\omega + \Omega + f] \sin j[\omega' + \Omega' + f']. \end{aligned} \tag{6.86}$$

From Eq. (2.88) we have

$$f = M + 2e \sin M + \frac{5}{4}e^2 \sin 2M + \mathcal{O}(e^3). \tag{6.87}$$

If we substitute this expression in $\cos j[\omega+\Omega+f]$ and $\sin j[\omega+\Omega+f]$, transform to longitudes as before, and carry out a Taylor series expansion we obtain

$$\begin{aligned}\cos j\theta \approx {} & (1 - j^2 e^2) \cos[j\lambda] \\ & + \left(\frac{1}{2}j^2 e^2 - \frac{5}{8}je^2\right) \cos[(2 - j)\lambda - 2\varpi] \\ & + \left(\frac{1}{2}j^2 e^2 + \frac{5}{8}je^2\right) \cos[(2 + j)\lambda - 2\varpi] \\ & - je \cos[(1 - j)\lambda - \varpi] + je \cos[(1 + j)\lambda - \varpi] \end{aligned} \tag{6.88}$$

and

$$\sin j\theta \approx (1 - j^2e^2)\sin[j\lambda] + \left(\frac{5}{8}je^2 - \frac{1}{2}j^2e^2\right)\sin[(2-j)\lambda - 2\varpi] + \left(\frac{5}{8}je^2 + \frac{1}{2}j^2e^2\right)\sin[(2+j)\lambda - 2\varpi] + je\sin[(1-j)\lambda - \varpi] + je\sin[(1+j)\lambda - \varpi]. \tag{6.89}$$

By substituting unprimed quantities for primed ones we can easily obtain similar expressions for $\cos j\theta'$ and $\sin j\theta'$. The resulting expression for $\cos j[\theta - \theta']$ is

$$\begin{aligned}\cos j[\theta - \theta'] \approx\ & (1 - j^2e^2 - j^2e'^2)\cos[j(\lambda - \lambda')] \\ & + \left(\frac{5}{8}je^2 + \frac{1}{2}j^2e^2\right)\cos[(2+j)\lambda - j\lambda' - 2\varpi] \\ & + \left(\frac{1}{2}j^2e^2 - \frac{5}{8}je^2\right)\cos[(2-j)\lambda + j\lambda' - 2\varpi] \\ & + je\cos[(1+j)\lambda - j\lambda' - \varpi] - je\cos[(1-j)\lambda + j\lambda' - \varpi] \\ & + \left(\frac{1}{2}j^2e'^2 - \frac{5}{8}je'^2\right)\cos[j\lambda + (2-j)\lambda' - 2\varpi'] \\ & + \left(\frac{5}{8}je'^2 + \frac{1}{2}j^2e'^2\right)\cos[j\lambda - (2+j)\lambda' + 2\varpi'] \\ & - je'\cos[j\lambda + (1-j)\lambda' - \varpi'] + je'\cos[j\lambda - (1+j)\lambda' + \varpi'] \\ & - j^2ee'\cos[(1+j)\lambda + (1-j)\lambda' - \varpi - \varpi'] \\ & - j^2ee'\cos[(1-j)\lambda + (1+j)\lambda' - \varpi - \varpi'] \\ & + j^2ee'\cos[(1+j)\lambda - (1+j)\lambda' - \varpi + \varpi'] \\ & + j^2ee'\cos[(1-j)\lambda - (1-j)\lambda' - \varpi + \varpi'].\end{aligned} \tag{6.90}$$

Although the summation over j in Eq. (6.66) is over all values, in practice we do not need to carry out this summation (see Sect. 6.9 for an example).

From Eqs. (6.60) and (2.81) we have

$$\begin{aligned}\varepsilon = \frac{r}{a} - 1 &\approx -e\cos M + \frac{1}{2}e^2(1 - \cos 2M) \\ &= -e\cos[\lambda - \varpi] + \frac{1}{2}e^2\left(1 - \cos[2\lambda - 2\varpi]\right)\end{aligned} \tag{6.91}$$

and hence

$$\varepsilon^2 \approx \frac{1}{2}e^2 + \frac{1}{2}e^2\cos 2M = \frac{1}{2}e^2 + \frac{1}{2}e^2\cos[2\lambda - 2\varpi], \tag{6.92}$$

with similar expressions for ε' and ε'^2. No powers beyond the second are necessary for this expansion since ε is of $\mathcal{O}(e)$.

Finally, before carrying out the summation in Eq. (6.66), we need to calculate the derivatives of the Laplace coefficients given by the $A_{i,j,m,n}$ function. This task can be simplified by noting that, for a given value of i in the summation, we need to calculate

$$a^i a'^{i+1} A_{i,j,m,n} = a^{i+m} a'^{i+n+1} \frac{\partial^{m+n}}{\partial a^m \partial a'^m}\left(a'^{-(2i+1)} b^{(j)}_{i+\frac{1}{2}}(a/a')\right). \tag{6.93}$$

The result of the differentiation is to leave a function of a/a' alone. In our case the required values of $A_{i,j,m,n}$ are:

$$a^i a'^{i+1} A_{i,j,0,0} = \alpha^i b^{(j)}_{i+\frac{1}{2}}(\alpha), \tag{6.94}$$

$$a^i a'^{i+1} A_{i,j,1,0} = \alpha^{i+1} D b^{(j)}_{i+\frac{1}{2}}(\alpha), \tag{6.95}$$

$$a^i a'^{i+1} A_{i,j,0,1} = -\alpha^{i+1} D b^{(j)}_{i+\frac{1}{2}}(\alpha) - (2i+1)\alpha^i b^{(j)}_{i+\frac{1}{2}}(\alpha), \tag{6.96}$$

$$a^i a'^{i+1} A_{i,j,2,0} = \alpha^{i+2} D^2 b^{(j)}_{i+\frac{1}{2}}(\alpha), \tag{6.97}$$

$$a^i a'^{i+1} A_{i,j,1,1} = -\alpha^{i+2} D^2 b^{(j)}_{i+\frac{1}{2}}(\alpha) - 2\alpha^{i+1}(i+1) D b^{(j)}_{i+\frac{1}{2}}(\alpha), \tag{6.98}$$

and

$$\begin{aligned} a^i a'^{i+1} A_{i,j,2,2} = {} & \alpha^{i+2} D^2 b^{(j)}_{i+\frac{1}{2}}(\alpha) + 4\alpha^{i+1}(i+1) D b^{(j)}_{i+\frac{1}{2}}(\alpha) \\ & + 2\alpha^i (2i^2 + 3i + 2) b^{(j)}_{i+\frac{1}{2}}(\alpha), \end{aligned} \tag{6.99}$$

where i will take the values 0 and 1; higher values can be ignored because of the presence of the Ψ^i term in Eq. (6.66).

We are now in a position to carry out the summation over i. To second order in the eccentricity and inclination we have

$$\begin{aligned} \mathcal{R}_\mathrm{D} = \Bigg(& \frac{1}{2}\big[a' A_{0,j,0,0} + \varepsilon a' A_{0,j,1,0} + \varepsilon' a' A_{0,j,0,1} \\ & + \varepsilon^2 a' A_{0,j,2,0} + \varepsilon\varepsilon' a' A_{0,j,1,1} + \varepsilon'^2 a' A_{0,j,0,2}\big] \\ & + \left(\frac{1}{2}\frac{r}{a}\frac{r'}{a'}\Psi\right) a a'^2 A_{1,j,0,0}\Bigg) \cos j[\theta - \theta']. \end{aligned} \tag{6.100}$$

Using the series that we have already derived for the quantities in this equation, we get an expansion with twenty-three cosine arguments. These can be categorised by the *order* of the argument, which is simply the sum of the coefficients of λ and λ'. If we write the second-order expansion as

$$\mathcal{R}_\mathrm{D} = \mathcal{R}^{(0)}_\mathrm{D} + \mathcal{R}^{(1)}_\mathrm{D} + \mathcal{R}^{(2)}_\mathrm{D}, \tag{6.101}$$

where $\mathcal{R}^{(i)}$ denotes the part of the expansion containing the arguments of order

i, then

$$
\begin{aligned}
\mathcal{R}_{\mathrm{D}}^{(0)} = & \left(\frac{1}{2}b_{\frac{1}{2}}^{(j)} + \frac{1}{8}(e^2+e'^2)\left[-4j^2+2\alpha D+\alpha^2D^2\right]b_{\frac{1}{2}}^{(j)}\right)\cos[j\lambda - j\lambda'] \\
& + \left(\frac{1}{8}ee'\left[2j+4j^2-2\alpha D-\alpha^2D^2\right]b_{\frac{1}{2}}^{(j)}\right) \\
& \quad \times \cos[(1+j)\lambda-(1+j)\lambda'-\varpi+\varpi'] \\
& + \left(\frac{1}{8}ee'\left[-2j+4j^2-2\alpha D-\alpha^2D^2\right]b_{\frac{1}{2}}^{(j)}\right) \\
& \quad \times \cos[(1-j)\lambda-(1-j)\lambda'-\varpi+\varpi'] \\
& + \left(\frac{1}{4}(s^2+s'^2)[-\alpha]b_{\frac{3}{2}}^{(j)}\right)\cos[(1+j)\lambda-(1+j)\lambda'] \\
& + \left(\frac{1}{4}(s^2+s'^2)[-\alpha]b_{\frac{3}{2}}^{(j)}\right)\cos[(1-j)\lambda-(1-j)\lambda'] \\
& + \left(\frac{1}{2}ss'[\alpha]b_{\frac{3}{2}}^{(j)}\right)\cos[(1+j)\lambda-(1+j)\lambda'-\Omega+\Omega'] \\
& + \left(\frac{1}{2}ss'[\alpha]b_{\frac{3}{2}}^{(j)}\right)\cos[(1-j)\lambda-(1-j)\lambda'-\Omega+\Omega'],
\end{aligned} \tag{6.102}
$$

$$
\begin{aligned}
\mathcal{R}_{\mathrm{D}}^{(1)} = & \left(\frac{1}{4}e[2j-\alpha D]b_{\frac{1}{2}}^{(j)}\right)\cos[(1+j)\lambda-j\lambda'-\varpi] \\
& + \left(\frac{1}{4}e[-2j-\alpha D]b_{\frac{1}{2}}^{(j)}\right)\cos[(1-j)\lambda+j\lambda'-\varpi] \\
& + \left(\frac{1}{4}e'[1+2j+\alpha D]b_{\frac{1}{2}}^{(j)}\right)\cos[j\lambda-(1+j)\lambda'+\varpi'] \\
& + \left(\frac{1}{4}e'[1-2j+\alpha D]b_{\frac{1}{2}}^{(j)}\right)\cos[j\lambda+(1-j)\lambda'-\varpi'],
\end{aligned} \tag{6.103}
$$

$$
\begin{aligned}
\mathcal{R}_{\mathrm{D}}^{(2)} = & \left(\frac{1}{16}e^2\left[5j+4j^2-2\alpha D-4j\alpha D+\alpha^2D^2\right]b_{\frac{1}{2}}^{(j)}\right) \\
& \quad \times \cos[(2+j)\lambda-j\lambda-2\varpi] \\
& + \left(\frac{1}{16}e^2\left[-5j+4j^2-2\alpha D+4j\alpha D+\alpha^2D^2\right]b_{\frac{1}{2}}^{(j)}\right) \\
& \quad \times \cos[(2-j)\lambda+j\lambda-2\varpi] \\
& + \left(\frac{1}{8}ee'\left[2j-4j^2-2\alpha D+2j\alpha D-\alpha^2D^2\right]b_{\frac{1}{2}}^{(j)}\right) \\
& \quad \times \cos[(1+j)\lambda+(1-j)\lambda'-\varpi-\varpi'] \\
& + \left(\frac{1}{8}ee'\left[-2j-4j^2-2\alpha D-2j\alpha D-\alpha^2D^2\right]b_{\frac{1}{2}}^{(j)}\right) \\
& \quad \times \cos[(1-j)\lambda+(1+j)\lambda'-\varpi-\varpi'] \\
& + \left(\frac{1}{16}e'^2\left[4+9j+4j^2+6\alpha D+4j\alpha D+\alpha^2D^2\right]b_{\frac{1}{2}}^{(j)}\right) \\
& \quad \times \cos[j\lambda-(2+j)\lambda'+2\varpi']
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{1}{16}e'^2\left[4-9j+4j^2+6\alpha D-4j\alpha D+\alpha^2D^2\right]b_{\frac{1}{2}}^{(j)}\right)\\
&\quad\times\cos[j\lambda+(2-j)\lambda'-2\varpi']\\
&+\left(\frac{1}{4}s^2\,[\alpha]\,b_{\frac{3}{2}}^{(j)}\right)\cos[(1-j)\lambda+(1+j)\lambda'-2\Omega]\\
&+\left(\frac{1}{4}s^2\,[\alpha]\,b_{\frac{3}{2}}^{(j)}\right)\cos[(1+j)\lambda+(1-j)\lambda'-2\Omega]\\
&+\left(\frac{1}{2}ss'\,[-\alpha]\,b_{\frac{3}{2}}^{(j)}\right)\cos[(1-j)\lambda+(1+j)\lambda'-\Omega-\Omega']\\
&+\left(\frac{1}{2}ss'\,[-\alpha]\,b_{\frac{3}{2}}^{(j)}\right)\cos[(1+j)\lambda+(1-j)\lambda'-\Omega-\Omega']\\
&+\left(\frac{1}{4}s'^2\,[\alpha]\,b_{\frac{3}{2}}^{(j)}\right)\cos[(1-j)\lambda+(1+j)\lambda'-2\Omega']\\
&+\left(\frac{1}{4}s'^2\,[\alpha]\,b_{\frac{3}{2}}^{(j)}\right)\cos[(1+j)\lambda+(1-j)\lambda'-2\Omega'].
\end{aligned}
\tag{6.104}
$$

The arguments in this expansion are not all unique and further simplification is possible. This is clear from an inspection of the different terms since, apart from the first term in $\mathcal{R}_{\mathrm{D}}^{(0)}$, they occur in pairs with similar form. Because the summation in j in Eq. (6.66) is over all values, we can always carry out a transformation of j of the form $j \to \pm j + k$ where k is an integer, provided that we apply it to the argument and its associated term. Also, since only cosines appear in the expansion we can always change the sign of the argument. We can then use these procedures to reduce the arguments in the expansion to some arbitrary standard form. In our case we have decided to make j the coefficient of λ' in each argument.

As an example, consider the two terms in ee' in $\mathcal{R}_{\mathrm{D}}^{(0)}$. These can be transformed to the same cosine argument,

$$
j\lambda' - j\lambda + \varpi' - \varpi, \tag{6.105}
$$

by changing j to $-j$ in the first, and then applying the transformation $j \to j+1$ in each. The resulting term associated with this argument is then

$$
\frac{1}{4}ee'\left[2+6j+4j^2-2\alpha D-\alpha^2D^2\right]b_{\frac{1}{2}}^{(j+1)}. \tag{6.106}
$$

Similar procedures can be carried out for the other arguments and the total number of arguments can be reduced from twenty-three to eleven. In such transformations we have made use of the fact that $b_s^{(-j)} = b_s^{(j)}$, as given in Eq. (6.69). We also point out that, even with our decision to express all the arguments in a form where the coefficient of λ' is j, the final form of our expansion is not unique and transformations of the form $j \to -j$ followed by reversal of the argument produce arguments of a different form.

The final form of our second-order expansion of the direct part is

$$
\begin{aligned}
\mathcal{R}_{\mathrm{D}} = {} & \left(\frac{1}{2}b_{\frac{1}{2}}^{(j)} + \frac{1}{8}\left(e^2 + e'^2\right)\left[-4j^2 + 2\alpha D + \alpha^2 D^2\right] b_{\frac{1}{2}}^{(j)} \right. \\
& \left. + \frac{1}{4}\left(s^2 + s'^2\right)\left([-\alpha]\, b_{\frac{3}{2}}^{(j-1)} + [-\alpha]\, b_{\frac{3}{2}}^{(j+1)}\right)\right) \\
& \times \cos[j\lambda' - j\lambda] \\
& + \left(\frac{1}{4}ee'\left[2 + 6j + 4j^2 - 2\alpha D - \alpha^2 D^2\right] b_{\frac{1}{2}}^{(j+1)}\right) \\
& \times \cos[j\lambda' - j\lambda + \varpi' - \varpi] \\
& + \left(ss'\,[\alpha]\, b_{\frac{3}{2}}^{(j+1)}\right)\cos[j\lambda' - j\lambda + \Omega' - \Omega] \\
& + \left(\frac{1}{2}e\,[-2j - \alpha D]\, b_{\frac{1}{2}}^{(j)}\right)\cos[j\lambda' + (1-j)\lambda - \varpi] \\
& + \left(\frac{1}{2}e'\,[-1 + 2j + \alpha D]\, b_{\frac{1}{2}}^{(j-1)}\right)\cos[j\lambda' + (1-j)\lambda - \varpi'] \\
& + \left(\frac{1}{8}e^2\left[-5j + 4j^2 - 2\alpha D + 4j\alpha D + \alpha^2 D^2\right] b_{\frac{1}{2}}^{(j)}\right) \\
& \times \cos[j\lambda' + (2-j)\lambda - 2\varpi] \\
& + \left(\frac{1}{4}ee'\left[-2 + 6j - 4j^2 + 2\alpha D - 4j\alpha D - \alpha^2 D^2\right] b_{\frac{1}{2}}^{(j-1)}\right) \\
& \times \cos[j\lambda' + (2-j)\lambda - \varpi' - \varpi] \\
& + \left(\frac{1}{8}e'^2\left[2 - 7j + 4j^2 - 2\alpha D + 4j\alpha D + \alpha^2 D^2\right] b_{\frac{1}{2}}^{(j-2)}\right) \\
& \times \cos[j\lambda' + (2-j)\lambda - 2\varpi'] \\
& + \left(\frac{1}{2}s^2\,[\alpha]\, b_{\frac{3}{2}}^{(j-1)}\right) \\
& \times \cos[j\lambda' + (2-j)\lambda - 2\Omega] \\
& + \left(ss'\,[-\alpha]\, b_{\frac{3}{2}}^{(j-1)}\right)\cos[j\lambda' + (2-j)\lambda - \Omega' - \Omega] \\
& + \left(\frac{1}{2}s'^2\,[\alpha]\, b_{\frac{3}{2}}^{(j-1)}\right)\cos[j\lambda' + (2-j)\lambda - 2\Omega'].
\end{aligned}
\tag{6.107}
$$

Generating the indirect parts of the disturbing function defined in Eqs. (6.47) and (6.48) is relatively simple since we have already derived an expression for $\cos\psi$. Expressions for r/a and $(a'/r')^2$ can be obtained from the elliptical expansions given in Sect. 2.5. To second order,

$$
\frac{r}{a} = 1 - e\cos[\lambda - \varpi] + \frac{1}{2}e^2(1 - \cos[2\lambda - 2\varpi]) \tag{6.108}
$$

and

$$
\left(\frac{a'}{r'}\right) = 1 + 2e'\cos[\lambda' - \varpi'] + \frac{1}{2}e'^2\left(1 + 5\cos[2\lambda' - 2\varpi']\right) \tag{6.109}
$$

and hence

$$
\begin{aligned}
\mathcal{R}_{\mathrm{E}} &= -\frac{r}{a}\left(\frac{a'}{r'}\right)^2 \cos\psi \\
&\approx \left(-1+\frac{1}{2}e^2+\frac{1}{2}e'^2+s^2+s'^2\right)\cos[\lambda'-\lambda] \\
&\quad - ee'\cos[2\lambda'-2\lambda-\varpi'+\varpi] - 2ss'\cos[\lambda'-\lambda-\Omega'+\Omega] \\
&\quad - \frac{1}{2}e\cos[\lambda'-2\lambda+\varpi] + \frac{3}{2}e\cos[\lambda'-\varpi] - 2e'\cos[2\lambda'-\lambda-\varpi'] \\
&\quad - \frac{3}{8}e^2\cos[\lambda'-3\lambda+2\varpi] - \frac{1}{8}e^2\cos[\lambda'+\lambda-2\varpi] \\
&\quad + 3ee'\cos[2\lambda-\varpi'-\varpi] - \frac{1}{8}e'^2\cos[\lambda'+\lambda-2\varpi'] \\
&\quad - \frac{27}{8}e'^2\cos[3\lambda'-\lambda-2\varpi'] - s^2\cos[\lambda'+\lambda-2\Omega] \\
&\quad + 2ss'\cos[\lambda'+\lambda-\Omega'-\Omega] - s'^2\cos[\lambda'+\lambda-2\Omega'],
\end{aligned}
\tag{6.110}
$$

where we have changed the sign of the argument in some cases in order to adopt the same convention used to derive $\mathcal{R}_{\mathrm{D}}$.

We can use similar methods to derive an expression for $\mathcal{R}_{\mathrm{I}}$. Alternatively we can reverse the primed and unprimed quantities in our expression for $\mathcal{R}_{\mathrm{E}}$. We obtain

$$
\begin{aligned}
\mathcal{R}_{\mathrm{I}} &= -\frac{r'}{a'}\left(\frac{a}{r}\right)^2 \cos\psi \\
&\approx \left(-1+\frac{1}{2}e^2+\frac{1}{2}e'^2+s^2+s'^2\right)\cos[\lambda'-\lambda] \\
&\quad - ee'\cos[2\lambda'-2\lambda-\varpi'+\varpi] - 2ss'\cos[\lambda'-\lambda-\Omega'+\Omega] \\
&\quad - 2e\cos[\lambda'-2\lambda+\varpi] + \frac{3}{2}e'\cos[\lambda-\varpi'] - \frac{1}{2}e'\cos[2\lambda'-\lambda-\varpi'] \\
&\quad - \frac{27}{8}e^2\cos[\lambda'-3\lambda+2\varpi] - \frac{1}{8}e^2\cos[\lambda'+\lambda-2\varpi] \\
&\quad + 3ee'\cos[2\lambda-\varpi'-\varpi] - \frac{1}{8}e'^2\cos[\lambda'+\lambda-2\varpi'] \\
&\quad - \frac{3}{8}e'^2\cos[3\lambda'-\lambda-2\varpi'] - s^2\cos[\lambda'+\lambda-2\Omega] \\
&\quad + 2ss'\cos[\lambda'+\lambda-\Omega'-\Omega] - s'^2\cos[\lambda'+\lambda-2\Omega'].
\end{aligned}
\tag{6.111}
$$

NO

6.6 Terms Associated with a Specific Argument

The method outlined in Sect. 6.5 permits the calculation of a full expansion of the planetary disturbing function to any specified order in the eccentricities and inclinations. Its major disadvantage is that to find the terms associated with a specific argument one has to carry out a complete expansion to the order of that argument. However, in Sect. 6.3 we showed that in this respect there are distinct advantages to using Kaula's form of the expansion. The major disadvantage of Kaula's formulae is the lack of Laplace coefficients.

Ellis & Murray (1999) derived a variation on Kaula's expansion that incorporates the best features of both approaches. Furthermore, they give explicit formulae for the *finite* series associated with a specific argument expanded to a specific order. Let the argument have the form

$$\varphi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega \tag{6.112}$$

and let $N_{\max}$ be the maximum order of the expansion. Ellis & Murray showed that the expression for $\mathcal{R}_{\mathrm{D}}$ associated with φ is

$$\begin{aligned}
\mathcal{R}_{\mathrm{D}} = & \sum_{i=0}^{i_{\max}} \frac{(2i)!}{i!}\frac{(-1)^i}{2^{2i+1}}\alpha^i \\
& \times \sum_{s=s_{\min}}^{i} \sum_{n=0}^{n_{\max}} \frac{(2s-4n+1)(s-n)!}{2^{2n}n!(2s-2n+1)!} \sum_{m=0}^{s-2n} \kappa_m \frac{(s-2n-m)!}{(s-2n+m)!} \\
& \times (-1)^{s-2n-m} F_{s-2n,\,m,\,p}(I)\, F_{s-2n,\,m,\,p'}(I') \sum_{l=0}^{i-s} \frac{(-1)^s 2^{2s}}{(i-s-l)!l!} \\
& \times \sum_{\ell=0}^{\ell_{\max}} \frac{(-1)^\ell}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \alpha^\ell \frac{\mathrm{d}^\ell}{\mathrm{d}\alpha^\ell} b^{(j)}_{i+\frac{1}{2}}(\alpha) \\
& \times X^{i+k,\,-j_2-j_4}_{-j_2}(e)\, X^{-(i+k+1),\,j_1+j_3}_{j_1}(e') \\
& \times \cos\left[j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega\right]
\end{aligned} \tag{6.113}$$

where, as before, $\kappa_0 = 1$ and $\kappa_m = 2$ for $m \neq 0$.

The following relationships hold throughout the calculation:

$$q = j_4, \tag{6.114}$$

$$q' = -j_3, \tag{6.115}$$

$$\ell_{\max} = N_{\max} - |j_5| - |j_6|, \tag{6.116}$$

$$p_{\min} = -(j_5 + j_6)/2, \quad p'_{\min} = 0 \quad \text{if} \quad j_5 + j_6 < 0, \tag{6.117}$$

$$p_{\min} = 0, \quad p'_{\min} = (j_5 + j_6)/2 \quad \text{if} \quad j_5 + j_6 \geq 0, \tag{6.118}$$

$$s_{\min} = \max\left(p_{\min},\ p'_{\min},\ j_6 + 2p_{\min},\ -j_5 + 2p'_{\min}\right), \tag{6.119}$$

$$i_{\max} = \left[\left(N_{\max} - |j_3| - |j_4|\right)/2\right], \tag{6.120}$$

where the square brackets in Eq. (6.120) denote the integer part of the expression. A number of intermediate definitions are required for the summation. These are:

$$n_{\max} = \left[\left(s - s_{\min}\right)/2\right], \tag{6.121}$$

$$m_{\min} = 0 \quad \text{if} \quad s,\, j_5 \quad \text{are both even or both odd}, \tag{6.122}$$

$$m_{\min} = 1 \quad \text{if} \quad s,\, j_5 \quad \text{are neither both even nor both odd}, \tag{6.123}$$

$$p = (-j_6 - m + s - 2n)/2 \quad \text{with } p \le s - 2n \text{ and } p \ge p_{\min}, \tag{6.124}$$
$$p' = (j_5 - m + s - 2n)/2 \quad \text{with } p' \le s - 2n \text{ and } p' \ge p'_{\min}, \tag{6.125}$$
$$j = \left| j_2 + i - 2\bar{a} - 2n - 2p + q \right| . \tag{6.126}$$

Note that q and q' are determined directly from φ and remain fixed over all the summations. However, p and p' change with s, n, and m but the relationships given in Eqs. (6.124) and (6.125) always hold.

Ellis & Murray (1999) showed that the summations involved in the definitions of the functions of eccentricity and inclination in Eq. (6.113) need only be evaluated to a finite order that is at most equal to $N_{\max}$. The Hansen coefficient in e need only include terms up to order $N_{\max} - |j_3| - |j_5| - |j_6|$ in e; similarly the Hansen coefficient in e' need only include terms up to order $N_{\max} - |j_4| - |j_5| - |j_6|$ in e'. The F inclination function in I need only include terms up to order $N_{\max} - |j_3| - |j_4| - |j_5|$ in I; similarly the F function in I' need only include terms up to order $N_{\max} - |j_3| - |j_4| - |j_6|$ in I'.

For the indirect parts we have

$$\mathcal{R}_{\mathrm{E}} = -\kappa_m \frac{(1-m)!}{(1+m)!} F_{1,m,p}(I) F_{1,m,p'}(I') X_{-j_2}^{1,\,-j_2-j_4}(e)\, X_{j_1}^{-2,\,j_1+j_3}(e')$$
$$\times \cos\left[j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega \right] \tag{6.127}$$

and

$$\mathcal{R}_{\mathrm{I}} = -\kappa_m \frac{(1-m)!}{(1+m)!} F_{1,m,p}(I) F_{1,m,p'}(I') X_{-j_2}^{-2,\,-j_2-j_4}(e)\, X_{j_1}^{1,\,j_1+j_3}(e')$$
$$\times \cos\left[j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega \right], \tag{6.128}$$

where each of the quantities p, p', and m must be integers and equal to 0 or 1. If these conditions are not satisfied then the given argument does not appear in the expansion of the indirect part. As with $\mathcal{R}_{\mathrm{D}}$ we can reduce the extent of the series expansions in powers of the eccentricity and inclination, and the same modifications apply. An analysis of the integers involved in the expansion of this indirect part gives the following relationships:

$$q = j_4, \tag{6.129}$$
$$q' = -j_3, \tag{6.130}$$
$$p = (j_2 + j_4 + 1)/2, \tag{6.131}$$
$$p' = -(j_1 + j_3 - 1)/2, \tag{6.132}$$
$$m = j_5 - 2p' + 1. \tag{6.133}$$

6.7 Use of the Disturbing Function

The complete expansions for $\mathcal{R}_{\mathrm{D}}$, $\mathcal{R}_{\mathrm{E}}$, and $\mathcal{R}_{\mathrm{I}}$ contain an infinite number of cosine arguments. However, in practice we are only interested in certain cosine arguments and we can neglect all others. Our basis for doing this is the *averaging* *principle* whereby we assume (with some justification) that all the unimportant terms will be of short period and therefore their effects will average out to zero over the longer-period motion. This concept is illustrated in Sect. 6.9. All that concerns us here is that the averaging principle allows us to isolate those terms in the disturbing function that are appropriate for a particular problem and to ignore the infinite number of remaining terms. Effectively we move from a consideration of the infinite series of the full disturbing function, $\mathcal{R}$, to a finite series of the averaged disturbing function, $\langle\mathcal{R}\rangle$. This concept is the basis of our analysis of secular perturbations in Chapter 7, resonant perturbations in Chapter 8, and their applications to chaotic motion in Chapter 9 and planetary rings in Chapter 10. This approach to the use of the planetary disturbing function permits us to carry out analytical studies when we move beyond the simplicity of the two-body problem.

The procedure for determining the appropriate term, $\langle\mathcal{R}\rangle$ or $\langle\mathcal{R}'\rangle$, in the disturbing function is as follows:

1) Decide which combination of angles, φ, is applicable to the problem at hand. This requires knowledge of the physical problem and will be discussed in Sect. 6.9.
2) Determine the "order", N, of the argument. This is equal to the absolute value of the sum of the coefficients of λ and λ' in φ.
3) By looking at the appropriate order terms in the expansion of $\mathcal{R}_{\mathrm{D}}$, determine the value of the integer j that gives agreement with the desired argument, φ.
4) Calculate the combination of Laplace coefficients for that value of j to give the explicit form of the term of interest, $\langle\mathcal{R}_{\mathrm{D}}\rangle$ say.
5) Decide whether an external or an internal perturbation is being considered. This is determined by the nature of the problem.
6) If the perturbation is external, then look at the appropriate order terms in the expansion of the indirect part, $\mathcal{R}_{\mathrm{E}}$, and isolate a matching argument, if it exists, and read off the corresponding indirect term $\langle\mathcal{R}_{\mathrm{E}}\rangle$.
7) If the perturbation is internal, then look at the appropriate order terms in the expansion of the indirect part, $\mathcal{R}_{\mathrm{I}}$, and isolate a matching argument, if it exists, and read off the corresponding indirect term $\langle\mathcal{R}_{\mathrm{I}}\rangle$.
8) If the perturbation is external then

$$\langle\mathcal{R}\rangle = \frac{\mu'}{a'}\left(\langle\mathcal{R}_{\mathrm{D}}\rangle + \alpha\langle\mathcal{R}_{\mathrm{E}}\rangle\right). \tag{6.134}$$

9) If the perturbation is internal then

$$\langle\mathcal{R}'\rangle = \frac{\mu}{a}\left(\alpha\langle\mathcal{R}_{\mathrm{D}}\rangle + \frac{1}{\alpha}\langle\mathcal{R}_{\mathrm{I}}\rangle\right). \tag{6.135}$$

Use of the explicit expansion of the indirect part in steps 6 and 7 can be avoided altogether since the averaged indirect part of the disturbing function, $\langle\mathcal{R}_{\mathrm{E}}\rangle$ or

$\langle\mathcal{R}_\mathrm{I}\rangle$, can be obtained from the averaged direct part, $\langle\mathcal{R}_\mathrm{D}\rangle$. The procedure is as follows: To obtain $\langle\mathcal{R}_\mathrm{E}\rangle$ replace every occurrence of $\alpha^n D^n A_1$ $(n = 0, 1)$ in $\langle\mathcal{R}_\mathrm{D}\rangle$ by -1 and replace every occurrence of $\alpha^n D^n B_0$ $(n = 0, 1)$ in $\langle\mathcal{R}_\mathrm{D}\rangle$ by -2; all other terms are ignored. To obtain $\langle\mathcal{R}_\mathrm{I}\rangle$ replace every occurrence of $\alpha^n D^n A_1$ $(n = 0, 1, 2, \ldots)$ in $\langle\mathcal{R}_\mathrm{D}\rangle$ by $(-1)^{n+1}(n + 1)!$ and replace every occurrence of $\alpha^n D^n B_0$ $(n = 0, 1, 2, \ldots)$ in $\langle\mathcal{R}_\mathrm{D}\rangle$ by $(-1)^{n+1}(2n + 2)n!$, ignoring all other terms.

Throughout this analysis we have assumed that $r' > r$ (i.e., that the orbits do not intersect). The convergence of the resulting series will therefore depend on how close the orbits are to intersection. Obviously if the orbits intersect there will be a singularity since $\mathbf{r} = \mathbf{r}'$ at some longitude and the first term in Eq. (6.16) or (6.18) becomes undefined. Hence an approximate condition for convergence is

$$a(1 + e) < a'(1 - e'), \tag{6.136}$$

or that the apocentric distance of the inner orbit has to be less than the pericentric distance of the outer orbit.

Another advantage of the Legendre-type expansion given in Eq. (6.113) is that it is easy to see the form of the lowest order terms in the expansion. We have already stated in Sect. 6.3 (and shown in Sect. 6.5) that the potential associated with the perturbations of the orbit of the mass m by the mass m' can be written as

$$\mathcal{R} = \mu' \sum S \cos\varphi, \tag{6.137}$$

where S is a function of the semi-major axes, eccentricities, and inclinations of m and m'. From the definitions of mean longitude and longitude of pericentre, the general form of the argument φ can be written as

$$\begin{aligned} \varphi = (l - 2p' + q')\lambda' - (l - 2p + q)\lambda - q'\varpi' + q\varpi \\ + (m - l + 2p')\Omega' - (m - l + 2p)\Omega, \end{aligned} \tag{6.138}$$

where, in this case, l, m, p, p', q, and q' are all integers. We can calculate the valid arguments by using the property that the sum of the integer coefficients of the angle variables in each argument is zero. If we write the general form of an argument as

$$\varphi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega \tag{6.139}$$

then our condition on the coefficients implies that

$$\sum_{i=1}^{6} j_i = 0. \tag{6.140}$$

This is the *d'Alembert relation* and it does *not* apply to any choice of angles – we must use angles that are referred to a fixed direction (i.e., longitudes rather than anomalies). The longitudes λ, λ', ϖ, ϖ', Ω, and Ω' form an appropriate set of angles. Hamilton (1994) provides an overview of the d'Alembert rules that determine such relationships.

Now consider the form of S, the "strength" of an individual term. From the properties of $X_{l-2p+q}^{l,l-2p}(e)$ and $F_{lmp}(I)$ we can calculate the lowest order terms in the eccentricities and inclinations. Use of Eqs. (6.37)–(6.42) gives

$$X_{l-2p+q}^{l,l-2p}(e) = \mathcal{O}(e^{|q|}), \qquad X_{l-2p'+q'}^{-l-1,l-2p'}(e') = \mathcal{O}(e'^{|q'|}) \tag{6.141}$$

and

$$F_{lmp}(I) = \mathcal{O}(s^{|m-l+2p|}), \qquad F_{lmp'}(I') = \mathcal{O}(s'^{|m-l+2p'|}), \tag{6.142}$$

where $s = \sin\frac{1}{2}I$ and $s' = \sin\frac{1}{2}I'$. Therefore we can write

$$S \approx \frac{f(\alpha)}{a'} e^{|q|} e'^{|q'|} s^{|m-l+2p|} s'^{|m-l+2p'|} = \frac{f(\alpha)}{a'} e^{|j_4|} e'^{|j_3'|} s^{|j_6|} s'^{|j_5|}, \tag{6.143}$$

where $f(\alpha)$ can be expressed as a function of Laplace coefficients. Hence the lowest power of e, for example, in a given term is at least equal to the absolute value of the coefficient of ϖ. Similarly the lowest powers of e', $\sin\frac{1}{2}I$, and $\sin\frac{1}{2}I'$ are greater than or equal to the absolute value of the coefficients of ϖ', Ω, and Ω' respectively in φ. This property is clear from the second-order expansion given in Sect. 6.5 and the fourth-order expansion in Appendix B.

6.8 Lagrange's Planetary Equations

The expansion of the disturbing function gives us the dependence of the perturbing potential on the orbital elements. Now we need to quantify the resulting orbital variations of the perturbed body. To do this we make use *Lagrange's planetary equations.* These are best derived using a Hamiltonian formulation (see Sect. 2.10). Here we confine ourselves to a statement of the equations. Full derivations can be found in Brouwer & Clemence (1961) and Roy (1988).

The use of Lagrange's equations require the introduction of an additional angle. If we write

$$\lambda = M + \varpi = n(t - \tau) + \varpi = nt + \epsilon, \tag{6.144}$$

where λ is the mean longitude, M is the mean anomaly, ϖ is the longitude of pericentre, t is time, and τ is the time of pericentre passage, then the new angle ϵ denotes the *mean longitude at epoch* (i.e., the mean longitude of the mass m at the moment from which time is measured). Lagrange's equations for the variations of the orbital elements are

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial \mathcal{R}}{\partial \epsilon}, \tag{6.145}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{\sqrt{1-e^2}}{na^2e}(1-\sqrt{1-e^2})\frac{\partial \mathcal{R}}{\partial \epsilon} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial \mathcal{R}}{\partial \varpi}, \tag{6.146}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{na^2e}\frac{\partial \mathcal{R}}{\partial e} + \frac{\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\frac{\partial \mathcal{R}}{\partial I}, \tag{6.147}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial \mathcal{R}}{\partial I}, \tag{6.148}$$

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathcal{R}}{\partial e} + \frac{\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\frac{\partial \mathcal{R}}{\partial I}, \tag{6.149}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{-\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\left(\frac{\partial \mathcal{R}}{\partial \epsilon} + \frac{\partial \mathcal{R}}{\partial \varpi}\right) - \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial \mathcal{R}}{\partial \Omega}. \tag{6.150}$$

A problem arises if we consider the expression for $\dot{\epsilon}$ given in Eq. (6.147) (see, e.g., Brouwer & Clemence 1961). Since the right-hand side of the equation contains a factor $\partial\mathcal{R}/\partial a$ we have to be aware that the semi-major axis occurs explicitly in the Laplace coefficients of the disturbing function and implicitly in the arguments of the cosine terms as the mean motion since $\lambda = nt + \epsilon$. This gives rise to the time occurring as a factor when the partial derivative is taken. The problem can be overcome if we define a new mean longitude at epoch, ϵ^*, by

$$\frac{\mathrm{d}\epsilon^*}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon}{\mathrm{d}t} + t\frac{\mathrm{d}n}{\mathrm{d}t}. \tag{6.151}$$

Hence

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = n + \frac{\mathrm{d}\epsilon^*}{\mathrm{d}t} \tag{6.152}$$

and

$$\lambda = \int n\,\mathrm{d}t + \epsilon^*. \tag{6.153}$$

This can also be written as

$$\lambda = \rho + \epsilon^*, \tag{6.154}$$

where

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = n, \qquad \frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} = \frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{3}{2}\frac{n}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \tag{6.155}$$

or

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} = -\frac{3}{a^2}\frac{\partial \mathcal{R}}{\partial \epsilon}. \tag{6.156}$$

In this case we should consider any derivatives of $\partial/\partial\epsilon$, such as those that occur in the expressions for $\dot{a}$, $\dot{e}$, and $\dot{I}$, to mean $\partial/\partial\lambda$. In practice the variation of ϵ can usually be neglected since it is a small effect.

The variation of the orbital elements of the mass m' can be expressed by equations similar to Eqs. (6.145)–(6.150), with $\mathcal{R}$ replaced by $\mathcal{R}'$ and all unprimed variables exchanged for primed ones. The derivation of Lagrange's planetary equations does not assume that $\mathcal{R}$ arises from perturbations by an external mass. Therefore we can equally well use these equations to study the perturbations on the mass m due to, for example, a nonspherical central mass. This will be considered in Sect. 6.11. Similarly the equations are equally applicable if we use the averaged disturbing functions $\langle\mathcal{R}\rangle$ and $\langle\mathcal{R}'\rangle$.

We have already seen in Sect. 2.9 that the variations in the orbital elements can be expressed in terms of the radial, tangential, and orthogonal forces acting on an orbiting object. However, Lagrange's equations allow us to derive similar variations but based on the Fourier series expansion of the disturbing function discussed in this chapter. As such they provide the basis for most of the perturbation calculations that follow.

6.9 Classification of Arguments in the Disturbing Function

We can now approach the subject of the physical significance of the expansion of the disturbing function. So far we have expressed the perturbing potential as a series involving an infinite number of permissible combinations of angles. But which angles are important in any given problem? In other words, which of the infinite terms in the expansion are important and which can be ignored? To a large extent the answers to these questions depend on the semi-major axis of the perturbed orbit. We can classify all arguments by considering the frequencies or periods associated with the cosine arguments in the expansion.

Each cosine argument contains a linear combination of the angles λ', λ, ϖ', ϖ, Ω', and Ω. We know that in the unperturbed problem the mean longitudes, λ' and λ, increase linearly at rates n' and n respectively. In contrast, all the other angles are constant in the unperturbed problem. Therefore, when we consider the perturbed system λ' and λ are rapidly varying quantities, whereas all the other angles undergo slow variations. Therefore, any valid arguments that do not involve mean longitudes are slowly varying. These give rise to *secular* terms, from the Latin verb *saeculum* meaning century, or long period. This does not imply that all other arguments are of short period. Consider a general argument of the form $\varphi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega$ with

$$\lambda' \approx n't + \epsilon' \quad \text{and} \quad \lambda \approx nt + \epsilon \tag{6.157}$$

(see Eq. (6.144)). Therefore $j_1\lambda' + j_2\lambda \approx (j_1n' + j_2n)t +$ constant and so, if the semi-major axes are such that

$$j_1n' + j_2n \approx 0, \tag{6.158}$$

then this argument also has a period longer than either orbital period. Equation (6.158) is satisfied when there is a commensurability between the two mean motions or orbital periods (see Sect. 1.7). We classify such arguments as giving rise to *resonant* terms in the expansion. If we consider the semi-major axes, the equivalent condition is

$$a \approx (|j_1|/|j_2|)^{\frac{2}{3}}\, a'. \tag{6.159}$$

Because of the dependence on semi-major axis, resonant terms are localised. Whereas a particular combination of angles may be slowly varying at one semi-major axis of the perturbed body, the same combination would be varying rapidly at another. In contrast the secular terms can be considered as global.

Any argument that is neither secular nor resonant is considered to give rise to a *short-period* term. In practise the application of the averaging principle mentioned in Sect. 6.7 allows us to ignore the infinite number of short-period terms in the expansion and accept that the dynamics is dominated by the appropriate secular and resonant terms.

Below we provide predictions of motion under secular and resonant terms in the context of the elliptical restricted three-body problem with small inclination, and we compare the answers with the results of numerical integrations. Here we assume that the mass m is negligible and that the orbit of m' is a fixed ellipse in the reference plane. Our starting point is a set of the lowest order form of Lagrange's equations for $\dot{a}$, $\dot{e}$, $\dot{\varpi}$, and $\dot{\Omega}$ derived from inspection of Eqs. (6.145), (6.146), (6.149), and (6.148). The equations of motion are

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial\langle\mathcal{R}\rangle}{\partial\lambda}, \tag{6.160}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{1}{na^2e}\frac{\partial\langle\mathcal{R}\rangle}{\partial\varpi}, \tag{6.161}$$

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = +\frac{1}{na^2e}\frac{\partial\langle\mathcal{R}\rangle}{\partial e}, \tag{6.162}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = +\frac{1}{na^2\sin I}\frac{\partial\langle\mathcal{R}\rangle}{\partial I}, \tag{6.163}$$

where $\langle\mathcal{R}\rangle$ is the averaged part of the disturbing function for an external perturber.

6.9.1 Secular Terms

Secular terms arise from those arguments that do not contain the mean longitudes. Inspection of the direct part of the second-order expansion in Eq. (6.107) shows that secular terms are obtained by setting $j = 0$ in those cosine arguments containing $j\lambda' - j\lambda$. This gives

$$\langle\mathcal{R}_\mathrm{D}\rangle = C_0 + C_1(e^2 + e'^2) + C_2 s^2 + C_3 ee'\cos(\varpi' - \varpi), \tag{6.164}$$

where

$$C_0 = \frac{1}{2}b_{\frac{1}{2}}^{(0)}(\alpha), \tag{6.165}$$

$$C_1 = \frac{1}{8}\left[2\alpha D + \alpha^2 D^2\right] b_{\frac{1}{2}}^{(0)}(\alpha), \tag{6.166}$$

$$C_2 = -\frac{1}{2}\alpha b_{\frac{3}{2}}^{(1)}(\alpha), \tag{6.167}$$

$$C_3 = \frac{1}{4}\left[2 - 2\alpha D - \alpha^2 D^2\right] b_{\frac{1}{2}}^{(1)}(\alpha). \tag{6.168}$$

Note that there are no ss' or s'^2 terms in $\langle\mathcal{R}_D\rangle$ because we are taking $s' = 0$ and that C_0 is a function of α only. Furthermore, inspection of the terms in $\mathcal{R}_E$ (Eq. (6.110)) shows that all the arguments contain at least one mean longitude and so there are no secular contributions from the indirect part of the disturbing function. Hence the low-order version of Lagrange's equations becomes

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_{\mathrm{sec}} = 0, \tag{6.169}$$

$$\left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_{\mathrm{sec}} = n\alpha(m'/m_c)C_3e'\sin(\varpi - \varpi'), \tag{6.170}$$

$$\left(\frac{\mathrm{d}\varpi}{\mathrm{d}t}\right)_{\mathrm{sec}} = n\alpha(m'/m_c)\left[2C_1 + C_3(e'/e)\cos(\varpi - \varpi')\right], \tag{6.171}$$

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)_{\mathrm{sec}} = n\alpha(m'/m_c)(C_2/2), \tag{6.172}$$

where we have used the fact that $\mu' = \mathcal{G}m' \approx n^2a^3(m'/m_c)$, where m_c is the mass of the central object. If we assume that $e \gg e'$ then the approximate solutions to these equations are

$$a = a_0, \tag{6.173}$$

$$e = e_0 - \frac{n\alpha}{\dot{\varpi}}(m'/m_c)C_3e'\left[\cos\varpi_0 - \cos\varpi\right], \tag{6.174}$$

$$\varpi = \varpi_0 + n\alpha(m'/m_c)2C_1t, \tag{6.175}$$

$$\Omega = \Omega_0 + n\alpha(m'/m_c)(C_2/2)t, \tag{6.176}$$

where the subscript 0 denotes the initial ($t = 0$) value of a quantity, and we have taken $\varpi' = 0$. These solutions predict that there is no secular change in a, that e varies sinusoidally with an amplitude of

$$(\Delta e)_{\mathrm{sec}} = \left|(n\alpha/\dot{\varpi})(m'/m_c)C_3e'\right|, \tag{6.177}$$

and that ϖ and Ω will either increase or decrease linearly with time depending on the signs of C_1 and C_2.

Figures 6.3a–d show the results of a numerical integration of the full equations of motion of the elliptical restricted three-body problem with $a' = 1$, $e' = 0.048$, $\varpi' = 0$, $I' = 0$, and $m'/m_c = 1/1047.355$ with starting conditions $a_0 = 0.192$, $e_0 = 0.1$, $\varpi_0 = 130°$, $\Omega_0 = 200°$, $\lambda_0 = 300°$, and $\lambda' = 0°$. Substitution of $\alpha = a/a' = 0.192$ in Eqs. (6.166)–(6.168) gives $C_1 = 0.0148335$, $C_2 = -0.0593339$, and $C_3 = -0.00708688$; note that $2C_1 = -C_2/2$. Since the mass ratio is that of the Jupiter–Sun ratio, the integration was designed to mimic the motion of an asteroid perturbed by Jupiter, and so the time units in the plots are given as Jupiter periods. However, the semi-major axis was deliberately chosen to be far away from Jupiter in order to avoid proximity to strong resonances. In these circumstances the secular perturbations alone should provide a good approximation to the motion.

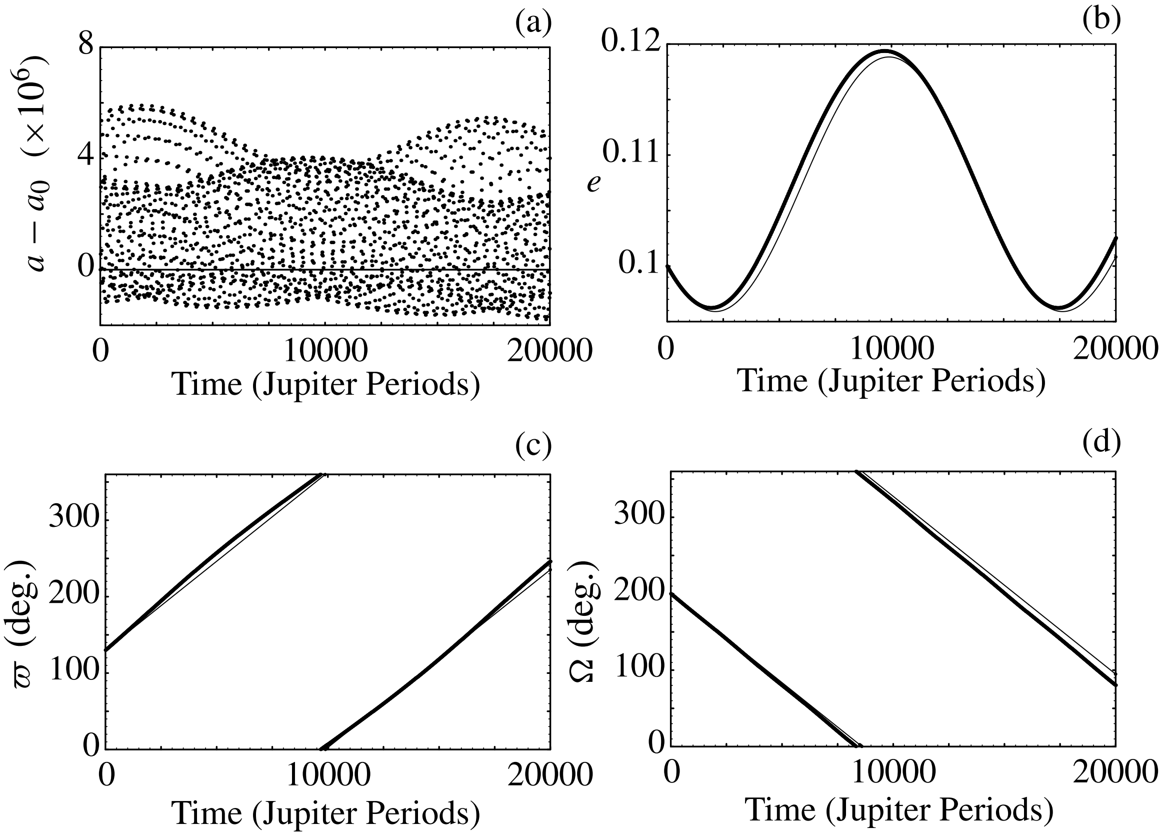

Fig. 6.3. A comparison of the results of a full numerical integration (thick line) with predictions from analytical theory (thin line) for the variation of (a) semi-major axis, (b) eccentricity, (c) longitude of perihelion, and (d) longitude of ascending node for a test particle undergoing predominantly secular perturbations from Jupiter.

The results show that the agreement is excellent over the 20,000 Jupiter periods of the integration. There are variations in a but these are extremely small; note that the scale in Fig. 6.3a is enlarged. The fact that the semi-major axis is almost constant justifies the evaluation of the Laplace coefficients for a fixed value of α. The eccentricity does vary as predicted, and while ϖ increases linearly with time (since $C_1 > 0$), Ω is decreasing linearly at the same rate (since $2C_1 = -C_2/2$; cf. Eqs. (6.175) and (6.176)). Prograde motion of the pericentre (or node) is called *precession* and retrograde motion is called *regression.* The behaviour of ϖ and Ω is a natural consequence of the secular terms in the disturbing function.

Because of the infinite number of short-period terms in the disturbing function, which we have neglected, there should be differences between the results of a full integration and the predictions of our analytical theory. We can see this already in the Fig. 6.3a, where there are small, but detectable short-period changes in the semi-major axis from the constant value predicted by theory. Figure 6.4 shows the difference between the "observed" eccentricity (i.e., the one determined by the numerical integration) and the calculated value from theory, as a function of time for the first 1,000 Jupiter periods of the integration. Here again we can see the effect of the short-period terms inherently included in any full integration.

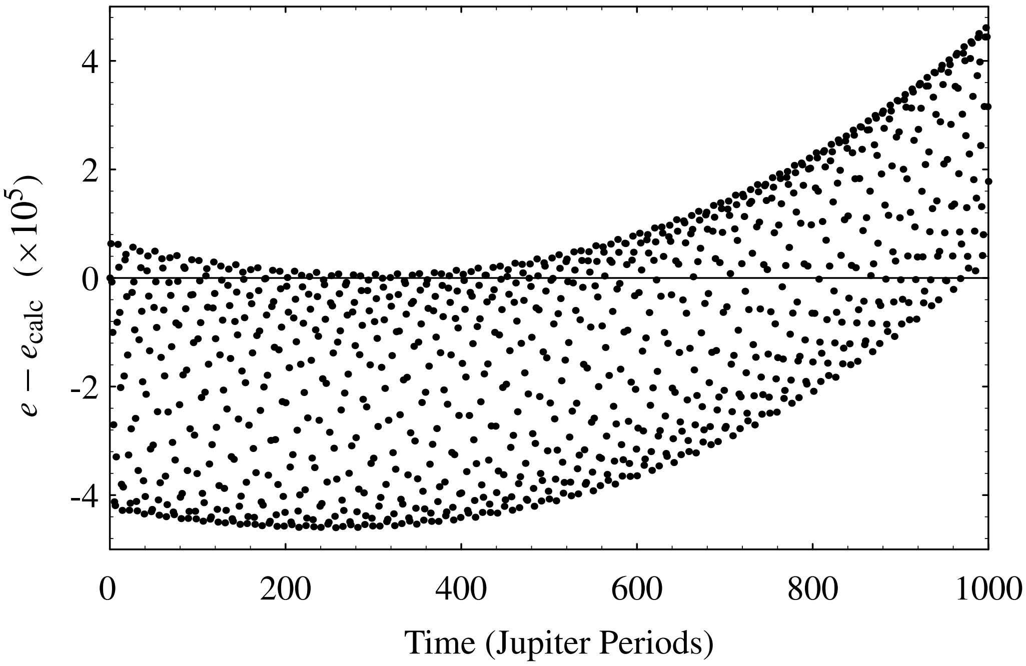

Fig. 6.4. Differences between the observed and calculated values of the test particle's eccentricity as a function of time. The data are sampled every Jupiter period and show the short-period variations in e.

6.9.2 Resonant Terms

Now suppose, for example, that we want to study an asteroid's motion at 3.27 AU, under the perturbing effect of Jupiter. Since Jupiter's semi-major axis is 5.20 AU we have, using Kepler's third law, that the ratio of their periods is $(3.27/5.20)^{3/2} \approx 0.499$. Hence, we have the relation $2n' \approx n$ and we would expect resonant terms to be important. Therefore, in the vicinity of the 2:1 resonance, as well as the secular terms discussed above, we also need to consider those terms in the expansion of the disturbing function that contain $2\lambda' - \lambda$ (i.e., the resonant terms for this location).

Inspection of Eq. (6.107) shows that in a second-order expansion there are two terms in $\langle \mathcal{R}_{\rm D} \rangle / a'$ that have a cosine argument containing $2\lambda' - \lambda$ for specific values of j. The relevant direct part of the averaged disturbing function is

$$\langle \mathcal{R}_{\rm D} \rangle = C_0 + C_1(e^2 + e'^2) + C_2(s^2 + s'^2) + C_3 e e' \cos(\varpi - \varpi') \\ + C_4 e \cos(2\lambda' - \lambda - \varpi) + C_5 e' \cos(2\lambda' - \lambda - \varpi'), \quad (6.178)$$

where the additional constants C_4 and C_5 are given by

$$C_4 = \frac{1}{2}\,[-4 - \alpha D]\, b_{\frac{1}{2}}^{(2)}(\alpha), \quad (6.179)$$

$$C_5 = \frac{1}{2}\,[3 + \alpha D]\, b_{\frac{1}{2}}^{(1)}(\alpha). \quad (6.180)$$

The second of these two resonant arguments makes no contribution to $\dot{e}$, $\dot{\varpi}$, and $\dot{\Omega}$ but does contribute a term to $\dot{a}$. Inspection of Eq. (6.110) shows that there is also a $-2\alpha e'$ contribution to the same argument from the indirect part.

Application of the approximate form of Lagrange's equations gives

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_{\mathrm{res}} = 2n\alpha a(m'/m_{\mathrm{c}})C_4 e\sin(2\lambda' - \lambda - \varpi) + 2n\alpha a(m'/m_{\mathrm{c}})\left(C_5 - 2\alpha\right)e'\sin(2\lambda' - \lambda - \varpi'), \tag{6.181}$$

$$\left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_{\mathrm{res}} = n\alpha(m'/m_{\mathrm{c}})C_4\sin(2\lambda' - \lambda - \varpi), \tag{6.182}$$

$$\left(\frac{\mathrm{d}\varpi}{\mathrm{d}t}\right)_{\mathrm{res}} = n\alpha(m'/m_{\mathrm{c}})(C_4/e)\cos(2\lambda' - \lambda - \varpi), \tag{6.183}$$

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)_{\mathrm{res}} = 0 \tag{6.184}$$

for the variations in a, e, ϖ, and Ω due to the 2:1 resonance. If we consider approximate solutions for these resonant equations alone we obtain

$$a = a_0 - \frac{2n\alpha a(m'/m_{\mathrm{c}})C_4 e}{2n' - n - \dot{\varpi}}\left[\cos(2\lambda' - \lambda - \varpi) - \cos(\lambda_0 + \omega_0)\right] - \frac{2n\alpha a(m'/m_{\mathrm{c}})(C_5 - 2\alpha)e'}{2n' - n}\left[\cos(2\lambda' - \lambda - \varpi') - \cos\lambda_0\right], \tag{6.185}$$

$$e = e_0 + \frac{n\alpha(m'/m_{\mathrm{c}})C_4}{2n' - n - \dot{\varpi}}\left[\cos(2\lambda' - \lambda - \varpi) - \cos(\lambda_0 + \omega_0)\right], \tag{6.186}$$

$$\varpi = \varpi_0 + \frac{n\alpha(m'/m_{\mathrm{c}})(C_4/e)}{2n' - n - \dot{\varpi}}\left[\sin(2\lambda' - \lambda - \varpi) + \sin(\lambda_0 + \omega_0)\right], \tag{6.187}$$

$$\Omega = \Omega_0. \tag{6.188}$$

To derive these solutions we have assumed that the only time-varying quantities on the right-hand side of the equations for $\dot{a}$, $\dot{e}$, and $\dot{\varpi}$ are in the cosine arguments and that ϖ increases linearly with time at a constant rate $\dot{\varpi}$ determined by secular theory. These equations suggest that a, e, and ϖ will experience sinusoidal variations with maximum amplitudes

$$(\Delta a)_{\mathrm{res}} = 2n\alpha a(m'/m_{\mathrm{c}})\left(\left|\frac{C_4 e}{2n' - n - \dot{\varpi}}\right| + \left|\frac{(C_5 - 2\alpha)e'}{2n' - n}\right|\right), \tag{6.189}$$

$$(\Delta e)_{\mathrm{res}} = \left|\frac{n\alpha(m'/m_{\mathrm{c}})C_4}{2n' - n - \dot{\varpi}}\right|, \tag{6.190}$$

$$(\Delta\varpi)_{\mathrm{res}} = \left|\frac{n\alpha(m'/m_{\mathrm{c}})(C_4/e)}{2n' - n - \dot{\varpi}}\right|. \tag{6.191}$$

These are only approximate solutions, particularly in the case of the semi-major axis where we have just combined the amplitudes of the terms associated with each of the two resonant arguments.

Figures 6.5a–d show the result of a full integration of the equations of motion and a comparison with the predicted variations from the combined secular and resonant theory outlined above. The calculations were done with the same

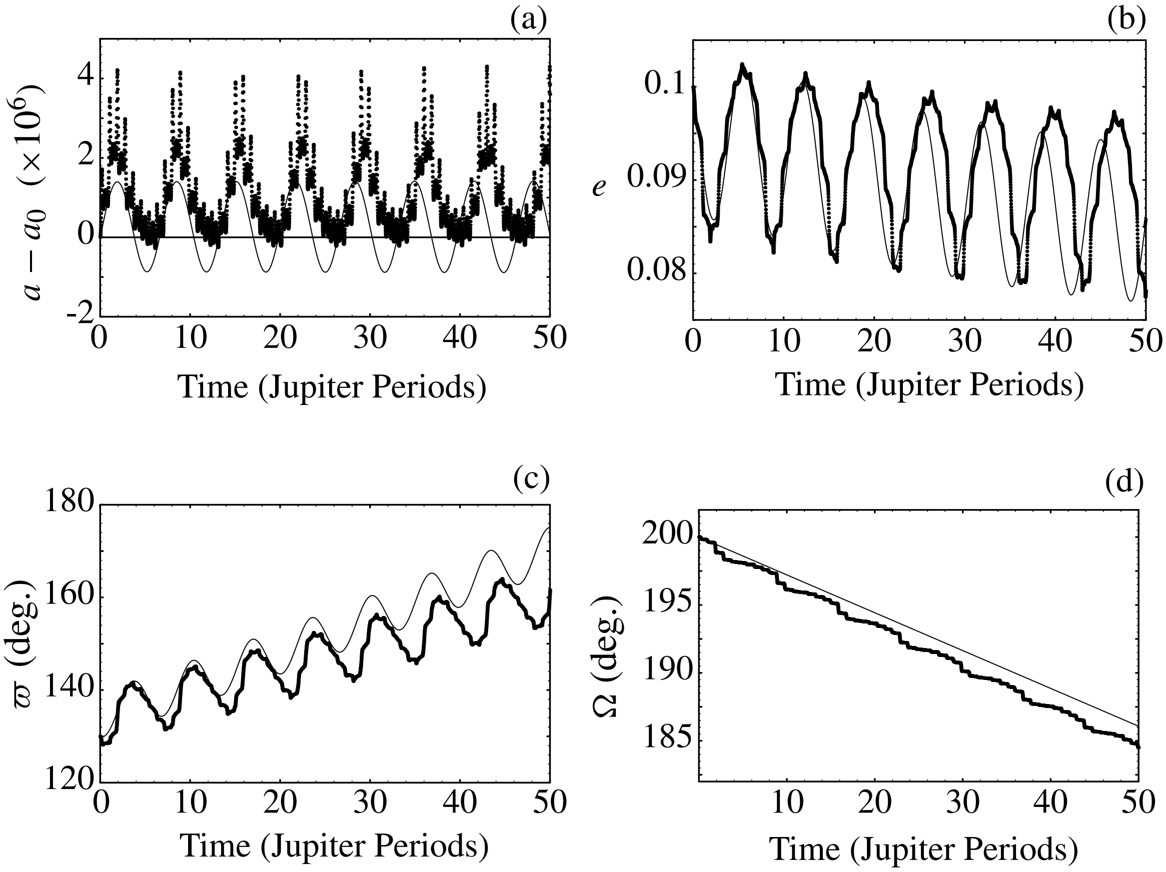

Fig. 6.5. A comparison of the results of a full numerical integration (points or thick line) with predictions from analytical theory (thin line) for the variation of (a) semi-major axis, (b) eccentricity, (c) longitude of perihelion, and (d) longitude of ascending node for a test particle near the 2:1 resonance undergoing resonant and secular perturbations from Jupiter.

starting values as in Sect. 6.9.1, but with $a = 0.6$ in order to place the test particle close to (but not in) the 2:1 jovian resonance. The relevant constants are now $C_1 = 0.314001$, $C_2 = -1.25600$, $C_3 = -0.447005$, $C_4 = -1.04332$, and $C_5 = 1.55230$. Note that the magnitudes of C_1 and C_2 have increased by a factor of ~ 20 over those in our secular example with $a = 0.192$. This is because the separation from Jupiter has decreased, thereby increasing the size of the secular effects. Examination of Fig. 6.5 shows that there is good agreement between the predictions and the numerical results, with the amplitudes and frequencies of the variations in a, e, and ϖ being close to their predicted values. We would expect there to be some differences, partly due to our approximate form of Lagrange's equations and partly due to the fact that in order to integrate the differential equations we took the quantities a and e on the right-hand side of Eqs. (6.181)–(6.183) to be constant, whereas clearly they are varying due to the resonance.

Note from Eqs. (6.189)–(6.191) that all the amplitudes contain a divisor of the form $2n' - n - \dot{\varpi}$ (i.e., the time derivative of the resonant argument $2\lambda' - \lambda - \varpi$). This implies that the changes in the elements will become even larger as the exact resonance is approached. However, it is in such circumstances that the assumptions in our simple analytical model break down. We consider a more complete model of resonance in Sect. 8.

6.9.3 Short-Period and Small-Amplitude Terms

Knowledge of the form of the disturbing function allows us to isolate the permissible secular arguments and the resonant arguments that are likely to be important. In effect, we are assuming that all other terms involving the mean longitudes are of short period and that their effect will average out to zero; this is the averaging principle. Our comparison of the analytical theory with the full integrations in Figs. 6.4 and 6.5 shows that this is a good approximation. Therefore, although short-period terms exist, their effects appear to be negligible, at least for the examples we chose.

In Sect. 6.9.2 we showed that if we want to know which terms are going to dominate the perturbed motion of the asteroid, we should find those terms for which $j_1 n' + j_2 n \approx 0$, where j_1 and j_2 are integers, because these will contribute to the creation of a small divisor in Eqs. (6.189)–(6.191). Therefore, in the vicinity of 3.27 AU the dominant terms are likely to be those with $j_1 = \pm 2$ and $j_2 = \mp 1$ since we will then have $2n' - n \approx 0$. However, this implies that we should also consider the terms with $j_1 = \pm 4, \pm 6, \ldots$ and $j_2 = \mp 2, \mp 3, \ldots$ since these will also give rise to small divisors. Can there be an infinite number of such terms, all contributing to motion at this resonance? Simple number theory tells us that we can always approximate the ratio of two real numbers (in our case the two mean motions) by a rational number, to arbitrary precision. Ought there be an infinite number of resonances that could contribute large-amplitude terms to the disturbing function at any semi-major axis?

We can resolve these paradoxes by considering our expression for S, the "strength" of the disturbing function (see Eq. (6.143)). For simplicity consider the case of a near commensurability of mean motions in the planar, circular, restricted, three-body problem. Let the resonant argument be

$$\varphi = (j+k)\lambda' - j\lambda - k\varpi \tag{6.192}$$

and let us assume that there is a near commensurability such that

$$(j+k)n' - jn \approx 0, \tag{6.193}$$

where j and k are integers. This means that arguments which contain expressions of the form $(j+k)\lambda' - j\lambda - k\varpi$ can vary slowly and produce long-period, large-amplitude perturbations. For example, in the case of the 2:1 resonance we have $j = \pm 1, \pm 2, \pm 3, \ldots$ and $k = \pm 1, \pm 2, \pm 3, \ldots$. However, although there is an infinite number of possible resonances for each pair of j and k, most of them are weak. This is because $S \propto e^{|k|}$ (see Eq. (6.143)) and $e < 1$. Therefore, as k increases the strength decreases. In effect these other terms exist but they are of small amplitude. By a similar argument we can overcome the difficulty of always being arbitrarily close to a resonance. For example, the 21:10 resonance is close to the 2:1 resonance, yet in this case $S \propto e^{11}$ and so the resonance is weak. Therefore the "nearly resonant" terms corresponding to higher orders in the eccentricities and inclinations can be discarded in the same way as all the other short-period terms.

The *order* k of a resonant term is identical to the order $N = |j_1 + j_2|$ of an argument in the disturbing function. Thus, if we require all the arguments that could contribute to a given second-order resonance, we should look at the arguments labelled D2 and E2 (or I2) in the expansion of the disturbing function given in Appendix B. We may need to consider other arguments as well, because if we require a fourth-order expansion in the orbital elements we should also look at the D4 and E4 (or I4) arguments. Similarly, because the secular terms do not contain mean longitudes, we only need to consider arguments labelled D0 in Appendix B.

6.10 Sample Calculations of the Averaged Disturbing Function

Here we consider the calculation of the appropriate terms in the disturbing function for two commensurabilities. In the first case, that of a second-order commensurability, we make use of the literal expansion given in Appendix B. In the second case the commensurability is of eleventh order and we resort to the form of the expansion for explicit arguments derived by Ellis & Murray (1999) and given in Sect. 6.6.

6.10.1 Terms Associated with the 3:1 Commensurability

Here we derive the terms required for a study of asteroid motion at 2.50 AU, close to the 3:1 commensurability with Jupiter. This will be used in our study of this resonance in Sect. 9.5.2. If we assume that the asteroid mass in negligible ($m \ll m'$) and that its eccentricity and inclination are small enough to allow us to use a second-order expansion of the disturbing function (i.e., we can ignore higher order terms in the fourth-order expansion given in Appendix B), then the necessary secular terms in the expansion are 4D0.1, 4D0.2, and 4D0.3 with $j = 0$, while the resonant terms are 4D2.1, 4D2.2, 4D2.3, 4D2.4, 4D2.5, 4D2.6 with $j = 3$, and 4E2.5. This gives an expression for the averaged disturbing function of the form

$$\begin{aligned} \frac{a'}{\mu'}\langle\mathcal{R}\rangle = & A_0 + A_1 e^2 + A_2 s^2 + A_3 e e' \cos(\varpi' - \varpi) + A_4 s s' \cos(\Omega' - \Omega) \\ & + A_5 e^2 \cos(3\lambda' - \lambda - 2\varpi) + A_6 e e' \cos(3\lambda' - \lambda - \varpi' - \varpi) \\ & + A_7 e'^2 \cos(3\lambda' - \lambda - 2\varpi') + A_8 s^2 \cos(3\lambda' - \lambda - 2\Omega) \\ & + A_9 s s' \cos(3\lambda' - \lambda - \Omega' - \Omega) + A_{10} s'^2 \cos(3\lambda' - \lambda - 2\Omega'), \end{aligned} \quad (6.194)$$

where the A_i ($i = 0, 1, \ldots, 10$) now denote combinations of Laplace coefficients and their derivatives. Note that there are other terms in the secular part of the expansion that contain expressions of second order in e' and s'. However, since we are interested in studying the motion of the asteroid (not Jupiter) we can take the orbital elements of Jupiter to be fixed and hence these expressions are effectively constants. The explicit forms of the constants A_i are

$$A_0 = \frac{1}{2} b_{\frac{1}{2}}^{(0)}(\alpha), \tag{6.195}$$

$$A_1 = \frac{1}{8}\left[2\alpha D + \alpha^2 D^2\right] b_{\frac{1}{2}}^{(0)}(\alpha), \tag{6.196}$$

$$A_2 = \frac{1}{2}\left[-\alpha\right] b_{\frac{3}{2}}^{(1)}(\alpha), \tag{6.197}$$

$$A_3 = \frac{1}{4}\left[2 - 2\alpha D - \alpha^2 D^2\right] b_{\frac{1}{2}}^{(1)}(\alpha), \tag{6.198}$$

$$A_4 = \left[\alpha\right] b_{\frac{3}{2}}^{(1)}(\alpha), \tag{6.199}$$

$$A_5 = \frac{1}{8}\left[21 + 10\alpha D + \alpha^2 D^2\right] b_{\frac{1}{2}}^{(3)}(\alpha), \tag{6.200}$$

$$A_6 = \frac{1}{4}\left[-20 - 10\alpha D - \alpha^2 D^2\right] b_{\frac{1}{2}}^{(2)}(\alpha), \tag{6.201}$$

$$A_7 = \frac{1}{8}\left[17 + 10\alpha D + \alpha^2 D^2\right] b_{\frac{1}{2}}^{(1)}(\alpha) - \frac{27}{8}\alpha, \tag{6.202}$$

$$A_8 = \frac{1}{2}\left[\alpha\right] b_{\frac{3}{2}}^{(2)}(\alpha), \tag{6.203}$$

$$A_9 = \left[-\alpha\right] b_{\frac{3}{2}}^{(2)}(\alpha), \tag{6.204}$$

$$A_{10} = \frac{1}{2}\left[\alpha\right] b_{\frac{3}{2}}^{(2)}(\alpha). \tag{6.205}$$

Note that, for the reasons given above, we have excluded the terms in 4D0.1 that only contained primed quantities. The $-(27/8)\alpha$ term in A_7 comes from the indirect term 4E2.5.

A numerical value for each of the A_i shown above can be calculated at a given value of α, the ratio of the semi-major axes. It is customary to fix this value of α when the asteroid is known to be in close proximity to the resonance such that the resonant terms in the expansion dominate. This is a good approximation, especially when the asteroid is actually inside the resonance. We can find the value of α for the nominal location of the resonance from the formula

$$\alpha_{3:1} = \frac{a_{3:1}}{a'} = \left(\frac{1}{3}\right)^{2/3} \left(\frac{m_c}{m_c + m'}\right)^{1/3}, \tag{6.206}$$

where m_c is the mass of the Sun and m' is the mass of Jupiter. This gives $\alpha_{3:1} \approx 0.480597$. The values of the constants A_i in this case are given in Table 6.1.

Table 6.1. The values of the constants A_i for the 3:1 jovian commensurability.

i	A_i
0	1.06671
1	0.142097
2	−0.568387
3	−0.165406
4	1.13677
5	0.598100
6	−2.21124
7	0.362954
8	0.330812
9	−0.661625
10	0.330812

By fixing a and a', the term in our expression for $\langle\mathcal{R}\rangle$ associated with A_0 becomes effectively a constant and can be neglected since ultimately we will be taking partial derivatives of $\langle\mathcal{R}\rangle$.

6.10.2 Terms Associated with the 18:7 Commensurability

Consider one of the terms relevant to a study of the motion of minor planet (2) Pallas. If n' and n denote the mean motions of Jupiter and Pallas respectively, then observations show that

$$18n' - 7n = -0.45^\circ\mathrm{y}^{-1}. \tag{6.207}$$

Therefore Jupiter and Pallas are close to a 18:7 resonance. In an eleventh-order expansion of the disturbing function there are 182 arguments associated with this resonance. Here we follow the example of Ellis & Murray (1999) and derive the terms associated with one of these arguments, namely

$$\varphi = 18\lambda' - 7\lambda - 5\varpi - 6\Omega. \tag{6.208}$$

Applying the definitions given in Eq. (6.114)–(6.120) gives $q = -5$, $q' = 0$, $\ell_{\max} = 5$, $p_{\min} = 3$, $p'_{\min} = 0$, $s_{\min} = 3$, and $i_{\max} = 3$. Since $s_{\min} = i_{\max}$ the only contribution will come from $i = s = 3$ and hence $l = 0$. Similarly, since $n_{\max} = [(s-3)/2] = 0$ we must have $n = 0$. Hence, from Eq. (6.124) the only valid value of p is $p = 3$; hence $m = 3$ and so from Eq. (6.125) $p' = 0$; we also have $j = 15$. We can now write the simplified form of Eq. (6.113) as

$$\begin{aligned}\langle\mathcal{R}_{\mathrm{D}}\rangle_+ = \frac{\alpha^3}{720}\sum_{\ell=0}^{5}\frac{(-1)^\ell}{\ell!}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\alpha^\ell D^\ell b_{\frac{7}{2}}^{(15)}(\alpha)\,F_{3,3,3}(I)\,F_{3,3,0}(I')\\ \times X_7^{3+k,\,12}(e)\,X_{18}^{-(4+k),\,18}(e')\cos\left[18\lambda' - 7\lambda - 5\varpi - 6\Omega\right].\end{aligned} \tag{6.209}$$

To complete the calculation we need to investigate the possibility that there are terms associated with the negative of our original argument, namely $\varphi = -(18\lambda' - 7\lambda - 5\varpi - 6\Omega)$. In this case inspection of Eq. (6.114)–(6.126) shows that there are no contributions and $\langle \mathcal{R}_{\mathrm{D}} \rangle_{-} = 0$.

We only require two evaluations of the inclination function and twelve evaluations of Hansen coefficients. Although our expansion is to eleventh order, according to the approximations given in Eq. (6.141)–(6.142) the function $F_{3,3,3}(I)$ will produce terms of $\mathcal{O}(I^6)$ and $X_7^{3+k,\,12}(e)$ will produce terms of $\mathcal{O}(e^5)$. Thus we are only concerned with the lowest order terms in all function evaluations. This means we can ignore the higher order terms in $F_{3,3,0}(I') = 15 + \mathcal{O}(I'^2)$ and $X_{18}^{-(4+k),\,18}(e') = 1 + \mathcal{O}(e'^2)$ for $k = 0, 1, \ldots 5$. We have

$$F_{3,3,3}(I) = 15s^6, \tag{6.210}$$

$$X_7^{3,12}(e) = -\frac{1577149}{1280}e^5, \tag{6.211}$$

$$X_7^{4,12}(e) = -\frac{1473703}{960}e^5, \tag{6.212}$$

$$X_7^{5,12}(e) = -\frac{7280077}{3840}e^5, \tag{6.213}$$

$$X_7^{6,12}(e) = -\frac{1486337}{640}e^5, \tag{6.214}$$

$$X_7^{7,12}(e) = -\frac{10842187}{3840}e^5, \tag{6.215}$$

$$X_7^{8,12}(e) = -\frac{409031}{120}e^5, \tag{6.216}$$

and the resulting expression for $\langle \mathcal{R}_{\mathrm{D}} \rangle$ is

$$\begin{aligned} \langle \mathcal{R}_{\mathrm{D}} \rangle = -\frac{e^5 s^6}{12288} \Big[& 4731447\alpha^3 + 1163365\alpha^4 D + 110950\alpha^5 D^2 \\ & + 5130\alpha^6 D^3 + 115\alpha^7 D^4 + \alpha^8 D^5 \Big] b_{\frac{7}{2}}^{(15)}(\alpha) \\ & \times \cos\left[18\lambda' - 7\lambda - 5\varpi - 6\Omega\right]. \end{aligned} \tag{6.217}$$

Application of the algorithm given in Sect. 6.6 for the indirect parts shows that none exist in this case; furthermore, there are no indirect terms associated with any of the 182 possible arguments to eleventh order at this resonance. Therefore the averaged part of the disturbing function associated with this argument is given by $\langle \mathcal{R} \rangle = \left(\mathcal{G}m'/a'\right) \langle \mathcal{R}_{\mathrm{D}} \rangle$.

6.11 The Effect of Planetary Oblateness

When the disturbing function was introduced in Sect. 6.2 it was in the context of the perturbing potential experienced by an orbiting mass due to the gravitational effect of another body. More generally it can be thought of as the terms in the