$\Lambda = L$

$\xi = \sqrt{-2V} \sin v \qquad \alpha = \sqrt{-2Z} \sin z$

$\eta = \sqrt{-2V} \cos v \qquad \beta = \sqrt{-2Z} \cos z$

Chapter 5

$V = C - L, \; Z = D - C, \; v = \varpi, \; z = \Omega$

SECULAR PERTURBATION THEORY

5.1 The secular perturbation Hamiltonian

As shown in Section 2.6, after removal of the short periodic terms the Hamiltonian would be reduced to the normal form:

$$H' = H'_0 + \epsilon\, H'_1 + \epsilon^2\, H'_2 + \mathcal{O}(\epsilon^3) \ ,$$

where all the H'_j are functions of the mean elements.

If we assume the Hamiltonian in mean elements is expressed by means of the mean Poincaré variables $(\boldsymbol{\Lambda}', \boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\lambda}', \boldsymbol{\eta}', \boldsymbol{\beta}')$ then

$$H' = H'_0(\boldsymbol{\Lambda}') + \epsilon\, H'_1(\boldsymbol{\Lambda}', \boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}') + \epsilon^2\, H'_2(\boldsymbol{\Lambda}', \boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}') + \mathcal{O}(\epsilon^3) \ .$$

This series contains the mean mean longitudes $\boldsymbol{\lambda}'$ only in the $\mathcal{O}(\epsilon^3)$ term. Thus, if we truncate the series after the $\mathcal{O}(\epsilon^2)$ term, we obtain a **secular Hamilton function**, not depending at all on the variables $\boldsymbol{\lambda}'$, that is such that all the variables $\boldsymbol{\Lambda}'$ are integrals at this level of approximation. In other words, we can claim that if we were able to compute mean variables $\boldsymbol{\Lambda}'$ from which all the short periodic perturbations are removed[1], then they would be constant, thus each corresponding mean a'_i would be also an **analytic proper semimajor axis**. Of course this goal can actually be achieved only up to some level of approximation; it needs to be explicitly described by the rules by which the terms in the determining function $\chi = \epsilon\, \chi_1 + \epsilon^2\, \chi_2$ are selected to be explicitly computed.

As a consequence, the components of $\boldsymbol{\Lambda}'$ are not anymore dynamical variables, but they appear just as parameters in H', which can be reinterpreted as the Hamiltonian providing the equations of motion in the space of the $4N + 4$ variables $(\boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}')$.

This would be a big step in the direction of computing an approximate solution for the equation of motion for $N + 1$ planets (asteroid included) problem, but there is a problem. In such a

[1]As pointed out in Section 2.10, secular perturbations in the semimajor axes (thus also $\boldsymbol{\Lambda}'$) $\mathcal{O}(\epsilon^2)$ are introduced by beats between short periodic perturbations $\mathcal{O}(\epsilon)$, and these need to be removed to compute accurate integrals.

context, the order zero portion $H_0'(\mathbf{\Lambda}')$ is just a constant and has no dynamical effect. Thus we can use as Hamilton function just

$$H' = \epsilon\, H_1' + \epsilon^2\, H_2' \tag{5.1}$$

which cannot be described as a "perturbed" Hamiltonian, that is it does not appear as the sum of an integrable portion plus a perturbation containing a small parameter. Indeed, ϵ is not a small parameter, it just sets the order of magnitude for the entire function H'.

This is best expressed as a change in scale for the time: indeed, if the Hamiltonian (5.1) is divided by ϵ, this is equivalent to a change of the time coordinate by a factor ϵ, that is by setting $\tau = \epsilon\, t$, the **slow time**, the Hamiltonian equations with τ as independent variable have as Hamiltonian $H'/\epsilon = H_1' + \epsilon\, H_2'$:

$$\begin{aligned} \frac{d\xi'}{d\tau} &= -\frac{\partial(H'/\epsilon)}{\partial\eta'} \quad , \quad \frac{d\eta'}{d\tau} = \frac{\partial(H'/\epsilon)}{\partial\xi'} \\ \frac{d\alpha'}{d\tau} &= -\frac{\partial(H'/\epsilon)}{\partial\beta'} \quad , \quad \frac{d\beta'}{d\tau} = \frac{\partial(H'/\epsilon)}{\partial\alpha'} \ . \end{aligned} \tag{5.2}$$

Anyway the problem is: how can we use a perturbation approach to solve the problem defined by the secular Hamilton function? To answer to this question we can go back to the third D'Alembert rule, and to the approach already used in Section 2.9: we use $e_i, \sin I_i$ as small parameters, all considered to be $\mathcal{O}(\epsilon^{1/r})$ for some $r > 0$, and expand into homogeneous functions of these small parameters, e.g.,

$$\frac{H_1'}{\varepsilon} \longleftarrow \quad H_1' = K_{\cancel{0}\,1}^{(0)}(;\mathbf{\Lambda}') + K_1^{(2)}(\boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}'; \mathbf{\Lambda}') + K_1^{(4)}(\boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}'; \mathbf{\Lambda}') + \mathcal{O}(\epsilon^{6/r}) \ ,$$

where $K_1^{(k)}$ is the portion of H_1' homogeneous of degree k in the small parameters $e_i, \sin I_i$; odd k terms are not allowed by D'Alembert rules. * Now the integrable portion of H' has appeared again, because a quadratic homogeneous Hamiltonian implies linear Hamilton equations, and those can be integrable. As for $K_1^{(0)}$ it does not matter, unless the time evolution of $\mathbf{\Lambda}'$ has to be computed: indeed, the notation $;\mathbf{\Lambda}'$ indicates that the variables connected with the mean semimajor axes appear just as non-dynamical parameters.

An interesting property of the secular Hamiltonian H_1' is that it does not contain any contribution from the indirect part of the perturbing Hamiltonian H_1. If heliocentric canonical coordinates are used, this implies that T_1 of equation (1.51) is such that its Fourier series expansion in Delaunay variables has no secular part, that is $\overline{T}_1 = 0$. For a classical proof see [Brouwer and Clemence 1961][page 508]. [but for the heliocentric canonical coordinates this should be easier... or maybe not?] in H_2' there are terms arising from beats between indirect short periodic terms, see Section 5.6.

* $j_5 + j_6$ è pari (positivo, negativo, nullo) e $j_3 + j_4 + j_5 + j_6 = 0$

5.2 Linear secular perturbation theory

As dictated by the third D'Alembert rule, $K_1^{(2)}$ contains only expressions of the form $\xi_i\, \xi_j + \eta_i\, \eta_j$ and $\alpha_i\, \alpha_j + \beta_i\, \beta_j$, where i, j are indexes of the planets (and asteroids), not excluding $i = j$. The

$= 2\sqrt{Z_i Z_j}\, \cos(z_i - z_j)$ $\qquad$ $= 2\sqrt{V_i V_j}\, \cos(v_i - v_j)$

$\xi = O(e) \qquad \eta = O(e)$

$\alpha = O(\operatorname{sen} I) \qquad \beta = O(\operatorname{sen} I)$

coefficients of these expressions are just functions of the variables Λ_i, Λ_j, that is

$$
\begin{aligned}
K_1^{(2)}(\boldsymbol{\xi}', \boldsymbol{\alpha}', \boldsymbol{\eta}', \boldsymbol{\beta}') &= \frac{1}{2} \sum_{i,j=1}^{N+1} \left[A_{ij}(\Lambda_i', \Lambda_j') \left(\xi_i' \xi_j' + \eta_i' \eta_j'\right) + E_{ij}(\Lambda_i', \Lambda_j') \left(\alpha_i' \alpha_j' + \beta_i' \beta_j'\right)\right] = \\
&= \frac{1}{2} \left[\boldsymbol{\xi}' \cdot A \, \boldsymbol{\xi}' + \boldsymbol{\eta}' \cdot A \, \boldsymbol{\eta}' + \boldsymbol{\alpha}' \cdot E \, \boldsymbol{\alpha}' + \boldsymbol{\beta}' \cdot E \, \boldsymbol{\beta}'\right] \qquad (5.3)
\end{aligned}
$$

where $A(\boldsymbol{\Lambda}') = (A_{ij})$, $E(\boldsymbol{\Lambda}') = (E_{ij})$ $i, j = 1, N+1$ are real symmetric matrices. Then the Hamilton equations defined by the Hamiltonian $K_1^{(2)}$ are, for each $i = 1, N+1$

* si scrive t ma si intende τ

$$
\frac{d\xi_i'}{dt} = -\sum_{j=1}^{N+1} A_{ij} \, \eta_j' \quad , \quad \frac{d\eta_i'}{dt} = \sum_{j=1}^{N+1} A_{ij} \, \xi_j'
$$

$$
\frac{d\alpha_i'}{dt} = -\sum_{j=1}^{N+1} E_{ij} \, \beta_j' \quad , \quad \frac{d\beta_i'}{dt} = \sum_{j=1}^{N+1} E_{ij} \, \alpha_j'
$$

In vector form

$$
\begin{aligned}
\frac{d\boldsymbol{\xi}'}{dt} &= -A \, \boldsymbol{\eta}' \quad , \quad \frac{d\boldsymbol{\eta}'}{dt} = A \, \boldsymbol{\xi}' \\
\frac{d\boldsymbol{\alpha}'}{dt} &= -E \, \boldsymbol{\beta}' \quad , \quad \frac{d\boldsymbol{\beta}'}{dt} = E \, \boldsymbol{\alpha}' \, ,
\end{aligned} \qquad (5.4)
$$

or equivalently as second order equations

$$
\begin{aligned}
\frac{d^2\boldsymbol{\xi}'}{dt^2} &= -A^2 \, \boldsymbol{\xi}' \quad , \quad \frac{d^2\boldsymbol{\eta}'}{dt^2} = -A^2 \, \boldsymbol{\eta}' \\
\frac{d^2\boldsymbol{\alpha}'}{dt^2} &= -E^2 \, \boldsymbol{\alpha}' \quad , \quad \frac{d^2\boldsymbol{\beta}'}{dt^2} = -E^2 \, \boldsymbol{\beta}' \, .
\end{aligned}
$$

Note that the two sets of equations (5.4) are decoupled, thus we can solve separately first for the ξ_i', η_i', then for the α_j', β_i'. The solutions can be expressed by means of matrix exponentials: by assembling all the equations for ξ_i', η_i'

$$
\frac{d}{dt} \begin{bmatrix} \boldsymbol{\xi}' \\ \boldsymbol{\eta}' \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -A \\ A & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}' \\ \boldsymbol{\eta}' \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \boldsymbol{\xi}'(t) \\ \boldsymbol{\eta}'(t) \end{bmatrix} = \exp \begin{bmatrix} \mathbf{0} & -At \\ At & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}'(0) \\ \boldsymbol{\eta}'(0) \end{bmatrix} .
$$

Note that the matrix of coefficients of the system is antisymmetric, thus the exponential is a rotation: all the lenghts of the vectors are preserved. Exactly the same applies to the solutions for $\boldsymbol{\alpha}', \boldsymbol{\beta}'$.

Let R, S be matrices such that $R^T \, A \, R = -diag[g_1, \ldots, g_{N+1}]$ and $S^T \, E \, S = -diag[s_1, \ldots, s_{N+1}]$ are diagonal; the coordinate change is orthogonal, that is $R^T = R^{-1}$, $S^T = S^{-1}$; in particular we can assume that R, S are rotations. The eigenvalues of the matrices A, E are the frequencies g_i, s_i, respectively, and are secular, in that they appear in the solutions of the equations (5.2) with as independent variable the slow time τ; in practice, periods are of the order of tens of thousands to millions of years. Then by changes of coordinates

$$
\boldsymbol{\xi}'' = R^T \, \boldsymbol{\xi}' \quad , \quad \boldsymbol{\eta}'' = R^T \, \boldsymbol{\eta}' \quad , \quad \boldsymbol{\alpha}'' = S^T \, \boldsymbol{\alpha}' \quad , \quad \boldsymbol{\beta}'' = S^T \, \boldsymbol{\beta}'
$$

we can decouple the differential equations, that is the double-primed variables represent the **proper modes** of oscillation, solutions of uncoupled harmonic oscillators:

$$\frac{d\boldsymbol{\xi}''}{dt} = diag[g_1, \ldots, g_{N+1}]\, \boldsymbol{\eta}'' \quad , \quad \frac{d\boldsymbol{\eta}''}{dt} = -diag[g_1, \ldots, g_{N+1}]\, \boldsymbol{\xi}''$$

$$\frac{d\boldsymbol{\alpha}''}{dt} = diag[s_1, \ldots, s_{N+1}]\, \boldsymbol{\beta}'' \quad , \quad \frac{d\boldsymbol{\beta}''}{dt} = -diag[s_1, \ldots, s_{N+1}]\, \boldsymbol{\alpha}'' \ .$$

Then we can define action-angle variables V_i'', v_i'' and Z_i'', z_i'' such that

$$\xi_i'' = \sqrt{-2V_i''}\, \sin v_i'' \quad , \quad \eta_i'' = \sqrt{-2V_i''}\, \cos v_i''$$

$$\alpha_i'' = \sqrt{-2Z_i''}\, \sin z_i'' \quad , \quad \beta_i'' = \sqrt{-2Z_i''}\, \cos z_i'' \ ,$$

and the actions are integrals for the linear equations, while the angles circulate with frequencies g_i, s_i. Indeed, the expression for $K_1^{(2)}$ as a function of the new variables is

$$K_1^{(2)} = \frac{1}{2}\, \sum_i (-g_i)\, (\xi_i''\, \xi_i'' + \eta_i''\, \eta_i'') + \frac{1}{2}\, \sum_i (-s_i)\, (\alpha_i''\, \alpha_i'' + \beta_i''\, \beta_i'') = \sum_i (g_i\, V_i'' + s_i\, Z_i'')$$

and the Hamilton equations in the double primed variables are

$$\frac{dV_i''}{dt} = 0 \quad , \quad \frac{dZ_i''}{dt} = 0$$

$$\frac{dv_i''}{dt} = g_i \quad , \quad \frac{dz_i''}{dt} = s_i$$

and $(V_i'', Z_i'', v_i'', z_i'')$ are action-angle variables for the proper mode i. Thus the linear portion of the secular Hamilton equations is integrable, not only in the sense of having a solution expressed by elementary analytic functions, but also in the sense of having action-angle variables. These double primed variables can be called **linear proper elements**.

However, there is an important difference in the system of linear differential equations for $\boldsymbol{\alpha}', \boldsymbol{\beta}'$, with respect to the one for $\boldsymbol{\xi}', \boldsymbol{\eta}'$: namely one of the eigenvalues of E must be zero, for all values of $\boldsymbol{\Lambda}'$. [Brouwer and Clemence 1961][page 518] show this result for two planets only, by the argument that one possible solution needs to be the one in which the two planets have the same inclination and nodes, $I_1 = I_2$, $\Omega_1 = \Omega_2$, then the coplanar orbits remain coplanar. [B&C claim that no general proff is available at that time (1961)]

Our way of showing the same property for an arbitrary number of planets is to argue that the components of the angular momentum c_x, c_y in the reference plane are integrals, thus since this property applies for all e_i, I_i then it must apply already at the lowest order, that is

$$\left\{c_x, K_1^{(2)}\right\} = 0 \quad , \quad \left\{c_y, K_1^{(2)}\right\} = 0 \ .$$

The contribution to c_x from the body i is

$$c_x^{(i)} = C_i\, \sin I_i\, \sin \Omega_i = C_i\, 2\, \sin \frac{I_i}{2}\, \cos \frac{I_i}{2}\, \sin \Omega_i$$

and by using $\sqrt{(1-\cos I_i)} = \sqrt{2}\sin\frac{I_i}{2}$ and the relationships with α_i', β_i' eq.(2.44)

$$c_x^{(i)} = \sqrt{C_i/2}\,\cos\frac{I_i}{2}\,\beta_i' = \sqrt{L_i/2}\,\beta_i' + \mathcal{O}(e_i^2, I_i^2)$$

$$c_y^{(i)} = -\sqrt{C_i/2}\,\cos\frac{I_i}{2}\,\alpha_i' = -\sqrt{L_i/2}\,\alpha_i' + \mathcal{O}(e_i^2, I_i^2)\ .$$

Then we can compute the lowest order (in e_i, I_i) Poisson brackets

$$\left\{c_x, K_1^{(2)}\right\} = \frac{\partial c_x}{\partial \boldsymbol{\beta}'}\cdot\frac{\partial K_1^{(2)}}{\partial \boldsymbol{\alpha}'} = \frac{1}{\sqrt{2}}\sqrt{\boldsymbol{\Lambda}'}\cdot E\,\boldsymbol{\alpha}' = 0\ ,$$

and similarly for $\left\{c_y, {}_1^{(2)}\right\}$. Because the equation above must be true for all possible α', it follows that the vector forming a scalar product with $\boldsymbol{\alpha}'$ is $\mathbf{0}$, thus

$$E^T\,\sqrt{\boldsymbol{\Lambda}'} = E\,\sqrt{\boldsymbol{\Lambda}'} = \mathbf{0}\ ,$$

that is the symmetric matrix E has one zero eigenvalue, and the matrix of the linear dynamical system for $\boldsymbol{\alpha}', \boldsymbol{\beta}'$ has two zero eigenvalues [note: matrices linear Hamilton equations have an even equation for eigenvalue, being of the form JB with B symmetric; prove here?]. This implies that one of the frequencies s_k is zero; conventionally, this frequency is associated to Jupiter, that is $s_5 = 0$.

If the reference system for which the computations are performed is such that the reference plane (x, y) is orthogonal to the total angular momentum vector, then it coincides with **Laplace invariable plane** and $c_x = c_y = 0$. In this case the linear proper action Z_5'' corresponding to the frequency s_5 is zero. Then the amplitudes of all the terms containing the angle variable z_5'' is also zero. From this a new D'Alembert rule arises, namely, the Fourier terms of K_1, at all degrees, cannot contain z_5''; this applies to the transformed series appearing in Section 5.4.

Thus the solutions belong to invariant tori of dimension $N+1$ in the space of $\boldsymbol{\xi}', \boldsymbol{\eta}'$ and to invariant tori of dimension N in the space of $\boldsymbol{\alpha}', \boldsymbol{\beta}'$. [reference to next chapter, also a more extensive discussion is needed on why there are neither perturbative terms nor secular resonances containing s_5.]

[Problem: why are the frequencies, computed at this level of truncation, g_i all > 0, s_i all ≤ 0? This should be explained; see [Brouwer and Clemence 1961], where there is proof for 2 planets that $g_1, g_2 > 0$ and $s_2 < 0, s_1 = 0$, but only numerical results for more planets.]

5.3 Second order terms from mean motion resonances

Let us compute the second order terms ($\mathcal{O}(\epsilon^2)$, that is those containing μ_5^2) generated by $\chi_{2/1}$ in the second order transformed Hamiltonian

$$\begin{aligned}
\epsilon^2\,H_2' &= -\frac{1}{2}\left\{H_{2/1}, \chi_{2/1}\right\} = \\
&= -\frac{1}{2}\,\mu_5^2\,e^2\left\{g(a,a_5)\,\cos(\lambda-2\lambda_5+\varpi), \frac{g(a,a_5)}{n-2n_5}\,\sin(\lambda-2\lambda_5+\varpi)\right\} + \ldots\ ;
\end{aligned}$$

by the same argument as above, the largest contribution to the secular portion of the Poisson bracket is from the product of the $\partial/\partial\lambda$ of the first argument and the $\partial/\partial\Lambda$ of the second argument, which contains a $n/(n-2\,n_5)^2$ factor.

However, here the conclusion is opposite from the one about the secular terms in semimajor axis, that is the inclusion of these second order terms in the computation of the proper eccentricity and inclination is necessary, in particular near the 2/1 resonance with Jupiter, but also near the 3/1 one, because the additional terms change significantly the secular frequencies.

Thus the secular Hamiltonian H' should be computed by adding at least the largest second order terms, resulting from Poisson brackets of terms with small divisors:

$$H' = \left[K_1^{(0)} + K_2^{(0)}\right] + \left[K_1^{(2)} + K_2^{(2)}\right] + \left[K_1^{(4)} + K_2^{(4)}\right] + \left[K_1^{(6)} + K_2^{(6)}\right] + \ldots$$

As an example, (Brouwer and Van Woerkom 1950) added some terms belonging to $K_2^{(6)}$ arising from the small divisor $2\,n_5 - 5\,n_6$ of the "great inequality", that is near resonance, between Jupiter and Saturn to their otherwise linear secular perturbation theory for the major planets. [also others? [Knežević 1986] says that B&VW used also second order terms from secular perturbations $2g_5 - g_6$ and $2g_6 - g5$]

[[Knežević 1989] and maybe others for the specific terms added in our theory]

[need to mention that $s_5 = 0$ and reduction to the invariable plane also apply to $K_2^{(2)}$, and no terms with s_5 also applies to $K_j^(2s)$]

5.4 Lie series for proper elements

After solving for the linear proper elements, a perturbative approach to solve the equation of secular motion can again be based on the Lie series algorithm, with the difference that both the Hamiltonian H' and the determining function χ' need to be expanded in (even) powers of the Poincaré variables: the Hamiltonian begins with the integrable term of degree 2

$$H' = K_2 + K_4 + K_6 + \ldots$$

where the dots indicate terms of order 8 and more; note we have simplified the notation, removing the distinction between the terms arising from $\mathcal{O}(\epsilon)$ and those from $\mathcal{O}(\epsilon^2)$. The determining function begins with terms of order 4

$$\chi' = \chi'_4 + \chi'_6 + \ldots$$

and the transformed Hamiltonian can be described by the Lie series

$$\begin{aligned} H' \circ \Phi_{\chi'}^{-1} &= H' - \{H', \chi'\} + \frac{1}{2}\ \{\{H', \chi'\}, \chi'\} + \ldots \\ &= K_2 + K_4 + K_6 - \{K_2 + K_4, \chi'_4 + \chi'_6\} + \frac{1}{2}\ \{\{K_2, \chi'_4\}, \chi'_4\} + \ldots\ , \end{aligned}$$

where the terms included in $\ldots$ are all of degree ≥ 8 in eccentricities and inclinations.

Now we need to analyse the series above by separating the terms of homogeneous degree $2, 4, 6, \ldots$; to do this we use the simple rule that the degree of $\{K_r, \chi'_s\}$ is $r + s - 2$: this implies that the terms not listed in the formula above are of degree ≥ 8. Thus the homological equations are for degree 2 just $K_2 = K_2$, for degree 4

$$\overline{K}_4 = K_4 - \{K_2, \chi'_4\} \ , \tag{5.5}$$

where $\overline{K}_4$ is the portion of K_4 commuting with K_2, that is $\left\{K_2, \overline{K}_4\right\} = 0$.
For degree 4, the determining function can be searched for in Delaunay variables $(\mathbf{V}'', \mathbf{Z}'', \mathbf{v}'', \mathbf{z}'')$, starting from the expansion of the secular Hamiltonian H' in a Fourier series with respect to the linear proper angles $\mathbf{v}'', \mathbf{z}''$.

$$K_4 = \sum_{\mathbf{j},\mathbf{h}} K_{4\mathbf{jh}} \cos(\mathbf{j}\,\mathbf{v}'' + \mathbf{h}\,\mathbf{z}'') \ .$$

Then the determining function for the same degree is

$$\chi'_4 = \sum_{\mathbf{j},\mathbf{h}} \chi'_{4\mathbf{jh}} \sin(\mathbf{j}\,\mathbf{v}'' + \mathbf{h}\,\mathbf{z}'') \ ,$$

and the Poisson brackets can be computed term by term

$$\{K_2, \chi'_4\} = -\sum_{\mathbf{j},\mathbf{h}} (\mathbf{j}\cdot\mathbf{g} + \mathbf{h}\cdot\mathbf{s})\ \chi'_{4\mathbf{jh}} \cos(\mathbf{j}\,\mathbf{v}'' + \mathbf{h}\,\mathbf{z}'') \ .$$

Then the terms of K_4 containing the angles $\mathbf{v}'', \mathbf{z}''$ can be removed from the normal form $\overline{K}_4$ by setting the coefficients of χ'_4 as follows:

$$\chi'_{4\mathbf{jh}} = -\frac{K_{4\mathbf{jh}}}{\mathbf{j}\cdot\mathbf{g} + \mathbf{h}\cdot\mathbf{s}} \ .$$

If we can assume a non-resonance condition for secular resonances $\mathbf{j}\cdot\mathbf{g}+\mathbf{h}\cdot\mathbf{s} = 0 \Longrightarrow \mathbf{j} = \mathbf{h} = \mathbf{0}$, at least for degree 4, that is for combinations of ≤ 4 secular frequencies, then $\overline{K}_4$ consists of the term without angles

$$\overline{K}_4 = K_{4\mathbf{00}} \ .$$

[Problem: the non-resonance condition does not include the coefficient of s_5, which is zero by the additional D'Alembert rule.]

Higher order normalization

To apply the same method to the removal of terms of degree 6 is conceptually the same, but requires a little more patience. Given that $\overline{K}_4$ has no angles, while $\tilde{K}_4 = K_4 - \overline{K}_4$ contains $\mathbf{v}'', \mathbf{z}''$, and that $\tilde{K}_4 = \{K_2, \chi'_4\}$, we can analyse the portion of the homological equation of degree 6 as follows:

$$K_6 = \overline{K}_6 + \tilde{K}_6 \quad , \quad \frac{1}{2}\ \{\{K_2, \chi'_4\}, \chi'_4\} = \frac{1}{2}\ \left\{\tilde{K}_4, \chi'_4\right\}$$

$$W_6 = K_6 - \{K_2, \chi'_6\} - \left\{\overline{K}_4, \chi'_4\right\} - \frac{1}{2}\ \left\{\tilde{K}_4, \chi'_4\right\} \ ,$$

with the last term the only one which could contain terms without angles, due to beats of two terms with the same angles. Thus

$$W_6 = \tilde{W}_6 + \overline{W}_6 \quad , \quad \overline{W}_6 = \overline{-\frac{1}{2}\left\{\tilde{K}_4, \chi_4'\right\}}$$

while the portion with the angles W_6 can be removed, assuming the non-secular resonance condition to be true also for combinations of ≤ 6 secular frequencies:

$$\{K_2, \chi_6'\} = \tilde{K}_6 + \tilde{W}_6$$

by selecting suitable coefficients for the Fourier series of χ_6'. If this procedure is executed for all the degree 6 terms, then a transformation is defined by $\chi_4' + \chi_6'$ to **proper elements**, that is canonical coordinates $(\mathbf{V}^*, \mathbf{Z}^*, \mathbf{v}^*, \mathbf{z}^*)$ in which the transformed Hamiltonian is a function of action variables only, apart from terms of degree ≥ 8:

$$K^* = K_2 + \overline{K}_4 + \overline{K}_6 + \overline{W}_6 = \ldots$$

Then the action variables are approximately constant, and the angle variables have approximately constant frequencies.

The homological equations and the procedures to solve them in this Section are the same as in Section 2.6, apart from the use of different small parameters. Thus we need to express the same doubts and make the same distinction mentioned in Section 2.9: can we consider this procedure of elimination of terms containing the variables $(\mathbf{v}'', \mathbf{z}'')$ as a way to remove them all, by means of a convergent perturbative series including all powers of eccentricities and inclinations, or as a method to remove a finite number of terms to obtain an approximate equation for secular motion? Also the answer needs to be the same: if the non-secular resonance condition $\mathbf{j} \cdot \mathbf{g} + \mathbf{h} \cdot \mathbf{s} = 0 \Longrightarrow \mathbf{j} = \mathbf{h} = \mathbf{0}$ had to apply to all integer combinations of secular frequencies (satisfying the first and second D'Alembert rules), then the subset of the linear proper elements space in which this condition is violated is dense[2].

The next question, suggested by the analogy with Section 2.11, is whether we should be computing the map

$$\Phi^1_{\chi'} : (\mathbf{V}'', \mathbf{Z}'', \mathbf{v}'', \mathbf{z}'') \mapsto (\mathbf{V}^*, \mathbf{Z}^*, \mathbf{v}^*, \mathbf{z}^*)$$

from linear proper to proper elements, or rather

$$\Phi^{-1}_{\chi'} : (\mathbf{V}^*, \mathbf{Z}^*, \mathbf{v}^*, \mathbf{z}^*) \mapsto (\mathbf{V}'', \mathbf{Z}'', \mathbf{v}'', \mathbf{z}'') \ .$$

The answer to this question requires some knowledge of KAM theory, which is discussed in the next Chapter.

[2]To prove this requires to check the non degeneracy condition, see in the next Chapter.

The results for three coefficients: $b^{(i)} = b^{(i)}_{1/2}, c^{(i)} = b^{(i)}_{3/2}, e^{(i)} = b^{(i)}_{5/2}$, are plotted in Figure 3.3. The curves indicate the values of i (as a function of α) needed to achieve the relative accuracy 10^{-8}. The plot clearly demonstrates that the convergence is quite slow in the high-α region, and that, for α larger than ≈ 0.3, one has to include coefficients up to much higher i than previously thought. The limiting i is slightly different for $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, but makes possible in all cases to easily estimate the number of coefficients necessary for an accurate calculation. For $\alpha > 0.86$ the number of necessary coefficients increases abruptly, and it is not possible to take into account all the terms that would guarantee an accurate result.

The values of i for several bins in α used for the practical work with analytical theory are given in Table 3.1.

Table 3.1: Value of i as a function of α in the range centered on the asteroid main belt.

α	< 0.3	$0.3 - 0.4$	$0.4 - 0.5$	$0.5 - 0.6$	$0.6 - 0.7$	> 0.7
i	11	21	31	43	61	81

Degree 4 terms in the indirect part of disturbing function

The indirect part of disturbing function produces in planetary case only very small short periodic perturbations, which even in the extreme cases do not contribute significantly to the total effect. This was, of course, known to Le Verrier who, as shown in Section 3.1, therefore provided the development of the indirect part of disturbing function up to degree 3 only. The higher order indirect terms, however, can give rise to significant long periodic perturbations of the second (and higher) order in perturbing mass, in particular near the resonant surfaces, due to the squares (and higher powers) of small divisors appearing in the solutions.

For asteroids, the situation is more complicated. The indirect 1/1 resonant term with critical argument $\lambda' - \lambda$ appears in the equations of motion even at degree one in eccentricity, thus the motion of Trojan asteroids can be affected by the first order indirect perturbations. Many asteroids with high eccentricities and/or inclinations are located so close to the mean motion resonances that even the higher degree indirect terms can produce non negligible effects either through the first order short periodic effects or via the second order long periodic ones.

As already mentioned in Section 3.2, Knežević (1993a) extended Le Verrier's development by adding terms of degree 4 and expressing the expansion with respect to the fixed reference plane (see eq. 3.34). To investigate the impact of the high degree indirect terms, several tests have been carried out of relative significance of terms of degrees 2,3, and 4, and of the accuracy of mean elements computed by taking into account higher degree terms. It was found that the indirect part of the disturbing function even in the most extreme cases gives rise to very small short periodic perturbations, with asteroids with high eccentricities/inclinations being most affected. Hence, indirect perturbations can in almost all cases be computed up to degree 2 terms only, without significant loss of accuracy of the asteroid mean elements. The higher degree indirect terms can become important because of their contribution to the higher order long periodic perturbations.

Tupikova et al. (1999) added to the development of the indirect part a few missing terms of degree 4 in inclinations I, I'.

To make the analytical computation of asteroid perturbations homogeneous in terms of the expansion used, Milani and Knežević (1994) included all the terms of degree 4 into their theory thus ending up with a total of 192 indirect terms, as reported in Section 2.9.

[Reminder: do not forget the changes to footnote 10, Section 2.9]

Asteroid proper elements for high e, I

The analytical theory of [Milani and Knežević, 1990], and its upgrades [Milani and Knežević, 1992, Milani and Knežević, 1994] make use of the literal expansion truncated at degree 4 in eccentricity and inclination in the first order with respect to the perturbing mass, and at degree 2 in the second order. Thus, the question is how accurate it is when applied to the asteroids with high orbital e, I.

As for the eccentricity, Fig. 3.2 provides the answer we look for: the convergence of the disturbing function in the region of the asteroid Main Belt occurs for the eccentricities up to about $e \simeq 0.3$, with somewhat higher limiting values for the inner belt, and lower limiting values for the outer belt and for the resonant objects beyond it.

For the inclination. however, a comprehensive test has been carried out to asses the limiting values for the application of Milani and Knežević analytical theory, above which a specially adapted semianalytical theory by [Lemaitre and Morbidelli 1994] should be used instead. The RMS values of the changes with time of proper elements computed by means of the two theories are taken as a measure of the instabiliy. The results of this test have shown that the analytical elements are more stable than the semianalytical ones below about 15° of inclination ($\sin I \simeq 0.27$), while above 17° ($\sin I \simeq 0.29$) the semianalytical ones have superior stability. In the transition region between 15° and 17° both theories provide proper elements of the same stability.

Hence, the proper elements computed by the analytical theory can be safely used up to $e, sinI \simeq 0.3$, and this limit has been respected in the subsequent computations of proper elements by means of the analytical theory, as well as in the catalog of asteroid proper elements maintained on AstSyS. Obviously, the situation changed dramatically when the synthetic proper elements (Chapter 7) became available.

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