4

THE N–BODY PROBLEM

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« Theory of Orbit Determination »

This chapter presents the basic theory of the gravitational N-body problem, the coordinate systems used for both theoretical investigations and practical applications, and how to select the dynamical model for a Solar System orbit.

4.1 Equation of motion and integrals

By $(N+1)$-body problem we mean the ordinary differential equation defining the motion of $N+1$ point masses with positions $\mathbf{r}_j$, velocities $\dot{\mathbf{r}}_j$, and masses m_j, interacting only through the mutual gravitational attraction

$$m_j \ddot{\mathbf{r}}_j = \sum_{i \neq j} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} (\mathbf{r}_i - \mathbf{r}_j), \qquad j = 0, \ldots, N \tag{4.1}$$

where the dots indicate time derivatives and G is the universal gravitational constant; this is the equation of motion in Newtonian form. We need to express it in another form, more suitable both to discuss symmetries and integrals and to perform coordinate changes. The mutual gravitational forces admit a potential, thus we can define the potential energy

$$V = - \sum_{0 \leq i < j \leq N} \frac{G\, m_i\, m_j}{|\mathbf{r}_i - \mathbf{r}_j|};$$

we introduce the kinetic energy T and the **Lagrange function** (or **Lagrangian**) L:

$$T = \frac{1}{2} \sum_{i=0}^{N} m_i\, |\dot{\mathbf{r}}_i|^2, \qquad L = T - V. \tag{4.2}$$

The Newton equation of motion is equivalent to the **Lagrange equation**

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}_j} \right) - \frac{\partial L}{\partial \mathbf{r}_j} = \mathbf{0} \tag{4.3}$$

with two important properties. The first one has to do with integrals of motion, the second is discussed in Section 4.2. A **first integral** of the Lagrange equation (4.3) is a function of all the positions and velocities

$$I = I(\mathbf{R}, \dot{\mathbf{R}}), \qquad \mathbf{R} = (\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_N), \qquad \dot{\mathbf{R}} = (\dot{\mathbf{r}}_0, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N)$$

such that the total time derivative along the solutions is identically zero:

$$\frac{dI}{dt} = \frac{\partial I}{\partial \mathbf{R}}\,\dot{\mathbf{R}} + \frac{\partial I}{\partial \dot{\mathbf{R}}}\,\ddot{\mathbf{R}} = 0;$$

thus the value of I is constant along the orbits.

Symmetries and integrals

A one-parameter **group of symmetries** of the Lagrange function L is a diffeomorphism F^s of the positions $\mathbf{R}$ depending (in a differentiable way) upon a parameter $s \in \mathbb{R}$ so that $F^s \circ F^z = F^{s+z}$ and the Lagrange function is invariant:

$$L\left(F^s(\mathbf{R}), \frac{d}{dt}F^s(\mathbf{R})\right) = L\left(F^s(\mathbf{R}), \frac{\partial F^s}{\partial \mathbf{R}}\dot{\mathbf{R}}\right) = L(\mathbf{R}, \dot{\mathbf{R}}).$$

F^0 is the identity transformation; we also assume the mixed derivatives $\partial^2 F^s/\partial \mathbf{R}\partial s$ are continuous. A *local one-parameter group of symmetries* of the Lagrange function is defined by the same properties for s in a neighborhood of 0. The main result we need is the **Noether theorem**, stating that if the Lagrange function L admits a local one-parameter group of symmetries F^s then

$$I(\mathbf{R}, \dot{\mathbf{R}}) = \frac{\partial L}{\partial \dot{\mathbf{R}}} \cdot \frac{\partial F^s(\mathbf{R})}{\partial s}\bigg|_{s=0} \tag{4.4}$$

is a first integral of the Lagrange equation (4.3).

To apply this theorem to the $(N+1)$-body problem we look for symmetries of the Lagrange function in (4.2), a function of the mutual distances $|\mathbf{r}_i - \mathbf{r}_j|$ and of the velocities $|\dot{\mathbf{r}}_j|$. Thus every **isometry** of the space of positions, preserving distances and independent of time, preserves the Lagrange function. The isometries of the Euclidean space $\mathbb{R}^3$ are the functions

$$G(\mathbf{x}) = R\,\mathbf{x} + \mathbf{q}, \qquad \frac{dG}{dt}(\mathbf{x}) = R\,\dot{\mathbf{x}},$$

where R is an orthogonal matrix ($R^T\ R = I$) and $\mathbf{q}$ a constant vector, both independent of time. The symmetry group of three-dimensional space has

dimension 6 and is generated by six one-parameter subgroups.[1] There are three one-parameter symmetry groups of translations ($R = I$):

$$F^s(\mathbf{x}) = \mathbf{x} + s\ \hat{\mathbf{v}}_h, \qquad \frac{\partial F^s(\mathbf{x})}{\partial s} = \hat{\mathbf{v}}_h$$

where $\hat{\mathbf{v}}_h$ is the unit vector along one coordinate axis, for $h = 1, 2, 3$. If equal translations are applied to all bodies, then the integral of (4.4) is

$$p_h = \hat{\mathbf{v}}_h \cdot \sum_{j=0}^{N} m_j\ \dot{\mathbf{r}}_j,$$

the component along the axis $\hat{\mathbf{v}}_h$ of the total **linear momentum p**. The latter is a vector integral, and the **center of mass** $\mathbf{b}_0$ moves with constant velocity:

$$\mathbf{b}_0 = \frac{1}{M_0} \sum_{j=0}^{N} m_j\ \mathbf{r}_j; \quad M_0 = \sum_{j=0}^{N} m_j \text{ (total mass)}; \quad \mathbf{b}_0(t) = \frac{t}{M_0}\ \mathbf{p} + \mathbf{b}_0(0). \tag{4.5}$$

In the above formula, $\mathbf{b}_0(0)$ is a constant vector which can be obtained as a combination of positions and velocities, but with coefficients depending upon time: each of its components is a **time-dependent first integral**.

The other three one-parameter symmetry groups are groups of rotations ($\mathbf{q} = \mathbf{0}$). A three-dimensional $\mathbf{x}$ rotates by an angle of s radians around an axis $\hat{\mathbf{v}}_h$; the rotation is counterclockwise for $s > 0$, as seen from the tip of $\hat{\mathbf{v}}_h$,

$$F^s(\mathbf{x}) = R_{s\hat{\mathbf{v}}_h}\ \mathbf{x}, \qquad \left.\frac{\partial F^s(\mathbf{x})}{\partial s}\right|_{s=0} = \hat{\mathbf{v}}_h \times \mathbf{x}$$

and the integral of the Noether theorem

$$c_h = \sum_{j=0}^{N} (\hat{\mathbf{v}}_h \times \mathbf{r}_j) \cdot m_j\ \dot{\mathbf{r}}_j = \hat{\mathbf{v}}_h \cdot \sum_{j=0}^{N} m_j\, (\mathbf{r}_j \times \dot{\mathbf{r}}_j)$$

is the component along $\hat{\mathbf{v}}_h$ of the total angular momentum

$$\mathbf{c} = \sum_{j=0}^{N} m_j\, (\mathbf{r}_j \times \dot{\mathbf{r}}_j)\,, \tag{4.6}$$

thus the motion preserves the angular momentum vector integral.

There is one additional integral, the total energy integral, which is not deduced from the Noether theorem.[2] By computing the total time deriva-

[1] The tangent space to the unit element, the *Lie Algebra*, is generated by the tangents to these subgroups. Only orientation preserving isometries are included in the one-parameter subgroups.

[2] It could be interpreted, with the Hamiltonian formalism, as a consequence of the invariance with respect to time, thus it corresponds to the symmetry $t \mapsto t + s$.

tives

$$\frac{dT}{dt} = \sum_{j=0}^{N} m_j \, \ddot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j, \qquad \frac{dV}{dt} = \sum_{j=0}^{N} \frac{\partial V}{\partial \mathbf{r}_j} \cdot \dot{\mathbf{r}}_j$$

and by eq. (4.1) they are opposite, thus $E = T + V$ is a first integral.

There is one additional symmetry in the $(N+1)$-body problem, which involves not only the coordinates but also the time and possibly the masses: the **change of scale**. It is also associated with a first integral, which is not independent of the previous ones. If the lengths are changed by a factor λ, the times by a factor τ, the masses by a factor μ, then

$$m_j \ddot{\mathbf{r}}_j \mapsto \frac{\mu\,\lambda}{\tau^2} \, m_j \ddot{\mathbf{r}}_j, \qquad \frac{\partial V}{\partial \mathbf{r}_j} \mapsto \frac{\mu^2}{\lambda^2} \, \frac{\partial V}{\partial \mathbf{r}_j},$$

and the equation of motion is satisfied by the scaled orbits if and only if

$$\lambda^3 = \mu \, \tau^2, \tag{4.7}$$

the dimensional version of Kepler's third law. If $\tau = 1$ it is possible to scale the lengths compensating with a scaling of the masses $\lambda^3 = \mu$; this may imply the impossibility of determining masses and lengths (see Section 6.2).

When a scaling with $\lambda^3 = \mu \, \tau^2$ is applied, the energy integral is scaled

$$T \mapsto \frac{\mu\lambda^2}{\tau^2} \, T, \qquad V \mapsto \frac{\mu^2}{\lambda} \, V \Longrightarrow E \mapsto \frac{\mu\lambda^2}{\tau^2} \, E$$

and the angular momentum vector integral scales as $\mathbf{c} \mapsto \mu\lambda^2/\tau \, \mathbf{c}$, thus the combination $E \, c^2$, where $c = |\mathbf{c}|$, scales as

$$E \, c^2 \mapsto \frac{\mu\lambda^2}{\tau^2} \, \frac{\mu^2\lambda^4}{\tau^2} E \, c^2 = \mu^5 \, E \, c^2;$$

thus $E \, c^2$ is invariant if $\mu = 1$, that is, if masses are not scaled.

A deep result obtained by the celestial mechanicians of the late nineteenth century states that for $N \geq 3$ there are no first integrals in the $(N+1)$-body problem independent of the 10 classical ones of the linear and angular momentum and total energy (seven time independent and three time dependent).

4.2 Coordinate changes

The first integrals have to be exploited to reduce the dimensionality of the equation of motion, and this is for two reasons. First, the dimensions $3N+3$ of the configuration space, and $6N+6$ of the phase space (of the initial conditions), are too large to understand the properties of the solutions. Second, the symmetries associated with the integrals may result in degeneracy of the

orbit determination problem, as discussed in Chapter 6; one of the possible remedies is to decrease the number of variables. Also for the above purpose, we need to know how the equation of motion transforms under a coordinate change: this is easier for the Lagrange equation.

Let $\mathbf{B} = (\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n)$ be another set of coordinates for the positions of the $N+1$ bodies, and $\mathbf{R} = \mathbf{R}(\mathbf{B})$ a coordinate change which is a diffeomorphism (with continuous second derivatives) of the $(3N+3)$-dimensional space; we are thus assuming that the Jacobian matrix $A(\mathbf{B}) = \partial\mathbf{R}/\partial\mathbf{B}$ is invertible at each point $\mathbf{B}$. The corresponding change in the velocities is

$$\dot{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial \mathbf{B}}(\mathbf{B})\, \dot{\mathbf{B}} = A(\mathbf{B})\, \dot{\mathbf{B}}.$$

Let $L(\mathbf{R}, \dot{\mathbf{R}})$, $\mathcal{L}(\mathbf{B}, \dot{\mathbf{B}})$ be Lagrange functions corresponding by value:

$$\mathcal{L}(\mathbf{B}, \dot{\mathbf{B}}) = L\left(\mathbf{R}(\mathbf{B}), \dot{\mathbf{R}}(\mathbf{B}, \dot{\mathbf{B}})\right) = L\left(\mathbf{R}(\mathbf{B}), A(\mathbf{B})\, \dot{\mathbf{B}}\right);$$

then the left-hand side of the Lagrange equation is transformed as follows:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{B}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{B}} = \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{R}}}\right) - \frac{\partial L}{\partial \mathbf{R}}\right] A(\mathbf{B}). \tag{4.8}$$

The Lagrange equations in the two coordinate systems are equivalent

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{B}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{B}} = \mathbf{0} \iff \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{R}}}\right) - \frac{\partial L}{\partial \mathbf{R}} = \mathbf{0};$$

solutions of one are transformed by $\mathbf{R} = \mathbf{R}(\mathbf{B})$ into solutions of the other.

Reduction of the two-body problem

We shall start from the simplest case, the two-body problem, to get some ideas to be exploited in the general case. The Lagrange function is

$$L = \frac{1}{2} m_0\, |\dot{\mathbf{r}}_0|^2 + \frac{1}{2} m_1\, |\dot{\mathbf{r}}_1|^2 + \frac{G\, m_0\, m_1}{|\mathbf{r}_0 - \mathbf{r}_1|}.$$

We can change coordinates by using, in place of $\mathbf{r}_0, \mathbf{r}_1$, the coordinates of the center of mass and the relative position of $\mathbf{r}_1$ with respect to $\mathbf{r}_0$

$$\mathbf{b}_0 = \mu_1\, \mathbf{r}_1 + (1-\mu_1)\mathbf{r}_0, \quad \mu_1 = \frac{m_1}{m_0 + m_1}, \quad \mathbf{b}_1 = \mathbf{r}_1 - \mathbf{r}_0. \tag{4.9}$$

Then $V = \mathcal{V}(b_1) = -Gm_0m_1/b_1$, with $b_1 = |\mathbf{b}_1|$; to write T as a function of $\mathbf{b}_0, \mathbf{b}_1$ we express $\dot{\mathbf{r}}_0$ and $\dot{\mathbf{r}}_1$ as a function of $\dot{\mathbf{b}}_0, \dot{\mathbf{b}}_1$ and substitute in T:

$$2T = m_0\, \dot{\mathbf{r}}_0^2 + m_1\, \dot{\mathbf{r}}_1^2 = (m_0 + m_1)\, \dot{\mathbf{b}}_0^2 + \frac{m_0 m_1}{m_0 + m_1}\, \dot{\mathbf{b}}_1^2$$

the mixed terms canceling. The Lagrange function as a function of $\mathbf{b}_0, \mathbf{b}_1$ is

$$L = \frac{1}{2}\, M_0\, \dot{\mathbf{b}}_0^2 + \frac{1}{2}\, M_1\, \dot{\mathbf{b}}_1^2 + \frac{G\, M_0\, M_1}{b_1}$$

with $M_0 = m_0 + m_1$ the total mass and M_1 the **reduced mass**:

$$M_1 = \frac{m_0 m_1}{m_0 + m_1}. \tag{4.10}$$

Then the Lagrange function L can be decomposed as the sum of two Lagrange functions $L = M_0\, L_0(\dot{\mathbf{b}}_0) + M_1\, L_1(\mathbf{b}_1, \dot{\mathbf{b}}_1)$, one containing only $\mathbf{b}_0$, the other containing only $\mathbf{b}_1$, and the Lagrange equation decouples:

$$M_0\, \ddot{\mathbf{b}}_0 = 0, \qquad M_1\, \ddot{\mathbf{b}}_1 = -\frac{\partial \mathcal{V}(\mathbf{b}_1)}{\partial \mathbf{b}_1}.$$

The first equation states that the center of mass moves with constant velocity along a straight line, the second equation is the **Kepler problem**, with a particle of mass M_1 attracted by a fixed center of mass M_0.

By repeating the same computations done for T, we find that also the angular momentum has a simple expression in the **B** coordinates:

$$\mathbf{c} = m_0\, \mathbf{r}_0 \times \dot{\mathbf{r}}_0 + m_1\, \mathbf{r}_1 \times \dot{\mathbf{r}}_1 = M_0\, \mathbf{b}_0 \times \dot{\mathbf{b}}_0 + M_1\, \mathbf{b}_1 \times \dot{\mathbf{b}}_1.$$

When $\mathbf{b}_0(t)$ from eq. (4.5) is substituted, the $\mathbf{b}_0$ contribution is constant

$$\mathbf{c}_0 = \mathbf{b}_0 \times \dot{\mathbf{b}}_0 = \frac{1}{M_0}\, \mathbf{b}_0(0) \times \mathbf{p}, \qquad \mathbf{c} = M_0\, \mathbf{c}_0 + M_1\, \mathbf{c}_1$$

and the contribution from $\mathbf{b}_1$ is $\mathbf{c}_1 = \mathbf{b}_1 \times \dot{\mathbf{b}}_1$, the angular momentum per unit (reduced) mass of $\mathbf{r}_1$ with respect to the center $\mathbf{r}_0$, which is also a vector first integral. Thus $\mathbf{b}_1, \dot{\mathbf{b}}_1$ will lie for each t in the orbital plane normal to $\mathbf{c}_1$.

Solution of the two-body problem

The two-body problem has another vector integral, not occurring in the $N \geq 3$-body problem: the **Laplace–Lenz vector**

$$\mathbf{e} = \frac{1}{G\, M_0}\, \dot{\mathbf{b}}_1 \times \mathbf{c}_1 - \frac{1}{b_1}\, \mathbf{b}_1. \tag{4.11}$$

This can be shown by using a reference frame formed by three mutually orthogonal unit vectors, $\mathbf{v}_z = \mathbf{c}_1/c_1$ ($c_1 = |\mathbf{c}_1|$), $\mathbf{v}_r = \mathbf{b}_1/b_1$, and $\mathbf{v}_\theta$ such that $\dot{\mathbf{b}}_1 \cdot \mathbf{v}_\theta > 0$. If θ is the angle between the vector $\mathbf{v}_r$ and a fixed direction in the orbital plane, and $r = b_1$, we have

$$\mathbf{c}_1 = r\, \mathbf{v}_r \times \frac{d}{dt}(r\, \mathbf{v}_r) = r\, \mathbf{v}_r \times (\dot{r}\, \mathbf{v}_r + r\dot{\theta}\, \mathbf{v}_\theta) = r^2\, \dot{\theta}\, \mathbf{v}_r \times \mathbf{v}_\theta = r^2\, \dot{\theta}\, \mathbf{v}_z,$$

$$G\,M_0\;\mathbf{e} = -r^2\;\dot r\;\dot\theta\;\mathbf{v}_\theta + (r^3\;\dot\theta^2 - G\,M_0)\;\mathbf{v}_r. \tag{4.12}$$

Along the solutions we have

$$\dot{\mathbf{c}}_1 = 0, \qquad 2\dot r\dot\theta + r\dot\theta^2 = 0, \qquad \ddot r = -\frac{GM_0}{r^2} + \frac{c_1^2}{r^3},$$

so that

$$G\,M_0\;\dot{\mathbf{e}} = \ddot{\mathbf{b}}_1 \times \mathbf{c}_1 - G\,M_0\;\dot\theta\;\mathbf{v}_\theta = -G\,M_0\;\dot\theta\;(\mathbf{v}_r \times \mathbf{v}_z + \mathbf{v}_\theta) = \mathbf{0}.$$

Thus $\mathbf{e}$ contains two integrals independent of $\mathbf{c}_1$ (not three because $\mathbf{e}\cdot\mathbf{c}_1 = 0$). We define the **true anomaly** v as the angle between $\mathbf{e}$ and $\mathbf{v}_r$ on the orbital plane, that is

$$e\cos v = \mathbf{e}\cdot\mathbf{v}_r = \frac{r^3\;\dot\theta^2}{G\,M_0} - 1 = \frac{c_1^2}{G\,M_0\;r} - 1$$

where $r^2\dot\theta = c_1$ is the (scalar) angular momentum of $\mathbf{b}_1$ and is constant. From this we find the familiar formula of a conic section

$$r = \frac{c_1^2/G\,M_0}{1 + e\cos v}$$

and the interpretation of the two additional **two-body integrals** as **eccentricity** $e = |\mathbf{e}|$ and **argument of pericenter** ω, that is the angle of $\mathbf{e}$ with a fixed direction in the orbital plane, in such a way that $\theta = v + \omega$. The eccentricity e is an integral depending upon angular momentum and energy. The energy integral of the two-body problem in $(\mathbf{b}_0, \mathbf{b}_1)$ coordinates is

$$E(\mathbf{B}, \dot{\mathbf{B}}) = M_0\;E_0 + M_1\;E_1, \quad E_0 = \frac{1}{2}\;|\dot{\mathbf{b}}_0|^2, \quad E_1 = \frac{1}{2}\;|\dot{\mathbf{b}}_1|^2 - \frac{G\,M_0}{|\mathbf{b}_1|}$$

and the eccentricity squared, computed from eq. (4.12), is

$$e^2 = \mathbf{e}\cdot\mathbf{e} = \frac{r^4\;\dot\theta^2\;\dot r^2 + \left(r^3\;\dot\theta^2 - G\;M_0\right)^2}{G^2\;M_0^2} = 1 + \frac{2\,E_1\,c_1^2}{G^2\;M_0^2}.$$

If the energy of the relative motion E_1 is negative, then $e < 1$ and the trajectory of $\mathbf{b}_1$ is an ellipse with semimajor axis

$$a = \frac{q+Q}{2} = \frac{1}{2}\left[\frac{c_1^2/G\,M_0}{1+e} + \frac{c_1^2/G\,M_0}{1-e}\right] = \frac{G\,M_0}{-2\,E_1},$$

where q, Q are the pericenter and apocenter distances, and the scalar angular momentum of the relative motion is $c_1 = \sqrt{G\,M_0\;a\;(1-e^2)}$. Formulae to express explicitly the solutions of the two-body problem are available in Appendix A.

4.3 Barycentric and heliocentric coordinates

The set of positions of the $N+1$ bodies can be represented in different coordinates; we are interested in the linear coordinate changes of the form

$$\mathbf{b}_j = \sum_{i=0}^{N} a_{ji}\,\mathbf{r}_i, \qquad A = (a_{ji}),\ i,j = 0,N \tag{4.13}$$

where the matrix A is a function of the masses only. The purpose is to exploit the integrals of the center of mass to reduce the number of equations, generalizing the results of the two-body case. A natural choice is to use the center of mass as $\mathbf{b}_0$, thus by (4.5) the first row of the matrix A is

$$a_{0i} = \frac{m_i}{M_0}, \qquad i = 0,N. \tag{4.14}$$

The choice of the other $\mathbf{b}_i, i = 1,N$, is not as simple as in the two-body case. Different choices have different advantages, and can be used for different purposes. We shall review in this and in the next section the most common coordinate systems used for the $(N+1)$-body problem.

Barycentric coordinates

The **barycentric coordinate** system uses the fact that a reference system with a constant velocity translation with respect to an inertial system is also inertial. Thus a reference system with $\mathbf{b}_0 = \mathbf{0}$ as origin and barycentric positions $\mathbf{b}_i = \mathbf{r}_i - \mathbf{b}_0$ for $i = 1,N$ is inertial, and the equation of motion is the same as eq. (4.1). The change to barycentric is not just a change of coordinates, but also a reduction of the dimension of the problem: we write three differential equations less. The barycentric coordinates of body 0 (e.g., the Sun) are not dynamical variables, but are deduced from the coordinates of the other bodies and $\mathbf{b}_0$, by eq. (4.5):

$$\mathbf{s} = \mathbf{s}(\mathbf{B}) = \mathbf{r}_0 - \mathbf{b}_0 = -\sum_{i=1}^{N} \frac{m_i}{m_0}\,\mathbf{b}_i, \tag{4.15}$$

where the first term is assumed to be zero. The equation of motion is

$$m_j\,\ddot{\mathbf{b}}_j = \sum_{i\neq j, i=1}^{N} \frac{G\,m_i\,m_j}{|\mathbf{b}_i - \mathbf{b}_j|^3}\,(\mathbf{b}_i - \mathbf{b}_j) + \frac{G\,m_0\,m_j}{|\mathbf{b}_j - \mathbf{s}|^3}\,(\mathbf{s} - \mathbf{b}_j) \qquad j = 1,\dots,N \tag{4.16}$$

and can be written in conservative form

$$m_j\,\ddot{\mathbf{b}}_j = -\frac{\partial \mathcal{V}(\mathbf{s},\mathbf{b}_1,\mathbf{b}_2,\dots,\mathbf{b}_n)}{\partial \mathbf{b}_j}, \qquad j = 1,N,$$

with the potential energy $\mathcal{V}(\mathbf{B}) = V(\mathbf{R}(\mathbf{B}))$, where the partial derivatives of $\mathcal{V}$ have to be computed before substituting $\mathbf{s} = \mathbf{s}(\mathbf{B})$. The integrals of energy and angular momentum have a less simple expression, including the contributions from $\dot{\mathbf{s}}$.

Barycentric coordinates are efficient to be used for numerical integrations: only the $3N$ equations (4.16) have to be integrated, and the only additional computation to be performed at each step is $\mathbf{s}$ according to (4.15). The computed orbit does not need to be used in barycentric coordinates: to change the output back to heliocentric coordinates is the normal procedure.

Barycentric coordinates need to be used when the inertial velocities are directly observable: this is the case when the radial velocity of some star is measured (either by radio astronomy, for pulsars, or by spectroscopy, for normal stars). This is used to detect the small velocity of the star as a result of the presence of a small companion, such as a planet, see Section 6.5. The measured radial velocity is the difference between $\dot{\mathbf{s}}$ of the star and $\dot{\mathbf{b}}_3$ of the Earth; to use heliocentric coordinates for the Earth would result in a serious mistake.[3] The barycentric coordinates also play a role in the general relativistic corrections to the Newton equation, see Section 6.6.

On the other hand, barycentric coordinates are seldom used in analytical developments and in theoretical discussions, because of the lack of symmetry of the equation and of the less simple expressions for the classical integrals.

Heliocentric coordinates

A possible choice to represent the motion of planets and asteroids is the use of **heliocentric coordinates**. These follow the same idea used in the two-body case, eq. (4.9), namely use the motion of bodies $j = 1, N$ relative to the one with index 0, usually the Sun. Since $m_0 \gg m_j,\ j = 1, N$, the Sun moves little, but this motion cannot be neglected in the differential equations. The positions are thus represented by the vectors $\mathbf{b}_i = \mathbf{r}_i - \mathbf{r}_0$ and the equation of motion can be simply derived from eq. (4.1), taking into account the non-inertial frame, that is adding the apparent force exactly opposite to the acceleration of the Sun times the mass of the body:

$$m_j\, \ddot{\mathbf{b}}_j = \sum_{i \neq j, i=0}^{N} \frac{G\, m_i\, m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} (\mathbf{r}_i - \mathbf{r}_j) - m_j \ddot{\mathbf{r}}_0 .$$

[3] It would lead to a pretended discovery of a companion with a period of one year!

The equation can be written in terms of the heliocentric vectors, since they contain only the differences $\mathbf{b}_i - \mathbf{b}_j = \mathbf{r}_i - \mathbf{r}_j$ and $\mathbf{b}_i = \mathbf{r}_i - \mathbf{r}_0$

$$m_j\,\ddot{\mathbf{b}}_j = -\frac{G\,m_0\;m_j}{|\mathbf{b}_j|^3}\,\mathbf{b}_j + \sum_{i\neq j,i=1}^{N} \frac{G\,m_i\;m_j}{|\mathbf{b}_i - \mathbf{b}_j|^3}\,(\mathbf{b}_i - \mathbf{b}_j) - m_j\ddot{\mathbf{r}}_0.$$

The value of the acceleration of the Sun, resulting from the gravitational attraction of all the planets, is obtained from eq. (4.1) for $j = 0$; by substituting into the equation and removing the common factor m_j

$$\ddot{\mathbf{b}}_j = -\frac{G\,m_0}{|\mathbf{b}_j|^3}\,\mathbf{b}_j + \sum_{i\neq j,i=1}^{N} \frac{G\,m_i}{|\mathbf{b}_i - \mathbf{b}_j|^3}\,(\mathbf{b}_i - \mathbf{b}_j) - \sum_{i=1}^{N} \frac{G\,m_i}{|\mathbf{b}_i|^3}\,\mathbf{b}_i. \tag{4.17}$$

The equations above allow us to compute a solution for each heliocentric vector $\mathbf{b}_i,\ i = 1, n$, without the need to compute the position of the Sun in an inertial frame. Taking into account that in the acceleration of the Sun there is also a component due to the same planet

$$\ddot{\mathbf{b}}_j = -\frac{G\,(m_0 + m_j)}{|\mathbf{b}_j|^3}\,\mathbf{b}_j + \sum_{i\neq j,i=1}^{N} \frac{G\,m_i}{|\mathbf{b}_i - \mathbf{b}_j|^3}\,(\mathbf{b}_i - \mathbf{b}_j) - \sum_{i\neq j,i=1}^{N} \frac{G\,m_i}{|\mathbf{b}_i|^3}\,\mathbf{b}_i. \tag{4.18}$$

In this way the equation of motion is split into the two-body part, with the planet orbiting around a fixed center with mass $m_0 + m_j$ (as in the reduction of a two-body problem with the Sun and the planet j only), the **direct perturbations** by the attraction of the other planets, and the **indirect perturbations**, resulting from the other planets accelerating the Sun.

The heliocentric coordinates are a natural choice for Solar System orbits. The relative positions $\mathbf{r}_j - \mathbf{r}_k = \mathbf{b}_j - \mathbf{b}_k$ generate the only quantities observable inside our Solar System, e.g., the direction angles in optical astrometry and the range and range-rate in radar observations. The center of mass $\mathbf{b}_0$ and the barycentric position $\mathbf{s}$ of the Sun are derived quantities containing the mass ratios m_j/m_0. Thus, a catalog of asteroid orbital elements, computed from Cartesian coordinates in a barycentric system, would contain values dependent upon the planetary masses: every time the masses are corrected, the catalog should be revised. If the orbital elements are computed from heliocentric coordinates, there is no need for revision when the estimated values of the planetary masses change, with the exception of the asteroids having close approaches to a planet whose mass has been revised.

4.4 Jacobian coordinates

The **Jacobian coordinates** are obtained by selecting, among the linear coordinate changes of the form (4.13), the ones with the center of mass as first vector, thus fulfilling eq. (4.14), with the simplest equation of motion. This requires a matrix A, thus a set of *Jacobian vectors* $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N$, and a set of reduced masses $M_0, M_1, M_2, \ldots, M_N$ with the properties

[1] the first vector $\mathbf{b}_0$ is the center of mass, M_0 is the total mass;

[2] the Lagrange equation in the $\mathbf{R}$ coordinates is transformed into the Lagrange equation in the Jacobian coordinates of the same form:

$$m_i\, \ddot{\mathbf{r}}_i = -\frac{\partial V}{\partial \mathbf{r}_i} \Longleftrightarrow M_i\, \ddot{\mathbf{b}}_i = -\frac{\partial \mathcal{V}}{\partial \mathbf{b}_i}$$

where $\mathcal{V}(\mathbf{B}) = V(\mathbf{R})$ is the potential energy in the Jacobian coordinates.

The conditions on A resulting from [1] are given in (4.14), the ones resulting from [2] require that the kinetic energy remains in diagonal form:

$$2T = \sum_{i=0}^{N} m_i\, |\dot{\mathbf{r}}_i|^2 = \sum_{j=0}^{N} M_j\, |\dot{\mathbf{b}}_j|^2;$$

then the Jacobian momentum is $M_j\, \dot{\mathbf{b}}_j$ and the equation is in the simple form required by [2]. By substituting eq. (4.13) in the above formula

$$2T = \sum_{i,k=0}^{N} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_k \sum_{j=0}^{N} a_{ji} M_j a_{jk} = \sum_{i,k=0}^{N} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_k\, m_i \delta_{ik}$$

where $\delta_{ik} = 1$ for $i = k$, and $\delta_{ik} = 0$ for $i \neq k$. Thus the equations for A are

$$m_i \delta_{ik} = \sum_{j=0}^{N} a_{ji} M_j a_{jk} \qquad i, k = 0, N. \tag{4.19}$$

In matrix form, if m, M are the diagonal matrices with the masses and the reduced masses, respectively, as coefficients

$$m = \mathrm{diag}[m_0, m_1, \ldots, m_N], \qquad M = \mathrm{diag}[M_0, M_1, \ldots, M_N]$$

then eq. (4.19) can be written with A^T, the transposed matrix

$$m = A^T\, M\, A. \tag{4.20}$$

The Jacobian coordinates have another property, which is a consequence of [2]: the total angular momentum (4.6) has also a simple expression

$$\mathbf{c} = \sum_{i=0}^{N} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i = \sum_{j=0}^{N} \mathbf{b}_j \times M_j \dot{\mathbf{b}}_j,$$

i.e., the total angular momentum of the $(N+1)$-body system is the angular momentum of the free motion of the center of mass $\mathbf{b}_0 \times M_0 \dot{\mathbf{b}}_0$ plus the sum of the angular momentum of the two-body subsystems $\mathbf{b}_j \times M_j \dot{\mathbf{b}}_j$, $j = 1, \ldots, N$.

Equation (4.20) implies $\det(m) = \det(M) \det(A)^2$, where the determinants of m, M are the product of all masses and the product of all reduced masses, respectively. Thus [1] and [2] allow rescaling of the masses; a change of orientation is also possible. To avoid this, two additional properties have to be added to the definition of Jacobian coordinates:

[3] the product of the masses is equal to the product of the reduced masses

$$\prod_{i=0}^{N} m_i = \prod_{j=0}^{N} M_j; \tag{4.21}$$

[4] the linear transformation defined by A preserves orientation: $\det(A) > 0$.

Properties [2], [3], and [4] imply $\det(A) = +1$.

Existence and conditional uniqueness of Jacobian coordinates

If the transformation (4.13) fulfills [1], [2], [3], and [4], it defines a system of Jacobian coordinates. Matrices A with all these properties exist but they are not unique for a given N and for the given set of masses m_i. To obtain a unique selection we proceed as follows.

Let $\mathbf{b}_0^N, \ldots, \mathbf{b}_N^N$ be a set of Jacobian vectors satisfying [1]–[4], with reduced masses $M_0^N, \ldots, M_N^N$. Let $m_{N+1}, \mathbf{r}_{N+1}$ be the mass and position of an additional body. Then there are unique Jacobian coordinates, satisfying [1]–[4], with N unchanged Jacobian vectors and N unchanged reduced masses

$$\mathbf{b}_j^{N+1} = \mathbf{b}_j^N, \qquad M_j^{N+1} = M_j^N \qquad j = 1, N.$$

The new reduced masses are

$$M_{N+1} = \frac{m_{N+1} M_0^N}{M_0^{N+1}}, \qquad M_0^{N+1} = M_0^N + m_{N+1} \tag{4.22}$$

and the new Jacobian vectors are

$$\mathbf{b}_{N+1} = \mathbf{r}_{N+1} - \mathbf{b}_0^N, \qquad \mathbf{b}_0^{N+1} = \frac{1}{M_0^{N+1}} \sum_{j=0}^{N+1} m_j \, \mathbf{r}_j. \tag{4.23}$$

This can be shown by comparing eqs. (4.19) and (4.21) for $N+1$ and $N+2$ bodies (Milani and Nobili 1983).

The solutions (4.23) and (4.22) can be described as follows. A Jacobian coordinate system is a way to decompose an $(N+1)$-body system into free motion of the center of mass and N two-body subsystems. To add a new body, the new Jacobian vector is the position of the new body $\mathbf{r}_{N+1}$ relative to the center of mass $\mathbf{b}_0^N$ of the previous system, and the new reduced mass is the harmonic mean of the new mass m_{N+1} and of the previous total mass M_0^N. This generalizes the reduction of the two-body problem (4.9), (4.10).

As for uniqueness, the reduction of the two-body problem to the central force problem gives the Jacobian coordinates for $N+1 = 2$ bodies. However, if the list of bodies was $\{\mathbf{r}_1, \mathbf{r}_0\}$ the Jacobian vector would be $\mathbf{b}_1 = \mathbf{r}_0 - \mathbf{r}_1$. For $N+1 = 3$ the standard solution is to first couple (m_0, m_1), that is

$$\mathbf{b}_1 = \mathbf{r}_1 - \mathbf{r}_0, \qquad M_1 = \frac{m_0 \, m_1}{m_0 + m_1}$$

then use the vector $\mathbf{b}_2$ relative to the center of mass of (m_0, m_1), that is

$$\mathbf{b}_2 = \mathbf{r}_2 - \frac{m_0}{m_0 + m_1} \, \mathbf{r}_0 - \frac{m_1}{m_0 + m_1} \, \mathbf{r}_1, \qquad M_2 = \frac{m_2 \, (m_0 + m_1)}{m_0 + m_1 + m_2}.$$

This solution is not unique: it is possible to form first the binary (m_2, m_0), that is $\mathbf{b}_1 = \mathbf{r}_0 - \mathbf{r}_2$ and then join $\mathbf{r}_1$ to the center of mass of (m_2, m_0). A third solution corresponds to the sequence of couplings $((m_1, m_2), m_0)$; there are three more solutions violating [4].

The choice of a solution depends upon the sequence of coupling operations, which can be represented by a symbol like $((m_0, m_1), m_2)$ for the standard three-body solution. At a purely formal level, each of the $(N+1)!$ ways to order the $N+1$ bodies results, by applying recursively the procedure above, in a set of Jacobian coordinates. When the relative size of the perturbation is computed, as in the next section, the solutions are found to be by no means equivalent. As an example, if m_0 corresponds to the Sun, m_1 to the Earth, m_2 to the Moon, the best Jacobian system is the one with $((m_1, m_2), m_0)$, that is the center of mass of the Earth–Moon system is orbiting around the Sun, while the Moon is orbiting around the Earth–Moon center of mass.

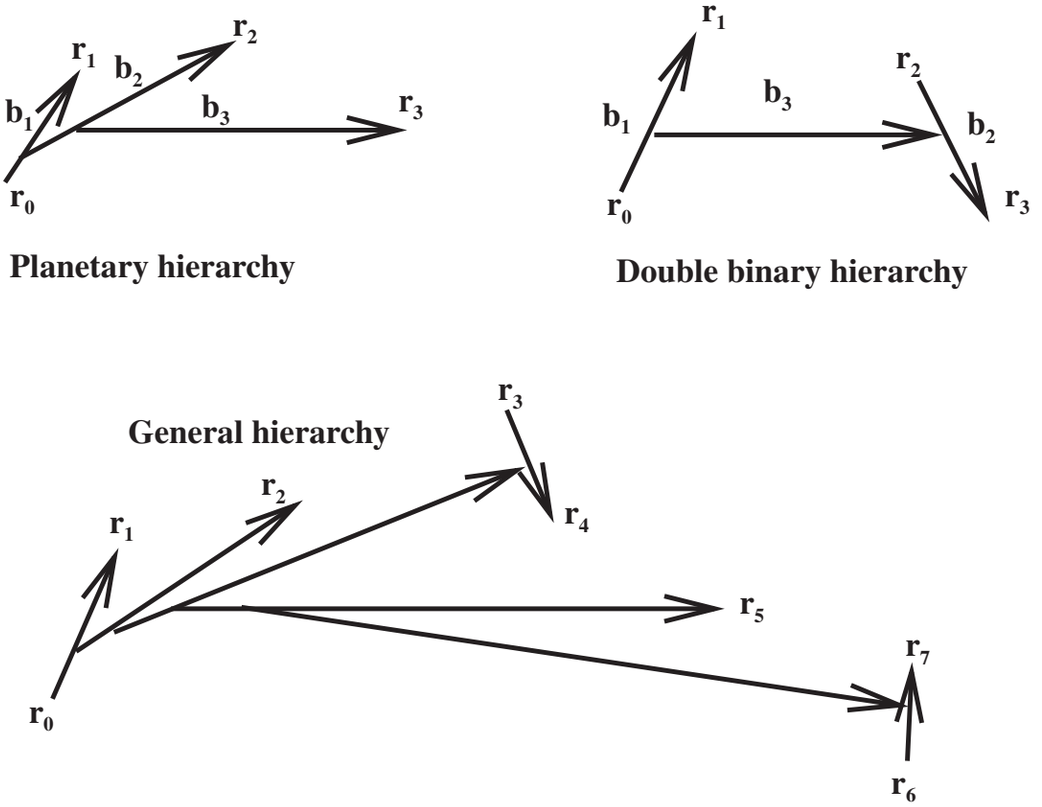

Fig. 4.1. Three examples of hierarchies and of the corresponding Jacobian vectors. The planetary hierarchy and the double binary hierarchy are described in the text. The hierarchy in the lower part of the figure could be used to describe a planetary system around the star $\mathbf{r}_0$, with planets $\mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_5$ and $\mathbf{r}_6$; planet $\mathbf{r}_3$ has one satellite, planet $\mathbf{r}_6$ has two satellites.

Planetary and binary type hierarchies

The non-uniqueness of the Jacobian coordinates becomes more significant for $N+1=4$ bodies. Let us assume some Jacobian coordinates have been selected for the first three bodies, e.g., according to the coupling order $((m_0,m_1),m_2)$. When a body $m_4, \mathbf{r}_4$ is added, there are two options. One is the recursive procedure of the previous subsection, that is $\mathbf{b}_3 = \mathbf{r}_3 - \mathbf{b}_0^3$. The other is to set $\mathbf{b}_2 = \mathbf{r}_3 - \mathbf{r}_2$ and to replace $\mathbf{r}_2$ with the center of mass of the binary (m_2, m_3), that is to use as $\mathbf{b}_3$ the vector joining the two centers of mass of the binary subsystems (m_0, m_1) and (m_2, m_3)

$$\mathbf{b}_1 = \mathbf{r}_1 - \mathbf{r}_0, \quad \mathbf{b}_2 = \mathbf{r}_3 - \mathbf{r}_2, \quad \mathbf{b}_3 = [(1-\mu_2)\,\mathbf{r}_2 + \mu_2\,\mathbf{r}_3] - [(1-\mu_1)\,\mathbf{r}_0 + \mu_1\,\mathbf{r}_1]$$

where $\mu_2 = m_3/(m_2+m_3)$. Then the reduced mass M_2 is the harmonic mean of the masses m_2 and m_3, M_3 is the harmonic mean of the masses (m_0+m_1) and (m_2+m_3):

$$M_1 = \frac{m_0\,m_1}{m_0+m_1}, \quad M_2 = \frac{m_2\,m_3}{m_2+m_3}, \quad M_3 = \frac{(m_0+m_1)\,(m_2+m_3)}{m_0+m_1+m_2+m_3}.$$

The first option is called a **planetary hierarchy** and is represented by the

coupling symbol $(((m_0, m_1), m_2), m_3)$; the second is a **double binary hierarchy** and is represented by $((m_0, m_1), (m_2, m_3))$. Formally, both choices are equivalent, in that both provide a Jacobian coordinate system satisfying [1], [2], [3], and [4]. The planetary hierarchy suggests that all the "planets" of masses m_1, m_2, m_3 orbit around the "star" of much larger mass m_0, at increasing distances $|\mathbf{r}_1 - \mathbf{r}_0|$, $|\mathbf{r}_2 - \mathbf{r}_0|$, and $|\mathbf{r}_3 - \mathbf{r}_0|$. The double binary hierarchy suggests that the "interior planet" m_1 orbits around the "star" m_0 at a smaller distance than the "exterior planet" m_2, the latter having a "satellite" m_3. To give rigorous meaning to this suggestion, we need to show that dynamical configurations, with different mass and distance ratios, are better represented in either one or the other hierarchy.

In general, given two subsystems with N' and N'' bodies, each with Jacobian coordinates, centers of mass $\mathbf{b}'_0$, $\mathbf{b}''_0$ and total masses M'_0, M''_0, respectively, there is a Jacobian system for the joint system of $N'+N''$ masses with a new Jacobian vector joining the two centers of mass, and a new reduced mass equal to the harmonic mean of the two total masses

$$\mathbf{b}_{N'+N''} = \mathbf{b}''_0 - \mathbf{b}'_0, \qquad M_{N'+N''} = \frac{M'_0\, M''_0}{M'_0 + M''_0};$$

$\mathbf{b}_0$ is the center of mass of all bodies, and the other $(N' - 1) + (N'' - 1)$ vectors coincide with the previously defined ones. This is the only way to combine the two subsystems, preserving $N' + N'' - 2$ Jacobian vectors (not including the centers of mass of the subsystems). In this way we can build a Jacobian system for an arbitrary coupling symbol. For example, for the hierarchy shown in the lower portion of Figure 4.1 the coupling symbol is $(((((m_0, m_1), m_2), (m_3, m_4)), m_5), (m_6, m_7))$.