dal libro di ANDREA MILANI e
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(in preparazione)

Chapter 1

THE EQUATION OF MOTION

1.1 The asteroid many-body problem

Given the point masses[1] m_i located at positions $\mathbf{x}_i$, with velocities $\dot{\mathbf{x}}_i$, for $i = 0, 1, \ldots, S$, the forces acting between the two bodies with indexes i and j, due to their mutual gravitational attraction, are given by

$$\mathbf{F}_{ij} = \frac{G\, m_j\, m_i}{x_{ij}^3}\,(\mathbf{x}_j - \mathbf{x}_i) \quad , \quad \mathbf{F}_{ji} = \frac{G\, m_i\, m_j}{x_{ji}^3}\,(\mathbf{x}_i - \mathbf{x}_j) \tag{1.1}$$

where $x_{ij} = |\mathbf{x}_j - \mathbf{x}_i| = x_{ji}$. They represent the attraction of body j by body i, and of body i by body j, respectively. Thus the attraction force is along the line joining the positions of the two bodies, of intensity inversely proportional to the square of the distance, and fulfills the **action-reaction principle**, that is $\mathbf{F}_{ij} + \mathbf{F}_{ji} = \mathbf{0}$. By the **superposition principle** the total force acting on each body i is just the sum of all the force vectors resulting from the attraction of all the other bodies, forming the **equation of motion** for body i

$$m_i\, \ddot{\mathbf{x}}_i = \sum_{j \neq i, j=0}^{S} \frac{G\, m_j\, m_i}{x_{ij}^3}\,(\mathbf{x}_j - \mathbf{x}_i)\ ; \tag{1.2}$$

this applies for each $i = 0, 1, \ldots, S$. Note that by the **Einstein equivalence principle** the m_i appearing in the left hand side is the same as the one on the right hand side of the equation. By the action-reaction principle the constant G appearing in $\mathbf{F}_{ij}$ and in $\mathbf{F}_{ji}$ must be the same, and by the equivalence principle it is also the same for all bodies. This implies that of the three possible definitions of mass (inertial, gravitational active and gravitational passive) only one appears in the equation of motion: to simlify the notation we shall use the gravitational active masses $\mu_j = G\, m_j$. After multiplying the equation of motion above by the constant G, all the m_i and G itself disappear:

$$\mu_i\, \ddot{\mathbf{x}}_i = \sum_{j \neq i, j=0}^{S} \frac{\mu_j\, \mu_i}{x_{ij}^3}\,(\mathbf{x}_j - \mathbf{x}_i) \tag{1.3}$$

[1] Point masses are approximations for the gravity field of an extended body, represented by the attraction of the total mass of the body concentrated in its center of mass. It can be shown that this approximation is good enough when the dimensions of the bodies are negligible with respect to their mutual distances, that is for long range perturbations such as the ones acting on asteroids.

Indeed the μ_j are the only masses which can be considered in Celestial Mechanics[2], thus we can solve for the accelerations:

$$\ddot{\mathbf{x}}_i = \sum_{j\neq i, j=0}^{S} \frac{\mu_j}{x_{ij}^3} (\mathbf{x}_j - \mathbf{x}_i). \tag{1.4}$$

In our solar system, if the index 0 stands for the Sun, the first N positive indexes are for the major planets, the last M for asteroids and other minor bodies (with $S = N+M$), and we have that $\mu_0 >> \mu_j$ $(j = 1, N) >> \mu_k$ $(k = N+1, S)$. In fact $\mu_j < 10^{-3}\,\mu_0$ and $\mu_k < 5\times10^{-10}\,\mu_0$, thus the ratios μ_i/μ_0 are **small parameters** and our problem belongs to a class of small perturbation theories. Of course there are two different levels of smallness, the asteroids masses being of higher order of smallness with respect to the planetary masses.

As a consequence of the dominace of the solar mass μ_0, it is convenient to use **heliocentric coordinates**:

$$\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_0 \quad , \quad r_i = |\mathbf{r}_i| \quad , \quad \mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i = \mathbf{x}_j - \mathbf{x}_i \quad , \quad r_{ij} = |\mathbf{r}_{ij}| = x_{ij} \,;$$

then, assuming the masses of the asteroids are negligible to the point that they do not count as sources of gravitational attraction, the equation of motion (1.4) becomes

$$\ddot{\mathbf{r}}_i = \sum_{j\neq i, j=0}^{N} \frac{\mu_j}{r_{ij}^3}\,\mathbf{r}_{ij} - \sum_{k=1}^{N} \frac{\mu_k}{r_k^3}\,\mathbf{r}_k \ ,$$

for $i = 1, S$. By isolating in the right hand side the largest term, the one with μ_0, and noting that in the heliocentric reference frame $\mathbf{r}_{i0} = \mathbf{r}_0 - \mathbf{r}_i = -\mathbf{r}_i$, we get

$$\ddot{\mathbf{r}}_i = -\frac{\mu_0}{r_i^3}\,\mathbf{r}_i + \sum_{j\neq i, j=1}^{N} \frac{\mu_j}{r_{ij}^3}\,\mathbf{r}_{ij} - \sum_{k=1}^{N} \frac{\mu_k}{r_k^3}\,\mathbf{r}_k \ . \tag{1.5}$$

The three terms on the right hand side are the **unperturbed 2-body acceleration**[3], the **direct perturbation**[4] and the **indirect perturbation**[5]. If the body i is a planet, the last sum contains also a $k = i$ term; it accounts for the fact that a heliocentric reference system is not an inertial one, that is the Sun is accelerated by all the planets (but not by the asteroids, because the corresponding accelerations are neglected).

1.2 Equation of motion for the restricted problem

The equation (1.2) refers to the hypothesis that the orbits of N planets and M asteroids have to be computed at once, which may be the case in a numerical integration. However, if we

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[2]The gravitational constant G, thus the masses m_i, cannot be determined from the planetary orbits. The only exception are the tests for the violation of the equivalence principle [Milani et al., 2010b].

[3]Acceleration due to gravitational attraction of the central body of the system (e.g. the Sun).

[4]Acceleration due to all other perturbing bodies in the system (e.g. solar system major planets).

[5]Acceleration $\ddot{\mathbf{x}}_0$ due to the change of position of the coordinate origin (e.g. heliocenter) due to the perturbations exerted on the central body by the other bodies in the system.

can indeed (because of the level of accuracy required) ignore the attraction from the asteroids[6], we are speaking of the **restricted problem**, in which the orbit of a given asteroid does not depend at all upon where the other asteroids are. Thus we can restrict ourselves to developing the theory of a single asteroid perturbed by the planets (hence the title of this book). If we are interested in computing the orbit of the asteroid $\mathbf{r} = \mathbf{r}_{N+1}$ only, then $S = N + 1$, and the equation of motion in heliocentric coordinates becomes

$$\ddot{\mathbf{r}} = -\frac{\mu_0}{r^3}\,\mathbf{r} + \sum_{i=1}^{N} \frac{\mu_i}{|\mathbf{r}_i - \mathbf{r}|^3}\,(\mathbf{r}_i - \mathbf{r}) - \sum_{i=1}^{N} \frac{\mu_i}{r_i^3}\,\mathbf{r}_i\ , \tag{1.6}$$

where $r = |\mathbf{r}|$: the direct and the indirect perturbations have sums of terms with the same indices, one for each planet.

The restricted problem is a good approximation because the asteroid mass is small, however by removing the terms with μ_{N+1} in the equations of motion for the planets, the action-reaction law by Newton is violated. In this way the asteroid does not contribute to the 10 classical scalar integrals of motion (energy, angular momentum, linear momentum and center of mass, see Section 1.3) and the equation (1.6) has no exact integral.

The equation (1.6) of the restricted problem can be derived from the Lagrange formalism: let the **kinetic energy** T, the **gravitational potential** $\mathcal{U}$ and the **Lagrange function** $\mathcal{L}$ for unit mass be

$$\mathcal{T} = \frac{1}{2}\,|\dot{\mathbf{r}}|^2 \quad , \quad \mathcal{U}_0 = \frac{\mu_0}{r} \tag{1.7}$$

$$\mathcal{U}_{DIR} = \sum_{i=1}^{N} \frac{\mu_i}{|\mathbf{r}_i - \mathbf{r}|} \quad , \quad \mathcal{U}_{IND} = -\sum_{i=1}^{N} \frac{\mu_i}{r_i^3}\,\mathbf{r}_i \cdot \mathbf{r} \tag{1.8}$$

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = \mathcal{T} + \mathcal{U} = \mathcal{T} + \mathcal{U}_0 + \mathcal{U}_{DIR} + \mathcal{U}_{IND} \tag{1.9}$$

Then the Lagrange equations of the restricted problem are defined by the conjugate momentum vector $\mathbf{p}$

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = \frac{\partial T}{\partial \dot{\mathbf{r}}} = \dot{\mathbf{r}} \quad , \quad \dot{\mathbf{p}} = \frac{\partial L}{\partial \mathbf{r}} = \frac{\partial U}{\partial \mathbf{r}} = \ddot{\mathbf{r}}\ . \tag{1.10}$$

The **Hamilton function** is defined by the **Legendre transform** (cite?)

$$\mathcal{H}(\mathbf{p}, \mathbf{r}) = \mathbf{p} \cdot \dot{\mathbf{r}} - \mathcal{L} = \frac{1}{2}\,|\mathbf{p}|^2 - \mathcal{U} = \mathcal{T} - \mathcal{U}$$

and can be decomposed into an unperturbed portion $\mathcal{H}_0$ and a perturbation part $\epsilon\,\mathcal{H}_1$ with small parameter $\epsilon \simeq Max_i(\mu_i/\mu_0)$

$$\mathcal{H} = \mathcal{H}_0 + \epsilon\mathcal{H}_1 \quad , \quad \mathcal{H}_0 = \frac{1}{2}\,|\mathbf{p}|^2 - \mathcal{U}_0 \quad , \quad \mathcal{H}_1 = -\frac{\mathcal{U}_{DIR} + \mathcal{U}_{IND}}{\epsilon}\ . \tag{1.11}$$

Thus equation (1.6), with its Lagrange and Hamilton equivalent

$$\ddot{\mathbf{r}} = \frac{\partial \mathcal{U}}{\partial \mathbf{r}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{r}}\ , \tag{1.12}$$

[6]This assumption may not be applicable in some extreme accuracy computation, such as the ones about predictions of impacts of an asteroid with a planet.

is the basic equation of motion we are discussing in this book. However, we need to assume that the equation of motion for the planets, that is (1.5) for $i = 1, N$, has been solved and the solution is available as a function of time t. Since this is by no means a trivial assumption, we need first to discuss the orbits of the planets. Thus in the following of this section we will give a general discussion of the complete equation (1.5) for $i = 1, N$, hence for the planets, which is an **autonomous equation**, that is it does not contain explicitly the time. Once the solution of the planetary motions is substituted in (1.6), the equation is not autonomous any more. If the two equations for the planets and the restricted one for the asteroid are considered as a single equation of motion for both, then it is autonomous.

1.3 First integrals for the planetary problem

1.3.1 Derivation from the equation of motion

To discuss the motion of the planets we have to return to the equation of motion in an inertial reference system (1.2), restricted to $N+1$ bodies (with $S = N$). There are $N+1$ such second order differential equations, one for each of the $i = 0, 1, ..., N$ bodies. Thus, their complete integration should give 2(N+1) 3-vector integrals, or 6(N+1) corresponding scalar integrals.

Independently of what is the number of bodies in the system, it is possible to find only three 3-vector integrals and one scalar integral, for a total of 10 scalar first integrals: these are the **first integrals** of the many-body problem.

Let us begin derivation of the general integrals by noting that the right hand sides of equations (1.2), for an arbitrary pair of bodies i and j, consist of "symmetrical" terms $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$, see eq. (1.1), which have zero vector sum, and that such couples exist for all the combinations of indexes. Hence, by summing all $(N+1)$ the equations (1.2) up, we get

$$\sum_{i=0}^{N} \mu_i \, \ddot{\mathbf{x}}_i = 0 \; . \tag{1.13}$$

Setting $M_0 = \sum_{i=0}^{N} \mu_i$, the **total mass**, the integration of (1.13) gives

$$\sum_{i=0}^{N} \mu_i \, \dot{\mathbf{x}}_i = M_0 \, \dot{\mathbf{b}}_0 \tag{1.14}$$

where $\dot{\mathbf{b}}_0$ is a constant vector independent from time. Integrating one more time

$$\sum_{i=0}^{N} \mu_i \mathbf{x}_i = M_0 \, (\mathbf{b}_0 + \dot{\mathbf{b}}_0 \, t) \tag{1.15}$$

where $\mathbf{b}_0$ is another constant vector. The two vectors represent the first two 3-vector integrals of the many-body problem: they are called the **center of mass integrals**. $\mathbf{b}_0$ is the position vector at $t = 0$ of the center of mass which moves with uniform velocity $\dot{\mathbf{b}}_0$. Note that all of the above is a consequence of the action-reaction principle only.

Multiplying vectorially equations (1.2) for $i = 0, 1, ..., N$ in turn with $\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_N$, and summing them up, each symmetrical couple on the right hand side gives

$$\frac{\mu_i\,\mu_j}{x_{ij}^3}\,\{\mathbf{x}_i \times (\mathbf{x}_j - \mathbf{x}_i) + \mathbf{x}_j \times (\mathbf{x}_i - \mathbf{x}_j)\} = \frac{\mu_i\,\mu_j}{x_{ij}^3}\,\{\mathbf{x}_i \times \mathbf{x}_j + \mathbf{x}_j \times \mathbf{x}_i\} = 0\,.$$

Hence

$$\sum_{i=0}^{N} \mu_i\,(\mathbf{x}_i \times \ddot{\mathbf{x}}_i) = 0\ . \tag{1.16}$$

Since, in general,

$$\frac{d}{dt}(\mathbf{x}_i \times \dot{\mathbf{x}}_i) = \mathbf{x}_i \times \ddot{x}_i\ ,$$

the equation (1.16) can be replaced with

$$\frac{d}{dt}\sum_{i=0}^{N} \mu_i\,(\mathbf{x}_i \times \dot{\mathbf{x}}_i) = 0\ ,$$

which is straightforwardly integrated to give

$$\sum_{i=0}^{N} \mu_i\,(\mathbf{x}_i \times \dot{\mathbf{x}}_i) = \mathbf{c}\,. \tag{1.17}$$

The obtained constant 3-vector **c** is called the **area integral**[7].

Multiplying scalarly equations (1.2) for $i = 0, 1, ..., N$ in turn with $d\mathbf{x}_0, d\mathbf{x}_1, ..., d\mathbf{x}_N$, and summing them up, the couples on the right hand sides give

$$\frac{\mu_i\,\mu_j}{x_{ij}^3}\,\{d\mathbf{x}_i \cdot (\mathbf{x}_j - \mathbf{x}_i) + d\mathbf{x}_j \cdot (\mathbf{x}_i - \mathbf{x}_j)\} = -\,\frac{\mu_i\,\mu_j}{x_{ij}^3}\,(\mathbf{x}_j - \mathbf{x}_i) \cdot (d\mathbf{x}_j - d\mathbf{x}_i)\,.$$

Since

$$(\mathbf{x}_j - \mathbf{x}_i) \cdot (d\mathbf{x}_j - d\mathbf{x}_i) = \mathbf{x}_{ij}\,d\mathbf{x}_{ij} = x_{ij} d\,x_{ij}\ ,$$

each couple reduces to

$$\frac{\mu_i\,\mu_j}{x_{ij}^2}\,dx_{ij}$$

and we end up with

$$\sum_{i=0}^{N} \mu_i\,\ddot{\mathbf{x}}_i \cdot d\mathbf{x}_i = -\sum_{i=0}^{N-1}\sum_{j=i+1}^{N} \frac{\mu_i\,\mu_j}{x_{ij}^2}\,dx_{ij}\,. \tag{1.18}$$

[7] $\mathbf{x}_i \times \dot{\mathbf{x}}_i$ in the two-body problem (Section 1.4) represents twice the oriented area which position vector $\mathbf{x}_i$ of the mass μ_i sweeps in unit time. Equation (1.17) states that the vectorial sum of these oriented areas, sweeped by vectors $\mathbf{x}_i$ and multipled with corresponding masses μ_i, is a constant vector.

In the right hand side of the equation above every combination of masses μ_i and μ_j appears only once, because each of the above couples provides a single term.

The equation (1.18) can also be easily integrated. Its right hand side is equal to the differential of the scalar expression

$$U = \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \frac{\mu_i \, \mu_j}{x_{ij}} \tag{1.19}$$

which is called the **force function** of the material system. Since

$$\frac{d^2 \mathbf{x}_i}{dt^2} \, d\mathbf{x}_i = \frac{d\dot{\mathbf{x}}_i}{dt} \, d\mathbf{x}_i = \dot{\mathbf{x}}_i \, d\dot{\mathbf{x}}_i = \dot{x}_i \, d\dot{x}_i$$

equation (1.18) can be replaced with

$$\sum_{i=0}^{N} \mu_i \, \dot{x}_i \, d\dot{x}_i = dU$$

which upon integration gives

$$\sum_{i=0}^{N} \frac{\mu_i \, \dot{x}_i^2}{2} = U + E \tag{1.20}$$

where E represents the integration constant.

The scalar integral (1.20) is called the **vis viva (energy) integral**. Together with 3-vectors $\mathbf{b}_0$ from (1.15), $\dot{\mathbf{b}}_0$ from (1.14), and $\mathbf{c}$ from (1.17), it completes the list of 10 scalar general integrals of the many-body problem.

1.3.2 Derivation via the Lagrange function

Again we have to begin from the equation of motion in an inertial reference system (1.2), for which we have to find the corresponding Lagrange function L, with kinetic energy T and gravitational potential U. Note that to compute the integrals of motion we assume all bodies have mass, that is either $S = N$, or if an asteroid is included $S = N + 1$, although the terms with μ_{N+1} give a negligible contribution.

$$T = \frac{1}{2} \sum_{j=0}^{N} \mu_j |\dot{\mathbf{x}}_j|^2 \quad , \quad U = \sum_{0 \le j < i \le N} \frac{\mu_i \, \mu_j}{|\mathbf{x}_i - \mathbf{x}_j|} \quad , \quad L = T + U \; , \tag{1.21}$$

where L is a function of all the positions $\mathbf{x}_j$ and all the velocities $\dot{\mathbf{x}}_j$. The momenta vectors and Lagrange equations are

$$\mathbf{p}_j = \frac{\partial T}{\partial \dot{\mathbf{x}}_j} = \mu_j \dot{\mathbf{x}}_j \quad , \quad \dot{\mathbf{p}}_j = \frac{\partial U}{\partial \mathbf{x}_j} = \mu_j \ddot{\mathbf{x}}_j \; . \tag{1.22}$$

It is easy to check that this Lagrangian is invariant with respect to a group of symmetries, namely the tranformations of the 3 dimensional space of each $\mathbf{x}_j$ that are **isometries**. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map $\mathbf{x} \mapsto A\,\mathbf{x} + \mathbf{d}$, with A a 3×3 matrix in the group $O(3)$ of orthogonal tranformations, that is $A^{-1} = A^T$, and $\mathbf{d}$ a constant vector. If this transformation is applied to the positions of all the $N+1$ bodies, and the map $\dot{\mathbf{x}} \mapsto A\,\dot{\mathbf{x}}$ to all veocities, then all the distances x_{ij} are conserved, and the length of the velocity vectors $\dot{\mathbf{x}}_i$ are conserved too; this implies that also the Lagrange function is invariant

$$L(\mathbf{x}_0, \ldots, \mathbf{x}_N, \dot{\mathbf{x}}_0, \ldots, \dot{\mathbf{x}}_N) = L(A\,\mathbf{x}_0 + \mathbf{d}, \ldots, A\,\mathbf{x}_n + \mathbf{d}, A\,\dot{\mathbf{x}}_0, \ldots, A\,\dot{\mathbf{x}}_N) \ .$$

A 1–parameter **group of symmetries** of the Lagrange function L is a diffeomorphism $\mathbf{F}^s$ of the positions $\mathbf{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_N)$ depending (in a differentiable way) upon a parameter $s \in \mathbb{R}$ so that $\mathbf{F}^s \circ \mathbf{F}^z = \mathbf{F}^{s+z}$, $\mathbf{F}^0$ is the identity transformation, and $\mathbf{F}^{-s} = [\mathbf{F}^s]^{-1}$. Moreover, the Lagrange function is invariant:

$$L\left(\mathbf{F}^s(\mathbf{X}), \frac{d}{dt}\mathbf{F}^s(\mathbf{X})\right) = L\left(\mathbf{F}^s(\mathbf{X}), \frac{\partial \mathbf{F}^s}{\partial \mathbf{X}}\dot{\mathbf{X}}\right) = L(\mathbf{X}, \dot{\mathbf{X}}) \ .$$

We also assume the mixed derivatives $\partial^2 F^s/\partial \mathbf{X}\partial s$ are continuous. A *local 1–parameter group of symmetries* of the Lagrange function is defined by the same properties for s in a neighborhood of 0. The main result we need is the **Noether theorem**, stating that if the Lagrange function L admits a local 1–parameter group of symmetries F^s then

$$I(\mathbf{X}, \dot{\mathbf{X}}) = \frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \left.\frac{\partial \mathbf{F}^s(\mathbf{X})}{\partial s}\right|_{s=0} \tag{1.23}$$

is a first integral of the Lagrange equation (1.22).

To prove this, let us compute the change in L because of $\mathbf{F}^s$ by a Taylor series expansion in s

$$L(\mathbf{F}^s(\mathbf{X}), \frac{d}{dt}\,\mathbf{F}^s(\mathbf{X})) - L(\mathbf{X}, \dot{\mathbf{X}}) = s\ \left[\frac{\partial L}{\partial \mathbf{X}} \cdot \left.\frac{\partial \mathbf{F}^s(\mathbf{X})}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \frac{\partial}{\partial s}\frac{d}{dt}\,\mathbf{F}^s(\mathbf{X})\Big|_{s=0}\right] + \mathcal{O}(s^2) \ .$$

Since this change in L is identically zero by hypothesis, the first order (in s) term must be zero: by exchanging the derivatives d/dt and $\partial/\partial s$

$$\begin{aligned} 0 \ &= \ \frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \frac{d}{dt}\left.\frac{\partial \mathbf{F}^s(\mathbf{X})}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial \mathbf{X}} \cdot \left.\frac{\partial \mathbf{F}^s(\mathbf{X})}{\partial s}\right|_{s=0} = \quad \text{(by the Lagrange equation)} \\ &= \ \frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \frac{d}{dt}\left.\frac{\partial \mathbf{F}^s(\mathbf{X})}{\partial s}\right|_{s=0} + \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{X}}}\left.\frac{\partial \mathbf{F}^s(\mathbf{X})}{\partial s}\right|_{s=0} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \left.\frac{\partial \mathbf{F}^s(\mathbf{X})}{\partial s}\right|_{s=0}\right] \end{aligned}$$

and the function defined in Eq. (1.23) is an integral.

Therefore Noether theorem applies to all the 1–parameter subgroups of the group of linear isometries. The simplest case is that of the 1–parameter groups of translations, e.g. the translations along one coordinate axis: $F^s(\mathbf{x}) = \mathbf{x} + s\,\mathbf{e}_h$, with $\mathbf{e}_h$ the unit vector along the axis $x_h, h = 1, 3$. If the symmetry group $\mathbf{F}^s(\mathbf{X})$ is defined by applying equal translations F^s to all bodies, then the **first integral** described by Neother's theorem is

$$P_h = \sum_{j=0}^{N} \left.\frac{\partial F^s(\mathbf{x}_j)}{\partial s}\right|_{s=0} \cdot \mathbf{p}_j = \hat{\mathbf{e}}_h \cdot \sum_{j=0}^{N} m_j\,\dot{\mathbf{x}}_j = \hat{\mathbf{e}}_h \cdot \mathbf{P}\ ,$$

that is the component along the axis h of the **total linear momentum P**. Thus $\mathbf{P}$ is a 3-vector integral, and the **center of mass** $\mathbf{b}_0$

$$\mathbf{b}_0 = \frac{1}{M_0}\sum_{j=0}^{N} \mu_j \mathbf{x}_j \ , \tag{1.24}$$

where M_0 is the total mass, moves with uniform velocity:

$$\dot{\mathbf{b}}_0 = \frac{1}{M_0}\mathbf{P} \ . \tag{1.25}$$

This leads to 3 scalar first integrals independent from time (the coordinates of $\dot{\mathbf{b}}_0$), plus 3 integrals dependent from time (the coordinates of $\mathbf{b}_0$).

Other 1–parameter subgroups of the group of isometries are the groups of rotations around a fixed axis. If F^s is the rotation by an angle of s radians around the unit vector $\mathbf{v}$

$$\left.\frac{\partial F^s(\mathbf{x})}{\partial s}\right|_{s=0} = \mathbf{v}\times\mathbf{x}$$

and the corresponding integral is:

$$c_h = \sum_{j=1}^{N} \left.\frac{\partial F^s(\mathbf{x}_j)}{\partial s}\right|_{s=0} \cdot \mathbf{p}_j = \sum_{j=1}^{N} (\mathbf{v}\times\mathbf{x}_j)\cdot\mathbf{p}_j = \mathbf{v}\cdot\sum_{j=1}^{N} \mathbf{x}_j\times\mathbf{p}_j = \mathbf{v}\cdot\sum_{j=1}^{N} \mu_j(\mathbf{x}_j\times\dot{\mathbf{x}}_j) \ ,$$

namely, the component along $\mathbf{v}$ of the total angular momentum

$$\mathbf{c} = \sum_{j=1}^{N} \mathbf{x}_j\times\mathbf{p}_j \ , \tag{1.26}$$

which is also a 3-vector integral, that is another 3 scalar integrals, for a total of 9 integrals deduced from the symmetry group of isometries.

The 10-th integral is the **energy integral** which can be computed as Hamilton function

$$H(\mathbf{p}_0,\ldots,\mathbf{p}_N,\mathbf{x}_0,\ldots,\mathbf{x}_N) = \sum_{j=0}^{N} \mathbf{p}_j\cdot\dot{\mathbf{x}}_j - L = \frac{1}{2}\sum_{j=0}^{N}\frac{|\mathbf{p}_j|^2}{\mu_j} - U(\mathbf{x}_0,\ldots,\mathbf{x}_N) = E \ . \tag{1.27}$$

It is well known (cite Poincaré) that besides these 10 integrals the $N+1$ body problem, as defined by either (1.21) or (1.27), has no other integrals. It also follows that the equation of motion of the restricted problem has no integrals, because the contribution of the asteroid to the integrals has been neglected (TBC?).

[Of course the Hamilton function defines the Hamilton equations, which are the equations of motion as a function of the time variable (taken as independent variable); this can be described by the statement that H, t are **conjugated variables**. Similarly, other integrals can be taken as Hamilton functions, and provide with the corresponding Hamilton equations the motion under the action of 1–parameter symmetry groups, e.g., c_h is the Hamiltonian of the rotation around the $\hat{\mathbf{e}}_h$ axis for $h=1,3$, with the rotation angle s (in radians) as independent variable, that is c_h, s are also conjugated variables; also $P_h, \mathbf{b}_0\cdot\hat{\mathbf{e}}_h$ for $h=1,3$.] [Maybe to be stated later in Chap.5, together with a formal definition of homogenization of time]

1.4 The 2-body problem

As the simplest example of the use of the first integrals to reduce the number of scalar equations in (1.22), and also for later reference, let us consider the 2-body problem with Lagrange function

$$L = \frac{1}{2}\mu_0\,|\dot{\mathbf{x}}_0|^2 + \frac{1}{2}\mu_1\,|\dot{\mathbf{x}}_1|^2 + \frac{\mu_0\,\mu_1}{|\mathbf{x}_1 - \mathbf{x}_0|}\ .$$

We can change coordinates by using, in place of $\mathbf{x}_0, \mathbf{x}_1$, the coordinates of the center of mass and the relative position of $\mathbf{x}_1$ with respect to $\mathbf{x}_0$

$$\mathbf{b}_0 = \epsilon_1\,\mathbf{x}_1 + (1-\epsilon_1)\mathbf{x}_0\ ,\quad \epsilon_1 = \frac{\mu_1}{\mu_0+\mu_1}\quad ,\quad \mathbf{b}_1 = \mathbf{x}_1 - \mathbf{x}_0\ . \tag{1.28}$$

Then $U(\mathbf{b}_1) = G\mu_0\mu_1/b_1$, with $b_1 = |\mathbf{b}_1|$; to write L as a function of $\mathbf{b}_0, \mathbf{b}_1$ we express $\dot{\mathbf{x}}_0$ and $\dot{\mathbf{x}}_1$ as a function of $\dot{\mathbf{b}}_0, \dot{\mathbf{b}}_1$ and substitute in T

$$\dot{\mathbf{x}}_0 = \dot{\mathbf{b}}_0 - \epsilon_1\dot{\mathbf{b}}_1 \quad , \quad \dot{\mathbf{x}}_1 = \dot{\mathbf{b}}_0 + (1-\epsilon_1)\dot{\mathbf{b}}_1$$

$$2T = \mu_0\,|\dot{\mathbf{x}}_0|^2 + \mu_1\,|\dot{\mathbf{x}}_1|^2 = (\mu_0+\mu_1)\,|\dot{\mathbf{b}}_0|^2 + \frac{\mu_0\,\mu_1}{\mu_0+\mu_1}\,|\dot{\mathbf{b}}_1|^2$$

the mixed terms canceling. The Lagrange function as a function of the new coordinates is

$$L(\mathbf{b}_0, \mathbf{b}_1, \dot{\mathbf{b}}_0, \dot{\mathbf{b}}_1) = \frac{1}{2}\,M_0\,|\dot{\mathbf{b}}_0|^2 + \frac{1}{2}\,M_1\,|\dot{\mathbf{b}}_1|^2 + \frac{M_0\,M_1}{b_1} \tag{1.29}$$

with $M_0 = \mu_0 + \mu_1$ the total mass, and M_1 the **reduced mass** (harmonic mean):

$$M_1 = \frac{\mu_0\,\mu_1}{\mu_0+\mu_1} \Longleftrightarrow \frac{1}{M_1} = \frac{1}{\mu_0} + \frac{1}{\mu_1}\ . \tag{1.30}$$

Then the Lagrange function L can be decomposed as the sum of two Lagrange functions $L = M_0\,L_0(\dot{\mathbf{b}}_0) + M_1\,L_1(\mathbf{b}_1, \dot{\mathbf{b}}_1)$, one containing only $\mathbf{b}_0$, the other containing only $\mathbf{b}_1$, and the Lagrange equations decouple:

$$M_0\,\ddot{\mathbf{b}}_0 = 0 \quad , \quad M_1\,\ddot{\mathbf{b}}_1 = \frac{\partial U(\mathbf{b}_1)}{\partial \mathbf{b}_1} = -\frac{M_0 M_1}{b_1^3}\mathbf{b}_1\ .$$

The first equation states that the center of mass moves with constant velocity along a straight line, the second equation is the **Kepler problem**, with a particle of mass M_1 attracted by a fixed center of mass M_0.

By repeating the same computations done above for T, we find that also the angular momentum has a simple expression in the $(\mathbf{b}_0, \mathbf{b}_1)$ coordinates:

$$\mathbf{c} = \mu_0\,\mathbf{x}_0 \times \dot{\mathbf{x}}_0 + \mu_1\,\mathbf{x}_1 \times \dot{\mathbf{x}}_1 = M_0\,\mathbf{b}_0 \times \dot{\mathbf{b}}_0 + M_1\,\mathbf{b}_1 \times \dot{\mathbf{b}}_1\ .$$

When $\mathbf{b}_0(t) = \dot{\mathbf{b}}_0\,t + \mathbf{b}_0(0)$ and eq. (1.25) are substituted, this reveals that the $\mathbf{b}_0$ contribution is constant and angular momentum decouples

$$\mathbf{c}_0 = \mathbf{b}_0 \times \dot{\mathbf{b}}_0 = \frac{1}{M_0}\,\mathbf{b}_0(0) \times \mathbf{P} \quad , \quad \mathbf{c} = M_0\,\mathbf{c}_0 + M_1\,\mathbf{c}_1\ .$$

The contribution from $\mathbf{b}_1$ is $\mathbf{c}_1 = \mathbf{b}_1 \times \dot{\mathbf{b}}_1$, the angular momentum per unit (reduced) mass of $\mathbf{x}_1$ with respect to the center $\mathbf{x}_0$; $\mathbf{c}_1$ is also a vector first integral, thus $\mathbf{b}_1, \dot{\mathbf{b}}_1$ will lie for each t in the orbital plane normal to $\mathbf{c}_1$.

The Laplace-Lenz vector and the energy integral

The 2-body problem has another vector integral, not occurring in the $N+1 \geq 3$-body problem: the **Laplace-Lenz vector**

$$\mathbf{e} = \frac{1}{M_0}\,\dot{\mathbf{b}}_1 \times \mathbf{c}_1 - \frac{1}{b_1}\,\mathbf{b}_1 \ . \tag{1.31}$$

which lies in the orbital plane and is directed towards the pericenter.

This can be shown by using a reference frame formed by three orthogonal unit vectors:

$$\mathbf{v}_z = \mathbf{c}_1/c_1 \quad , \quad \mathbf{v}_r = \mathbf{b}_1/b_1 \quad , \quad \mathbf{v}_\theta = \mathbf{v}_z \times \mathbf{v}_r \ ,$$

where $c_1 = |\mathbf{c}_1|$; we also use the time derivatives

$$\dot{\mathbf{v}}_r = \dot{\theta}\,\dot{\mathbf{v}}_\theta \quad , \quad \mathbf{v}_\theta = -\dot{\theta}\,\mathbf{v}_r \quad , \quad \dot{\mathbf{v}}_z = \mathbf{0} \ .$$

Let θ be the angle between the vector $\mathbf{v}_r$ and a fixed direction in the orbital plane, and $r = b_1$, then have

$$\mathbf{c}_1 = r\,\mathbf{v}_r \times \frac{d}{dt}(r\,\mathbf{v}_r) = r\,\mathbf{v}_r \times (\dot{r}\,\mathbf{v}_r + r\dot{\theta}\,\mathbf{v}_\theta) = r^2\,\dot{\theta}\,\mathbf{v}_r \times \mathbf{v}_\theta = r^2\,\dot{\theta}\,\mathbf{v}_z$$

$$\dot{\mathbf{b}}_1 = \dot{r}\,\mathbf{v}_r + r\,\dot{\theta}\,\mathbf{v}_\theta \ ,$$

which gives

$$M_0\,\mathbf{e} = (\dot{r}\,\mathbf{v}_r + r\,\dot{\theta}\,\mathbf{v}_\theta) \times r^2\,\dot{\theta}\,\mathbf{v}_z - M_0\,\mathbf{v}_r = -r^2\,\dot{r}\,\dot{\theta}\,\mathbf{v}_\theta + (r^3\,\dot{\theta}^2 - M_0)\,\mathbf{v}_r \ . \tag{1.32}$$

For $\mathbf{e}$ to be an integral its derivative must be equal to zero. Along the solutions we make use of the well know equations for the tangential and the radial acceleration, which give

$$\dot{\mathbf{c}}_1 = 0 \Longrightarrow 2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} = 0 \quad , \quad \ddot{r} = -\frac{M_0}{r^2} + \frac{c_1^2}{r^3} \ ,$$

so that

$$\begin{aligned} M_0\,\dot{\mathbf{e}} &= \frac{d}{dt}\left[\dot{\mathbf{b}}_1 \times \mathbf{c}_1 - M_0\,\mathbf{v}_r\right] = \ddot{\mathbf{b}}_1 \times \mathbf{c}_1 - M_0\,\dot{\theta}\,\mathbf{v}_\theta = \\ &= -M_0/r^2\,\mathbf{v}_r \times r^2\,\dot{\theta}\,\mathbf{v}_z - M_0\,\dot{\theta}\,\mathbf{v}_\theta = \\ &= -M_0\,\dot{\theta}\,(\mathbf{v}_r \times \mathbf{v}_z + \mathbf{v}_\theta) = \mathbf{0} \ . \end{aligned}$$

However, $\mathbf{e}$ corresponds to only two additional scalar integrals (not all components of $\mathbf{e}$ are independent from $\mathbf{c}_1$ because $\mathbf{e} \cdot \mathbf{c}_1 = 0$). We define the **true anomaly** f as the angle between $\mathbf{e}$ and $\mathbf{v}_r$ in the orbital plane, that is

$$e \cos f = \mathbf{e} \cdot \mathbf{v}_r = \frac{r^3\,\dot{\theta}^2}{M_0} - 1 = \frac{c_1^2}{M_0\,r} - 1$$

where $r^2\dot{\theta} = c_1$ is the (scalar) angular momentum of $\mathbf{b}_1$ and is constant[8]. From this there follows the familiar formula of a conic section

$$r = \frac{c_1^2/M_0}{1 + e\cos f} \ , \tag{1.33}$$

[8] Note that $r^2\,\dot{\theta}$ is equal to twice the sectorial velocity, thus $\mathbf{c}_1 = r^2\,\dot{\theta}\,\mathbf{v}_z$ represents the Kepler's second law in vector form.

and the interpretation of the two additional **two-body integrals** as **eccentricity** $e = |\mathbf{e}|$ and **argument of pericenter** ω, that is the angle of $\mathbf{e}$ (direction of pericenter) with a fixed direction in the orbital plane, in such a way that $\theta = f + \omega$.

The eccentricity e is an integral depending upon angular momentum and energy. The energy integral of the 2-body problem in $(\mathbf{b}_0, \mathbf{b}_1)$ coordinates is

$$E(\mathbf{b}_0, \mathbf{b}_1, \dot{\mathbf{b}}_0, \dot{\mathbf{b}}_1) = \frac{1}{2} M_0 |\dot{\mathbf{b}}_0|^2 + \frac{1}{2} M_1 |\dot{\mathbf{b}}_1|^2 - \frac{M_0 M_1}{|\mathbf{b}_1|} = M_0 E_0 + M_1 E_1 \tag{1.34}$$

$$E_0 = \frac{1}{2} |\dot{\mathbf{b}}_0|^2 \quad , \quad E_1 = \frac{1}{2} |\dot{\mathbf{b}}_1|^2 - \frac{M_0}{|\mathbf{b}_1|} = \frac{1}{2} (\dot{r}^2 + c_1 \dot{\theta}) - \frac{M_0}{r} \ , \tag{1.35}$$

and the eccentricity squared, computed from eq. (1.32), is

$$e^2 = \mathbf{e} \cdot \mathbf{e} = \frac{r^4 \dot{r}^2 \dot{\theta}^2 + \left(r^3 \dot{\theta}^2 - M_0 \right)^2}{M_0^2} = 1 + \frac{2 E_1 c_1^2}{M_0^2} \ . \tag{1.36}$$

If the energy of the relative motion E_1 is negative, then $e < 1$ and the trajectory of $\mathbf{b}_1$ is an ellipse with semimajor axis a; its relation with energy and angular momentum can be derived from (1.33) and (1.36:

$$a = \frac{q + Q}{2} = \frac{1}{2} \left[\frac{c_1^2 / M_0}{1 + e} + \frac{c_1^2 / M_0}{1 - e} \right] = \frac{c_1^2 / M_0}{1 - e^2} = \frac{M_0}{-2 E_1} \ , \tag{1.37}$$

where q, Q are the pericenter $(f = 0, q = a(1 - e))$ minimum and apocenter maximum $(f = \pi, Q = a(1 + e))$ distances. The scalar angular momentum per unit mass of the relative motion, from the same equation, is

$$c_1 = \sqrt{M_0 \, a \, (1 - e^2)} \ . \tag{1.38}$$

1.5 Barycentric coordinates

The set of positions of the $N + 1$ bodies can be represented in different coordinates; we are interested in the linear coordinate changes of the form

$$\mathbf{b}_i = \sum_{j=0}^{N} a_{ij} \, \mathbf{x}_j \quad , \quad A = (a_{ij}), \ i, j = 0, N \tag{1.39}$$

where the matrix A is a function of the masses only. The purpose is to exploit the integrals of the center of mass to reduce the number of equations, generalizing the results of the 2-body case. A natural choice is to use the center of mass as $\mathbf{b}_0$, thus by (1.24) the first row of the matrix A is

$$a_{0i} = \frac{\mu_i}{M_0} \quad , \quad i = 0, N \ , M_0 = \sum_{k=0}^{N} \mu_k \ . \tag{1.40}$$

The choice of the other $\mathbf{b}_i, i = 1, N$, is not as simple as in the 2-body case. Different choices have different advantages, and can be used for different purposes. We shall review in this and in the next section some coordinate systems especially useful for the $(N + 1)$-body problem.

The **barycentric coordinate** system uses the **Galileo equivalence principle**, stating that that two reference systems related by a constant velocity translation lead to the same equation of motion, implying that a system moving with a constant velocity translation with respect to an inertial system is also inertial. Thus a reference system with $\mathbf{b}_0 = \mathbf{0}$ as origin and barycentric positions $\mathbf{b}_i = \mathbf{x}_i - \mathbf{b}_0 = \mathbf{x}_i$ for $i = 1, N$ is inertial; the equation of motion is the same as eq. (1.2). However, in this approach the barycentric coordinates $\mathbf{s}$ of mass index 0 (i.e. the Sun) are not independent dynamical variables, but are deduced from the coordinates of the other bodies and $\mathbf{b}_0$, by eq. (1.24). Noting that in the barycentric system:

$$\sum_{i=0}^{N} \mu_i \, \mathbf{x}_i = \mathbf{0} \quad \Rightarrow \quad \mu_0 \mathbf{x}_0 = -\sum_{i=1}^{N} \mu_i \, \mathbf{x}_i$$

we get the equation for the barycentric coordinates of the Sun:

$$\mathbf{s} = \mathbf{s}_B(\mathbf{b}_1, \ldots, \mathbf{b}_N) = \mathbf{x}_0 - \mathbf{b}_0 = \mathbf{x}_0 = -\sum_{i=1}^{N} \frac{\mu_i}{\mu_0} \, \mathbf{b}_i \ . \tag{1.41}$$

The change to the barycentric system is not just a change of coordinates, but also a reduction of the dimension of the problem: we write 3 differential equations less (the ones for mass $i = 0$); moreover, the differential equations do not contain the indirect perturbation term. The reduced equation of motion is

$$\mu_i \, \ddot{\mathbf{b}}_i = \frac{\mu_0 \, \mu_i}{|\mathbf{b}_i - \mathbf{s}|^3} \, (\mathbf{s} - \mathbf{b}_i) + \sum_{j \neq i, j=1}^{N} \frac{\mu_j \, \mu_i}{|\mathbf{b}_j - \mathbf{b}_i|^3} \, (\mathbf{b}_j - \mathbf{b}_i) \qquad i = 1, \ldots, N \tag{1.42}$$

and can be expressed in conservative form

$$\mu_i \, \ddot{\mathbf{b}}_i = \frac{\partial U(\mathbf{s}, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N)}{\partial \mathbf{b}_j} \quad , \quad i = 1, N \ , \tag{1.43}$$

where U is the same potential defined in (1.21), and the partial derivatives of U have to be computed before substituting $\mathbf{s} = \mathbf{s}_B(\mathbf{b}_1, \ldots, \mathbf{b}_N)$. The integrals of energy and angular momentum have less simple expressions when written as a function of $(\mathbf{b}_1, \ldots, \mathbf{b}_N)$, because they include the contributions from $\dot{\mathbf{s}}$.

Barycentric coordinates are efficient to be used for numerical integrations[9]: only the $3N$ equations (1.42) have to be integrated, and the only additional computation to be performed at each step is $\mathbf{s}$ according to (1.41). On the other hand, barycentric coordinates are seldom used in analytical developments and in theoretical discussions, because of the lack of symmetry of the equation and of the less simple expressions for the classical integrals. This is not a problem because the numerically computed orbit does not need to be used in barycentric coordinates: to change the output back to heliocentric coordinates is the standard procedure.

[9] As an alternative approach, in a numerical integration it is possible to compute the full solution of eq. (1.2), then use $\mathbf{b}_0 = \dot{\mathbf{b}}_0 = \mathbf{0}$ as accuracy check. Besides the small increase in efficiency, which is not important with current computers, there are advantages in describing the general relativistic effects in barycentric coordinates, although the very definition of barycenter has to be modified to remain an integral.

1.6 Heliocentric Canonical Coordinates

To derive equation (1.5) from a single Lagrange (or Hamilton) function is not immediate, mostly because of the asymmetric indirect term. To solve this, Poincaré invented the **heliocentric canonical coordinates** $(\mathbf{r}_i, \mu_i\dot{\mathbf{b}}_i)$, in which the positions are heliocentric and the linear momenta are barycentric [Laskar 1989]. To show their properties, let us use a linear coordinate change

$$\mathbf{r}_i = \sum_{j=0}^{N} a_{ij}\mathbf{x}_j \quad , \quad A = (a_{ij}), i, j = 0, N$$

such that $\mathbf{r}_0 = \mathbf{x}_0 = \mathbf{s}$, that is $a_{0j} = \delta_{0j}$ and the others are heliocentric vectors: $\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_0$ for $i = 1, N$, that is $a_{ij} = \delta_{ij} - \delta_{i0}$ for $j = 0, N$ (the notation δ_{ij} stands for the Kronecker δ, $\delta_{ij} = 1$ if $i = j$, $= 0$ otherwise). To complete the transformation of the coordinates $\mathbf{x}_i$ with a linear change of the momenta $\mu_i\dot{\mathbf{x}}_i$ such that the new coordinates are canonical (see later in Section 2.1), we need to use the matrix B

$$\mathbf{p}_i = \sum_{j=0}^{N} b_{ij}\mu_j\dot{\mathbf{x}}_j \quad , \quad B = (b_{ij}), i, j = 0, N$$

such that $B = (A^{-1})^T$, that is $b_{0j} = 1$ and $b_{ij} = \delta_{ij}$ for $i \neq 0$. Then $\mathbf{p}_0 = \mathbf{P} = M_0\dot{\mathbf{b}}_0$ is the linear momentum integral. The other momentum vectors $\mathbf{p}_i = \mu_i\dot{\mathbf{x}}_i$, for $i = 1, N$, are unchanged with respect to the previous, inertial coordinate system..

To perform the reduction to $3N$ differential equations, we assume that the coordinates $\mathbf{x}_i$ had already been translated in such a way that $\mathbf{b}_0 = \mathbf{0}$ for all times t, thus also $\dot{\mathbf{b}}_0 = \mathbf{0} = \mathbf{p}_0$. Thus the momentum vectors $\mathbf{p}_i = \mu_i\dot{\mathbf{x}}_i$, for $i = 1, N$, are barycentric, and $\mathbf{r}_0 = \mathbf{s} = \mathbf{x}_0$ is given by a formula similar, but not the same as (1.41), because it is a function of heliocentric position vectors:

$$\mu_0\mathbf{s} = -\sum_{i=1}^{N} \mu_i\left(\mathbf{r}_i + \mathbf{s}\right) \ ,$$

by solving for $\mathbf{s}$:

$$\mathbf{s}_H(\mathbf{r}_1, \ldots, \mathbf{r}_N) = -\sum_{i=1}^{N} \frac{\mu_i}{M_0}\mathbf{r}_i \ . \tag{1.44}$$

The Lagrange function $L = T + U$ in the coordinates $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$ has to have the same value as the one in the $(\mathbf{x}_i, \dot{\mathbf{x}}_i)$ coordinates: for the kinetic energy

$$T = \frac{1}{2}\sum_{i=0}^{N} \mu_i|\dot{\mathbf{x}}_i|^2 = \frac{1}{2}\sum_{i=1}^{N} \mu_i|\dot{\mathbf{r}}_i + \dot{\mathbf{s}}|^2 + \frac{1}{2}\,\mu_0|\dot{\mathbf{s}}|^2 \ , \tag{1.45}$$

and by replacing $\dot{\mathbf{s}}$ with the value constrained by (1.44)

$$\dot{\mathbf{s}}_H(\dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N) = -\sum_{j=1}^{N} \frac{\mu_j}{M_0}\dot{\mathbf{r}}_j \tag{1.46}$$

we get $T = T(\dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N)$; we can check that

$$\mathbf{p}_i = \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \mu_i \left(\dot{\mathbf{r}}_i + \dot{\mathbf{s}}_H\right)$$

as claimed. U has the same expression as in the heliocentric coordinates, since $\mathbf{x}_i - \mathbf{x}_j = \mathbf{r}_i - \mathbf{r}_j$. Thus it is possible to derive the Lagrange equations and check that they are the same as (1.5): by collecting together the direct attraction from the Sun and the indirect term from the same planet being attracted:

$$\ddot{\mathbf{r}}_i = -\frac{\mu_0 + \mu_i}{r_i^3}\,\mathbf{r}_i + \sum_{j\neq i, j=1}^{N} \frac{\mu_j}{r_{ij}^3}\,\mathbf{r}_{ij} - \sum_{k\neq i, k=1}^{N} \frac{\mu_k}{r_k^3}\,\mathbf{r}_k \ . \tag{1.47}$$

[TBC: computations checking this from [Laskar 1989] pages 7-8, but the notations are different]

The improvement with respect to the conventional heliocentric variables is in the Hamiltonian formulation. To espress the kynetic energy $T(\dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N)$ as a function of the momenta $T_{\mathbf{p}}(\mathbf{p}_1, \ldots, \mathbf{p}_N)$ we substitute in T given by (1.45) the relationships

$$\mathbf{p}_i = \mu_i(\dot{\mathbf{r}}_i + \dot{\mathbf{s}}_H) \quad , \quad \sum_{i=1}^{N} \mathbf{p}_i = -\mu_0 \dot{\mathbf{s}}_H$$

(the first is a consequence of $\mathbf{p}_i = \mu_i \dot{\mathbf{x}}_i$, the second of the constraint $\dot{\mathbf{b}}_0 = \mathbf{0}$): we get

$$T_p(\mathbf{p}_1, \ldots, \mathbf{p}_N) = \frac{1}{2}\sum_{i=1}^{N} \frac{|\mathbf{p}_i|^2}{\mu_i} + \frac{1}{2\,\mu_0}\left|\sum_{i=1}^{N} \mathbf{p}_i\right|^2 = \frac{1}{2}\sum_{i=1}^{N} |\mathbf{p}_i|^2 \left[\frac{1}{\mu_i} + \frac{1}{\mu_0}\right] + \sum_{1\leq i<j\leq N} \frac{\mathbf{p}_i \cdot \mathbf{p}_j}{\mu_0} \ , \tag{1.48}$$

which is convenient because of the especially simple expression (just a sum of scalar products of the $\mathbf{p}_i$ vectors) for the indirect term, which has been moved in the T part.

Because the function $T_{\mathbf{p}}$ is homogeneous and quadratic in the variables $\mathbf{p}_i$, the Legendre transform has a simple expression:

$$H = \sum_{i=1}^{N} \mathbf{p}_i \cdot \dot{\mathbf{r}}_i - L = 2T_{\mathbf{p}} - T_{\mathbf{p}} - U = T_p - U \ ,$$

that is the value of the Hamiltonian is the total energy. The Hamilton equations (see Section 2.1) are

$$\dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{r}_i} = -\frac{\partial U}{\partial \mathbf{r}_i} \quad , \quad \dot{\mathbf{r}}_i = \frac{\partial H}{\partial \mathbf{p}_i} = \frac{\partial T}{\partial \mathbf{p}_i} \ ; \tag{1.49}$$

they are equivalent to the second order equation (1.5) and (1.47), with the indirect part arising from the kinetic energy rather than from the potential.

To decompose the Hamilton function into an unperturbed part H_0, given by the sum of N Hamiltonians of the 2 body problem, corresponding to Lagrangians as in eq. (1.29), and a perturbation H_1, we include the indirect portion of the kynetic energy in the perturbation:

$$H_0 = T_0 - U_0 \ = \ \frac{1}{2}\sum_{i=1}^{N} |\mathbf{p}_i|^2 \left[\frac{1}{\mu_i} + \frac{1}{\mu_0}\right] - \sum_{i=1}^{N} \frac{\mu_0 + \mu_i}{r_i} \tag{1.50}$$

$$H_1 = T_1 - U_1 \ = \ \sum_{1\leq i<j\leq N} \frac{\mathbf{p}_i \cdot \mathbf{p}_j}{\mu_0} - \sum_{1\leq i<j\leq N} \frac{\mu_i\,\mu_j}{|\mathbf{r}_i - \mathbf{r}_j|} \ . \ . \tag{1.51}$$

In this way the unperturbed part H_0 contains the sum of the 2-body Sun-planet relative motion energies, as in eq. (1.35); this we will need in Section (2.5).

1.6.1 The angular momentum integral

The heliocentric canonical coordinates have another advantage in an especially simple expression for the angular momentum integral: by starting from the expression of $\mathbf{c}$ in barycentric coordinates, then by using (1.44) and (1.46)

$$\begin{aligned} \mathbf{c} &= \sum_{i=0}^{N} \mathbf{x}_i \times \mu_i \, \dot{\mathbf{x}}_i = \mathbf{s}_H \times \mu_0 \, \dot{\mathbf{s}}_H + \sum_{i=1}^{N} \mathbf{s}_H \times \mu_i \, \dot{\mathbf{x}}_i + \sum_{i=1}^{N} \mathbf{r}_i \times \mu_i \, \dot{\mathbf{x}}_i \\ &= \mathbf{s}_H \times \left[\sum_{i=1}^{N} \mu_i \, \dot{\mathbf{x}}_i + \mu_0 \, \dot{\mathbf{s}}_H \right] + \sum_{i=1}^{N} \mathbf{r}_i \times \mu_i \, \dot{\mathbf{x}}_i \ , \end{aligned}$$

where the portion between square brackets is just $M_0 \, \dot{\mathbf{b}}_0 = \mathbf{0}$, thus

$$\mathbf{c} = \sum_{i=1}^{N} \mathbf{r}_i \times \mu_i \, \dot{\mathbf{x}}_i = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \ . \tag{1.52}$$

Hence the total angular momentum is just the sum of the ones of the 2-body Sun-planet subsystems; this we will need in (2.6). Note that it would be the same if the sum was to include $i = 0$, since $\mathbf{p}_0 = \mathbf{0}$.