STIEFEL & SCHEIFELE,
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Chapter IX

Classical Canonical Theory of the Perturbations of Elements

Introduction. The classical elements were introduced in Section 14, and one of the aims of this chapter is to establish the differential equations describing their variation under perturbations.

In Section 35 on Delaunay elements the Jacobian integration method is applied to the frame of the spherical coordinates. The differential equations for the Delaunay elements are established in Section 36, and later on they are reformulated in terms of the classical elements.

35. Delaunay Elements

Spherical Coordinates. In this section the potential acting per unit of mass is *assumed to depend only on the distance* of the particle

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{1}$$

and is thus denoted by $F(r)$. The relations (29,4) to (29,7) applied to this single particle ($n = 3$) produce the Hamiltonian

$$H(x_1, x_2, x_3, p_1, p_2, p_3) = \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) + F(r) \tag{2}$$

and consequently the Jacobi-equation

$$\frac{1}{2}\left(\frac{\partial S}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial x_2}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial x_3}\right)^2 + F\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right) = c\,. \tag{3}$$

The method of integrating this equation by separation, outlined in Section 33, required the special form

$$S(x_1, x_2, x_3) = S_1(x_1) + S_2(x_2) + S_3(x_3) \tag{4}$$

of the generating function. Eq. (3) is then transformed into [10]

$$\tfrac{1}{2}S_1'^2 + \tfrac{1}{2}S_2'^2 + \tfrac{1}{2}S_3'^2 + F\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right) = c\,. \tag{5}$$

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[10] S_j' is an abbreviation for $\dfrac{dS_j}{dx_j}$.

The first step of separation, isolating the couple S_1', x_1 from the remaining terms, is unlikely to succeed, unless the potential F is a sum of the type

$$F = F_1(x_1) + F_2(x_2) + F_3(x_3). \tag{6}$$

By virtue of

$$\frac{\partial F}{\partial x_1} = \frac{dF}{dr}\frac{x_1}{r}$$

there follows

$$\frac{1}{r}\frac{dF}{dr} = \frac{1}{x_1}\frac{\partial F}{\partial x_1}, \quad \frac{1}{r}\frac{dF}{dr} = \frac{1}{x_1}\frac{dF_1}{dx_1}.$$

The right-hand side of the last relation depends only on x_1. This leads immediately to the result that both expressions are equal to a constant a, hence it follows

$$\frac{1}{r}\frac{dF}{dr} = a, \quad F = \frac{a}{2}r^2 + b.$$

The additive constant b is irrelevant. The proposal of Eq. (6) leads therefore to the potential

$$F = \frac{a}{2}(x_1^2 + x_2^2 + x_3^2) \tag{7}$$

of the *harmonic oscillator*. Since our regularized theory considers the Keplerian motion as a harmonic oscillation the separation of the corresponding Jacobi differential equation is feasible in rectangular coordinates.

In contrast the classical Newtonian potential

$$F = -\frac{K^2}{r} \tag{8}$$

does not have this property and in order to achieve separability polar coordinates must be introduced. Since any polar coordinate system is singular at isolated points (north and south-poles) this fact prevents the introduction of elements that are everywhere regular.

We proceed now to the use of polar coordinates r, ϑ, ψ, where ϑ and ψ are geographical latitude and longitude respectively. The deeper reason for adopting polar coordinates is that then the potential depends only on one coordinate, namely r, and this fact thus facilitates the separation.

According to the rule (33,83) the transformation of the coordinates

$$\begin{aligned} x_1 &= r\cos\vartheta\cos\psi \\ x_2 &= r\cos\vartheta\sin\psi \\ x_3 &= r\sin\vartheta \end{aligned} \tag{9}$$

is supplemented to a canonical transformation

$$\begin{aligned} p_r &= \sum_{k=1}^{3} p_k \frac{\partial x_k}{\partial r} = p_1 \cos\vartheta \cos\psi + p_2 \cos\vartheta \sin\psi + p_3 \sin\vartheta \\ p_\vartheta &= \sum_{k=1}^{3} p_k \frac{\partial x_k}{\partial \vartheta} = -p_1 r \sin\vartheta \cos\psi - p_2 r \sin\vartheta \sin\psi + p_3 r \cos\vartheta \\ p_\psi &= \sum_{k=1}^{3} p_k \frac{\partial x_k}{\partial \psi} = -p_1 r \cos\vartheta \sin\psi + p_2 r \cos\vartheta \cos\psi , \end{aligned} \tag{10}$$

where p_r, p_ϑ, p_ψ are conjugated to r, ϑ, ψ respectively. In matrix-notation the set (10) may be written

$$\begin{pmatrix} p_r \\ \dfrac{p_\vartheta}{r} \\ \dfrac{p_\psi}{r\cos\vartheta} \end{pmatrix} = \begin{pmatrix} \cos\vartheta\cos\psi & \cos\vartheta\,\sin\psi & \sin\vartheta \\ -\sin\vartheta\cos\psi & -\sin\vartheta\,\sin\psi & \cos\vartheta \\ -\sin\psi & \cos\psi & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix},$$

and the orthogonality of this 3×3-matrix yields, firstly, the inverse transformation

$$\begin{aligned} p_1 &= p_r \cos\vartheta \cos\psi - \frac{p_\vartheta}{r} \sin\vartheta \cos\psi - \frac{p_\psi}{r\cos\vartheta} \sin\psi \\ p_2 &= p_r \cos\vartheta\, \sin\psi - \frac{p_\vartheta}{r} \sin\vartheta\, \sin\psi + \frac{p_\psi}{r\cos\vartheta} \cos\psi \\ p_3 &= p_r\, \sin\vartheta + \frac{p_\vartheta}{r} \cos\vartheta . \end{aligned} \tag{11}$$

Secondly, the relation

$$p_1^2 + p_2^2 + p_3^2 = p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\psi^2}{r^2\cos^2\vartheta} . \tag{12}$$

New Hamiltonian:

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\psi^2}{r^2\cos^2\vartheta}\right) + F(r) . \tag{13}$$

Corresponding canonical equations:

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r}, \quad \dot{\vartheta} = \frac{\partial H}{\partial p_\vartheta}, \quad \dot{\psi} = \frac{\partial H}{\partial p_\psi}, \\ \dot{p}_r &= -\frac{\partial H}{\partial r}, \quad \dot{p}_\vartheta = -\frac{\partial H}{\partial \vartheta}, \quad \dot{p}_\psi = -\frac{\partial H}{\partial \psi} . \end{aligned} \tag{14}$$

Jacobi's Integration Method. The corresponding Jacobian equation

$$\frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{2r^2}\left(\frac{\partial S}{\partial \vartheta}\right)^2+\frac{1}{2r^2\cos^2\vartheta}\left(\frac{\partial S}{\partial \psi}\right)^2+F(r)=c \tag{15}$$

is fit for separation: By putting

$$S(r,\vartheta,\psi)=S_1(r)+S_2(\vartheta)+S_3(\psi)$$

it is reduced to

$$\frac{1}{2}S_1'^2+\frac{1}{2r^2}S_2'^2+\frac{1}{2r^2\cos^2\vartheta}S_3'^2+F(r)=c\,. \tag{16}$$

For the first separation-step we propose to adopt as the first couple to be separated the couple S_3', ψ (this first choice is suggested by the fact that ψ does not appear explicitly)

$$S_3'^2=r^2\cos^2\vartheta[2c-S_1'^2-2F(r)]-\cos^2\vartheta S_2'^2\,.$$

According to (33,97) this relation splits up into

$$\begin{gathered}S_3'^2=c_1\,,\\ r^2\cos^2\vartheta[2c-S_1'^2-2F(r)]-\cos^2\vartheta S_2'^2=c_1\,.\end{gathered} \tag{17}$$

As next and last separation-step the couple S_1, r is put on the left-hand side, and the couple S_2, ϑ on the right-hand side of the equation

$$r^2[2c-S_1'^2-2F(r)]=S_2'^2+\frac{c_1}{\cos^2\vartheta}\,,$$

whence it follows

$$S_2'^2+\frac{c_1}{\cos^2\vartheta}=c_2\,, \tag{18}$$

$$r^2[2c-S_1'^2-2F(r)]=c_2\,. \tag{19}$$

By solving the ordinary differential Eqs. (17) (18) (19) for S_1, S_2, S_3 the results

$$S_1=\int\sqrt{2c-\frac{c_2}{r^2}-2F(r)}\,dr\,,\qquad S_2=\int\sqrt{c_2-\frac{c_1}{\cos^2\vartheta}}\,d\vartheta\,,\qquad S_3=\sqrt{c_1}\,\psi$$

appear, which lead to the generating function

$$S(r,\vartheta,\psi,c,c_1,c_2)=\int\sqrt{2c-\frac{c_2}{r}-2F(r)}\,dr+\int\sqrt{c_2-\frac{c_1}{\cos^2\vartheta}}\,d\vartheta+\sqrt{c_1}\,\psi\,. \tag{20}$$

According to the general theory (Section 33) there remains the choice of the functions

$$c=g_1(\bar p_1,\bar p_2,\bar p_3)\,,\qquad c_1=g_2(\bar p_1,\bar p_2,\bar p_3)\,,\qquad c_2=g_3(\bar p_1,\bar p_2,\bar p_3)\,.$$

In order to facilitate the introduction of the classical Keplerian elements it is practicable to put

$$c=\bar{p}_1, \quad c_1=\bar{p}_3^2, \quad c_2=\bar{p}_2^2. \tag{21}$$

Any other choice would lead, in principle, to the same final results, but would lead to elements unfit for geometrical interpretation.

The lower limits of the integrals in Eq. (20) are still open, and some comments on their choice are in order. Let r_0 and 0 be respectively these limits, such that

$$S(r,\vartheta,\psi,\bar{p}_1,\bar{p}_2,\bar{p}_3)=\int_{r_0}^{r}\sqrt{2\bar{p}_1-\frac{\bar{p}_2^2}{\varrho^2}-2F(\varrho)}\,d\varrho+\int_0^{\vartheta}\sqrt{\bar{p}_2^2-\frac{\bar{p}_3^2}{\cos^2\mu}}\,d\mu+\bar{p}_3\cdot\psi. \tag{22}$$

In the application to the perturbation theories some additional freedom is obtained by admitting that r_0 is a function of the parameters $\bar{p}_k$; this more general point of view does not destroy the fact that (22) is a solution of the Jacobi-equation (3).

As far as the lower limits of integration are concerned we admit that r_0 is not a universal constant but that it may depend on $\bar{p}_1, \bar{p}_2$. The generated canonical transformation is defined by Eqs. (33,92) and (33,103)

$$p_r=\frac{\partial S}{\partial r}=\sqrt{2\bar{p}_1-\frac{\bar{p}_2^2}{r^2}-2F(r)}, \tag{23}$$

$$p_\vartheta=\frac{\partial S}{\partial \vartheta}=\sqrt{\bar{p}_2^2-\frac{\bar{p}_3^2}{\cos^2\vartheta}}, \tag{24}$$

$$p_\psi=\frac{\partial S}{\partial \psi}=\bar{p}_3, \tag{25}$$

$$\bar{x}_1=\frac{\partial S}{\partial \bar{p}_1}=\int_{r_0}^{r}\frac{1}{\sqrt{2\bar{p}_1-\frac{\bar{p}_2^2}{\varrho^2}-2F(\varrho)}}\,d\varrho-k\cdot\frac{\partial r_0}{\partial \bar{p}_1} \tag{26}$$

$$\bar{x}_2=\frac{\partial S}{\partial \bar{p}_2}=-\int_{r_0}^{r}\frac{\bar{p}_2}{\varrho^2\sqrt{2\bar{p}_1-\frac{\bar{p}_2^2}{\varrho^2}-2F(\varrho)}}\,d\varrho+\int_0^{\vartheta}\frac{\bar{p}_2}{\sqrt{\bar{p}_2^2-\frac{\bar{p}_3^2}{\cos^2\mu}}}\,d\mu, \tag{27}$$

$$-k\frac{\partial r_0}{\partial \bar{p}_2}.$$

$$\bar{x}_3=\frac{\partial S}{\partial \bar{p}_3}=-\int_0^{\vartheta}\frac{\bar{p}_3}{\cos^2\mu\sqrt{\bar{p}_2^2-\frac{\bar{p}_3^2}{\cos^2\mu}}}\,d\mu+\psi \tag{28}$$

where

$$k = \sqrt{2\bar{p}_1 - \frac{\bar{p}_2^2}{r_0^2} - 2F(r_0)}\,.$$

The terms containing k are produced by the variability of the lower limit r_0.
The general theory requires the solution of Eqs. (26) (27) (28) for r, ϑ, ψ but this will be carried out later on, and only in the particular case of the Newtonian potential. At this stage a canonical transformation of the set $r, \vartheta, \psi, p_r, p_\vartheta, p_\psi$ into the set $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{p}_1, \bar{p}_2, \bar{p}_3$ is constructed, and the new Hamiltonian is according to (33, 105)

$$H = \bar{p}_1\,. \tag{29}$$

The new canonical equations become

$$\begin{aligned} &\frac{d\bar{x}_1}{dt} = 1\,, \quad \frac{d\bar{x}_2}{dt} = 0\,, \quad \frac{d\bar{x}_3}{dt} = 0\,, \\ &\frac{d\bar{p}_1}{dt} = 0\,, \quad \frac{d\bar{p}_2}{dt} = 0\,, \quad \frac{d\bar{p}_3}{dt} = 0\,. \end{aligned} \tag{30}$$

From these relations it follows that $\bar{x}_1$ is an element varying linearly, whereas the five remaining variables are constant elements. The motion of the particle in polar coordinates $r(t), \vartheta(t), \psi(t)$ is obtained from the transformation formulae (26) (27) (28) after having inverted them. The guiding principle for performing this task is the introduction of the classical symbols (14, 56) $\Omega, \omega, J, a, e, M, E$ (eccentric anomaly), φ (true anomaly) as auxiliary variables and the establishment of their relations with the canonical variables. This will be carried out in the next subsection.

Remark. The canonical transformation (23)...(28) depends, of course, on the choice of the function $r_0(\bar{p}_j)$.

Canonical and Classical Elements. Now we restrict ourselves to the Newtonian potential

$$F = -\frac{K^2}{r} \tag{31}$$

and we make use of the properties of the pure elliptic Kepler-motion explained in Section 10. Our aim is to establish the connections between the canonical elements defined above by the Jacobian method and the classical elements which were described in Section 14.

Fig. 15 is essentially a copy of Fig. 6 in Section 14. The text of this section, and in particular the list (14, 56) of symbols, should be consulted.

First, the canonical elements $\bar{p}_1, \bar{p}_2, \bar{p}_3$ are discussed. From (29) and (29,7) it is seen that $\bar{p}_1$ is the total energy

$$\bar{p}_1 = -h_k = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{K^2}{r} = -\frac{K^2}{2a} \tag{32}$$

where a is the semi major-axis. These relations follow from Eqs. (2,10) and (11,64). Next observe that $\bar{p}_2$ is a constant during the motion; its value may be determined, therefore, by inserting the distance r_p of the

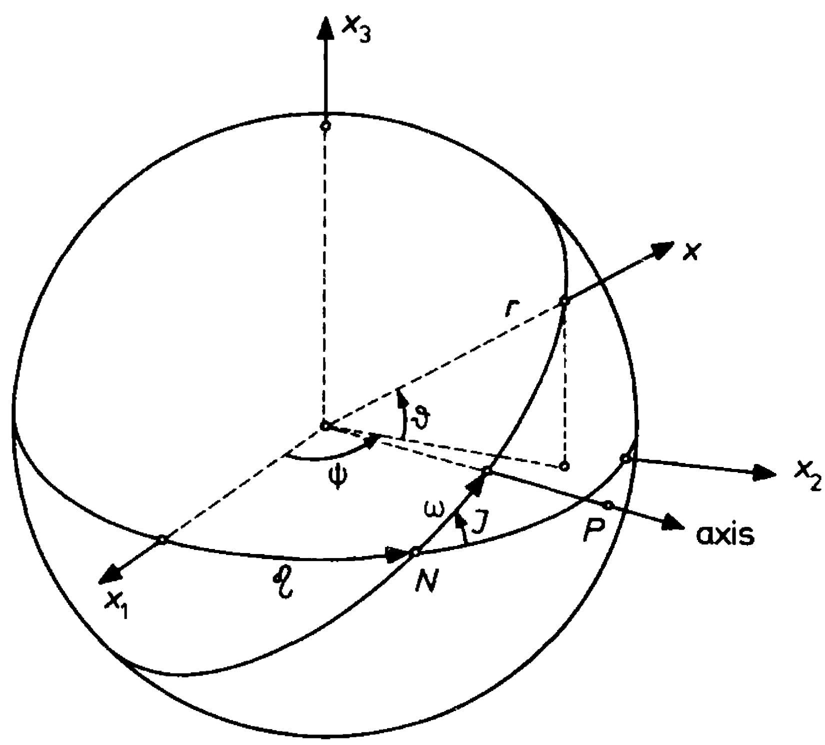

Fig. 15

pericentre[11] into Eq. (23). The radial velocity on the left-hand side vanishes, thus

$$2\bar{p}_1 - \frac{\bar{p}_2^2}{r_p^2} + 2\frac{K^2}{r_p} = 0, \tag{32a}$$

or

$$-\frac{K^2}{a} - \frac{\bar{p}_2^2}{r_p^2} + 2\frac{K^2}{r_p} = 0, \quad \bar{p}_2^2 = -r_p^2\frac{K^2}{a} + 2r_pK^2,$$

and by remembering the distance $r_p = a(1-e)$ of the pericentre

$$\bar{p}_2 = K\sqrt{a(1-e^2)} = K\sqrt{p}, \tag{33}$$

where p is the semi latus rectum. Similarly the significance of $\bar{p}_3$ is obtained by inserting the highest point of the orbit into (24). Here ϑ is the inclina-

[11] For the application to perturbation theory by "*pericentre*" is meant the pericentre of the osculating orbit, which varies under the influence of the perturbing forces. This is the reason why we admit a variable lower limit r_0 in the integral (22).

tion J and p_ϑ vanishes

$$\bar{p}_3 = \bar{p}_2 \cos J = K\sqrt{p}\cos J\,. \tag{34}$$

We add the remark, that $\bar{p}_2$ has the interpretation of the total angular momentum, whereas $\bar{p}_3$ is its component in the direction of the x_3-axis.

It still remains to evaluate the integrals in Eqs. (26) (27) (28). We begin with Eq. (28) and we observe that the lower limit 0 corresponds to the node N. Again the fact that $\bar{x}_3$ is constant is helpful, and thus the integral can be evaluated at any convenient point of the orbit. As such a point we choose again the node N where the upper limit of the integral vanishes. Hence the value of $\bar{x}_3$ is the longitude Ω of the node

$$\bar{x}_3 = \Omega\,. \tag{35}$$

By fixing the lower limit r_0 of the integrals in (26) (27) as the distance $r_0 = r_P$ of the pericentre it follows at first that the function $k(\bar{p}_j)$ by virtue of Eq. (32a) vanishes; secondly there results from Eq. (27)

$$\bar{x}_2 = -\int\limits_{r_P}^{r} \frac{\bar{p}_2}{\varrho^2\sqrt{2\bar{p}_1 - \frac{\bar{p}_2^2}{\varrho^2} + \frac{2K^2}{\varrho}}}\,d\varrho + \int\limits_{0}^{\vartheta} \frac{\bar{p}_2}{\sqrt{\bar{p}_2^2 - \frac{\bar{p}_3^2}{\cos^2\mu}}}\,d\mu\,,$$

This relation may be evaluated for instance at the node N where the second integral vanishes

$$\bar{x}_2 = -\int\limits_{r_P}^{r_N} \frac{\bar{p}_2}{\varrho^2\sqrt{2\bar{p}_1 - \frac{\bar{p}_2^2}{\varrho^2} + \frac{2K^2}{\varrho}}}\,d\varrho\,. \tag{36}$$

r_N is the distance of the node. Whereas the foregoing results used only the fact that the motion is planar and periodic, the computation of this integral needs information about the shape of the orbit. Therefore we make use of Eq. (11,66)

$$\varrho = \frac{p}{1 + e\cos\varphi}\,. \tag{37}$$

for substituting the integration variable ϱ by the true anomaly φ; also the values (32) (33) of $\bar{p}_1$ and $\bar{p}_2$ are used for this transformation of the integral. The square root reduces by virtue of the relation $p = a(1-e^2)$ to

$$\frac{K}{\sqrt{p}}\,e\sin\varphi\,,$$

and thus the value of the integral is

$$\bar{x}_2 = -\int\limits_{0}^{-\omega} d\varphi = \omega\,. \tag{38}$$

Here ω is the angular distance from the node to the pericentre.

Finally, the first equation of the set (30) is integrated

$$\bar{x}_1 = t - \text{const}.$$

and by virtue of Eq. (26) there follows:

$$\bar{x}_1 = t - \text{const} = \int_{r_P}^{r} \frac{d\varrho}{\sqrt{2\bar{p}_1 - \frac{\bar{p}_2^2}{\varrho^2} + \frac{2K^2}{\varrho}}}$$

This relation, evaluated at the pericentre yields

$$\text{const} = t_P$$

where t_P is the time of pericentre passage. Consequently we have

$$\bar{x}_1 = t - t_P. \tag{39}$$

By collating results there follows the list of canonical elements

$$\bar{p}_1 = -\frac{K^2}{2a}, \quad \bar{p}_2 = K\sqrt{p}, \quad \bar{p}_3 = K\sqrt{p}\cos J, \tag{40}$$

$$\bar{x}_1 = t - t_P, \quad \bar{x}_2 = \omega, \quad \bar{x}_3 = \Omega. \tag{41}$$

Comment. After Eq. (28) we mentioned that the complete theory of the canonical transformations at hand requires the solution of Eqs. (26) (27) (28) for r, ϑ, ψ. This is now feasible, in principle, by adopting the following detour. From the canonical elements $\bar{x}_i, \bar{p}_i$ the classical elements are computed by means of the Eqs. (40) (41). Then the elementary theory of the Kepler-motion (c.f. collection of formulae at the end of Section 14) enables us to compute the position of the particle by solving Kepler's equation. Some of the complications encountered in the foregoing theory are explained by the fact that Kepler's equation is a transcendental equation. Such difficulties are avoided in the canonical KS-theory which uses the eccentric anomaly as independent variable.

Delaunay Elements. The element $\bar{x}_1$ is the time elapsed since pericentre-passage. It is to our advantage to replace this element by the mean anomaly M which is, according to Eq. (10, 28)

$$M = \frac{K}{a^{\frac{3}{2}}}(t - t_P). \tag{42}$$

Another reason for this slight modification is explained in Section 42.

The introduction of M will be achieved by a further canonical transformation of the actual variables $\bar{x}_i, \bar{p}_i$. As always the new canonical set is composed of three variables of the first category, denoted by

$$l_D = M, \quad g_D, \quad h_D \tag{43}$$

and their conjugates of the second category which are denoted by

$$L_D\,, \quad G_D\,, \quad H_D\,. \tag{44}$$

Observe that we choose the mean anomaly as first variable of the first category and write Eq. (42) in terms of the canonical variables

$$l_D = \bar{x}_1 \frac{1}{K^2}(-2\bar{p}_1)^{\frac{3}{2}}\,. \tag{45}$$

At this stage, the special transformations considered at the beginning of Section 33 are helpful, they are characterized by two properties:

1. The variables of one category are transformed among themselves.
2. The variables of the other category are transformed linearly.

In relation (45) we are faced with a linear transformation of $\bar{x}_1$ into l_D which is a linear transformation of the variables of the first category. We thus try to find a transformation of the variables of the second category among themselves of the type

$$\bar{p}_1 = f(L_D)\,, \quad \bar{p}_2 = G_D\,, \quad \bar{p}_3 = H_D \tag{46}$$

which, supplemented to a canonical transformation, yields in particular the desired relation (45). This proposal is compatible with the generating function of type II (Table (32,71))

$$S = \bar{x}_1 f(L_D) + \bar{x}_2 G_D + \bar{x}_3 H_D\,,$$

since

$$\bar{p}_1 = \frac{\partial S}{\partial \bar{x}_1} = f(L_D)\,, \quad \bar{p}_2 = \frac{\partial S}{\partial \bar{x}_2} = G_D\,, \quad \bar{p}_3 = \frac{\partial S}{\partial \bar{x}_3} = H_D\,,$$

as was required. According to the rules (32.71) this implies

$$l_D = \frac{\partial S}{\partial L_D} = \bar{x}_1 \frac{df}{dL_D}\,, \quad g_D = \frac{\partial S}{\partial G_D} = \bar{x}_2\,, \quad h_D = \frac{\partial S}{\partial H_D} = \bar{x}_3\,.$$

By comparison with the requirement (45) the differential equation

$$\frac{df}{dL_D} = \frac{1}{K^2}(-2f)^{\frac{3}{2}}$$

is obtained, yielding

$$L_D = K^2(-2f)^{-\frac{1}{2}}\,, \quad f(L_D) = -\frac{K^4}{2L_D^2} = \bar{p}_1\,.$$

To sum up, the wanted transformation is

$$\bar{p}_1 = -\frac{K^4}{2L_D^2}, \qquad \bar{p}_2 = G_D, \qquad \bar{p}_3 = H_D, \tag{47}$$

$$\bar{x}_1 = \frac{1}{K^4} l_D L_D^3, \qquad \bar{x}_2 = g_D, \qquad \bar{x}_3 = h_D. \tag{48}$$

The new Hamiltonian is, according to Eq. (29),

$$H = -\frac{K^4}{2L_D^2}. \tag{49}$$

The six elements $L_D, G_D, H_D, l_D, g_D, h_D$ are the *Delaunay elements*, in terms of the classical elements they may be written

$$L_D = K\sqrt{a}, \qquad G_D = K\sqrt{p}, \qquad H_D = K\sqrt{p}\cos J, \tag{50}$$

$$l_D = M, \qquad g_D = \omega, \qquad h_D = \Omega, \tag{51}$$

where M is the mean anomaly and

$$p = a(1-e^2). \tag{52}$$

36. Perturbation of Elements

Perturbation of the Delaunay Elements. Bearing in mind the canonical perturbation theory of Section 33, we assume now that the moving particle is subjected to a perturbing potential V and a remaining perturbing force $\boldsymbol{P}$ with rectangular components P_1, P_2, P_3 (c.f. collection of formulae (3, 16)). The foregoing canonical transformations to polar coordinates and to Delaunay elements give rise to canonical forces

$$\begin{array}{lll} P_L, & P_G, & P_H, \\ P_l, & P_g, & P_h. \end{array} \tag{53}$$

In principle these forces should be computed from $\boldsymbol{P}$ by the rules (31, 48) and (31, 49) in the following way

$$\begin{array}{lll} P_L = \sum\limits_{j=1}^{3} P_j \dfrac{\partial x_j}{\partial l_D}, & P_G = \sum\limits_{j=1}^{3} P_j \dfrac{\partial x_j}{\partial g_D}, & P_H = \sum\limits_{j=1}^{3} P_j \dfrac{\partial x_j}{\partial h_D} \\ P_l = \sum\limits_{j=1}^{3} P_j \dfrac{\partial x_j}{\partial L_D}, & P_g = \sum\limits_{j=1}^{3} P_j \dfrac{\partial x_j}{\partial G_D}, & P_h = \sum\limits_{j=1}^{3} P_j \dfrac{\partial x_j}{\partial H_D}. \end{array} \tag{54}$$

We do not carry out the rather lengthy computation of the partial derivatives in this set.