BATTIN R.
An Introduction to the Mathematics and Methods of Astrodynamics, 1999

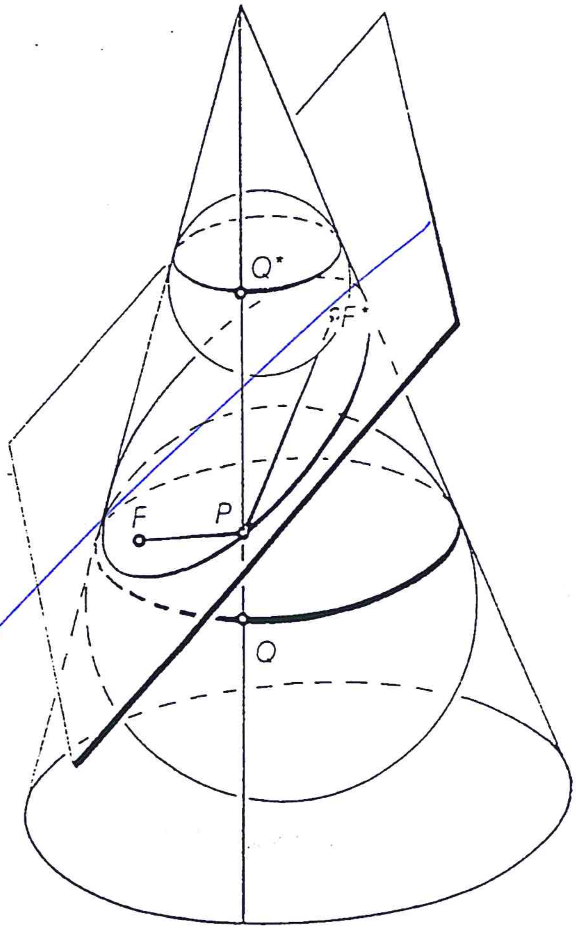

Fig. 4.4: Ellipse as a section of a cone.

between the two circles. But this distance is the same for all points P on the section. Therefore, the curve is an ellipse.

Problem 4–2
If the cutting plane has the same inclination to the base of the cone as the generators, there is a single sphere tangent to the cone along a circle and tangent to the plane at a point. Show that this point is the focus of the parabola and that the directrix is the line in which the plane of the circle cuts the plane of the parabola.

Furthermore, show that when the plane cuts both portions of the cone, the curve of intersection is a hyperbola. Note that one sphere is in each portion of the cone.

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4.2 Parabolic Orbits and Barker's Equation

Except for the circle, for which the true anomaly is proportional to the time, the position of a body in orbit at a given time is simplest for the parabola. The polar equation of a parabola is

$$r = \frac{p}{1+\cos f} = \frac{p}{2}(1+\tan^2 \tfrac{1}{2}f) \tag{4.9}$$

so that from the law of areas

$$r^2 \frac{df}{dt} = h = \sqrt{\mu p}$$

it follows that

$$4\sqrt{\frac{\mu}{p^3}}\, dt = \sec^4 \tfrac{1}{2} f \, df$$

Performing the integration, we obtain

$$\tan^3 \tfrac{1}{2} f + 3 \tan \tfrac{1}{2} f = 2B \qquad \text{where} \qquad B = 3\sqrt{\frac{\mu}{p^3}}(t - \tau) \tag{4.10}$$

and τ is the time of pericenter passage.

This relation between the true anomaly f and the time t is called *Barker's equation.*† The solution for f when t is given requires the root of a cubic equation in $\tan \frac{1}{2} f$ and it is easy to show that one and only one real root exists. To obtain it, we substitute

$$\tan \tfrac{1}{2} f = z - \frac{1}{z} \tag{4.11}$$

and derive, thereby, a quadratic equation in z^3

$$z^6 - 2Bz^3 - 1 = 0$$

for which

$$z = \left(B \pm \sqrt{B^2 + 1}\right)^{\frac{1}{3}}$$

Either sign produces the same solution for $\tan \frac{1}{2} f$. Therefore,

$$\tan \tfrac{1}{2} f = \left(B + \sqrt{1 + B^2}\right)^{\frac{1}{3}} - \left(B + \sqrt{1 + B^2}\right)^{-\frac{1}{3}} \tag{4.12}$$

which is in accord with the classic formula of Jerome Cardan.‡

Many variations of the solution of Barker's equation are considered in the following subsections which are both interesting and useful.

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† The parabolic form of Kepler's equation is called Barker's equation after Thomas Barker (1722–1809) who published extensive tables for its solution in 1757. It contained values of the expression $75 \tan \frac{1}{2} f + 25 \tan^3 \frac{1}{2} f$ for the true anomalies at intervals of five minutes of arc from 0° to 180°. Although, Halley (1705) and Euler (1744) did essentially the same thing; nevertheless, it is still referred to as Barker's equation.

‡ Gerolamo Cardano (1501–1576) published the method for solving cubic equations which he obtained from Niccolò Fontana of Brescia (1499?–1557). Fontana is better known as Tartaglia, which means "Stammerer"—an unfortunate name he acquired because of a speech defect. Tartaglia had a method for solving the cubic which he revealed to Cardan in 1539 after a pledge from Cardan to keep it secret. Despite the pledge, Cardan published his version of the method in his *Ars Magna* in 1545.

4.4 Hyperbolic Orbits and the Gudermannian

An analogous procedure for hyperbolic orbits can be formulated which parallels the discussion presented for elliptic orbits. We begin with the equation of the hyperbola expressed in parametric form as

$$x = a \sec \varsigma \qquad y = b \tan \varsigma \tag{4.44}$$

in an x, y cartesian coordinate system with origin at the center. Clearly, if ς is eliminated, the standard form of the hyperbola results.

To express the radius to a point P in terms of the parameter ς, we again use Eq. (4.4) to obtain†

$$r = a(1 - e \sec \varsigma) \tag{4.45}$$

Now, if in the equation of orbit

$$r + re \cos f = p = a(1 - e^2)$$

we express the first r in terms of ς using Eq. (4.45), it follows that

$$-a \sec \varsigma + r \cos f = -ae$$

Thus, the angle ς and the true anomaly f are related as shown in Fig. 4.12. Therefore, when ς is used in the analytical description of hyperbolic orbits it has a direct geometric analogy with the eccentric anomaly of the ellipse. In both cases auxiliary circles, whose centers are at the center of the orbit and whose radii are the semimajor axes of the orbits, play similar roles in the analysis.

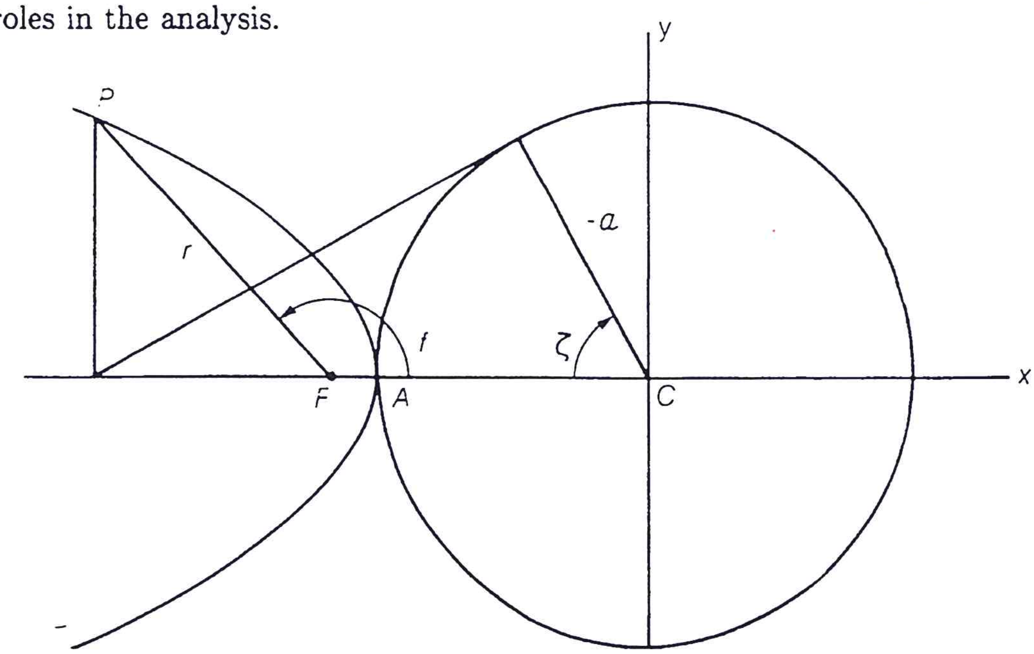

Fig. 4.12: Geometrical representation of the Gudermannian ς.

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† It is important to remember that a is a negative number and e is greater than one.

The identities relating ς and the true anomaly follow as before:

↘ si riferisce al caso ellittico

$$\cos f = \frac{e - \sec\varsigma}{e\sec\varsigma - 1} \qquad \sec\varsigma = \frac{e + \cos f}{1 + e\cos f}$$
$$\sin f = \frac{\sqrt{e^2-1}\tan\varsigma}{e\sec\varsigma - 1} \qquad \tan\varsigma = \frac{\sqrt{e^2-1}\sin f}{1 + e\cos f} \tag{4.46}$$

Then since

$$\sin^2 \tfrac{1}{2} f = -\frac{a(e+1)}{r\cos\varsigma}\sin^2 \tfrac{1}{2}\varsigma \qquad \cos^2 \tfrac{1}{2} f = -\frac{a(e-1)}{r\cos\varsigma}\cos^2 \tfrac{1}{2}\varsigma \tag{4.47}$$

we have

$$\tan \tfrac{1}{2} f = \sqrt{\frac{e+1}{e-1}}\tan \tfrac{1}{2}\varsigma \tag{4.48}$$

To derive the analog of Kepler's equation for hyperbolic motion, we calculate the differential of Eq. (4.48) to obtain

$$r\,df = b\sec\varsigma\,d\varsigma \tag{4.49}$$

Hence, in the same manner as for the ellipse, we have

$$N = e\tan\varsigma - \log\tan(\tfrac{1}{2}\varsigma + \tfrac{1}{4}\pi) \tag{4.50}$$

where the quantity N is analogous to the mean anomaly of elliptic motion and is defined as

$$N = \sqrt{\frac{\mu}{(-a)^3}}\,(t - \tau) \tag{4.51}$$

◇ Problem 4–14

The two straight lines

$$y = \pm\frac{b}{a}x = \pm(\tan\psi)x$$

through the center C are the *asymptotes* of the hyperbola where ψ is related to the eccentricity as

$$\tan\psi = \sqrt{e^2-1} \qquad \text{or} \qquad \sec\psi = e$$

The equation of orbit can then be written as

$$r = \frac{p}{1 + e\cos f} = \frac{p\cos\psi}{2\cos\frac{1}{2}(f+\psi)\cos\frac{1}{2}(f-\psi)}$$

which clearly displays the behaviour of the hyperbola in the vicinity of the asymptotes. Indeed, this equation defines the asymptotes.

Carl Friedrich Gauss 1809

◇ Problem 4–15

Define the quantity u as

$$u = \tan(\tfrac{1}{2}\varsigma + \tfrac{1}{4}\pi)$$

Equation (4.50) can then be written as

$$N = \frac{e}{2}\left(u - \frac{1}{u}\right) - \log u$$

with the radius and true anomaly expressible in terms of u as

$$r = a - \frac{ae}{2}\left(u + \frac{1}{u}\right) \qquad \text{and} \qquad \tan\tfrac{1}{2}f = \sqrt{\frac{e+1}{e-1}}\,\frac{u-1}{u+1}$$

Problem 4–16

With the angle ψ defined in Prob. 4–14, the quantity u, defined in Prob. 4–15, can be expressed in terms of the angles f and ψ as

$$u = \frac{\cos\frac{1}{2}(f - \psi)}{\cos\frac{1}{2}(f + \psi)} \qquad \text{where} \qquad 1 \le u < \infty \quad \text{for } f > 0$$

Carl Friedrich Gauss 1809

The Gudermannian Transformation

The analysis for hyperbolic orbits may be accomplished in terms of hyperbolic, rather than trigonometric, functions. Because of the familiar identity

$$\cosh^2 H - \sinh^2 H = 1$$

the parametric equations of the hyperbola can be written as

$$x = a\cosh H \qquad y = b\sinh H \tag{4.52}$$

and the radius vector magnitude becomes

$$r = a(1 - e\cosh H) \tag{4.53}$$

The identities between H and the true anomaly are found simply by substituting

$$\tan\varsigma = \sinh H \qquad \sec\varsigma = \cosh H \tag{4.54}$$

in Eqs. (4.46). We can also show that

$$\tan\tfrac{1}{2}\varsigma = \tanh\tfrac{1}{2}H \tag{4.55}$$

so that Eq. (4.48) becomes

$$\tan\tfrac{1}{2}f = \sqrt{\frac{e+1}{e-1}}\tanh\tfrac{1}{2}H \tag{4.56}$$

Applying the definition of the hyperbolic functions in terms of the exponential function, it follows from Eqs. (4.54) that

$$H = \log(\tan\varsigma + \sec\varsigma) = \log\tan(\tfrac{1}{2}\varsigma + \tfrac{1}{4}\pi) \tag{4.57}$$

Hence, the relation between time and the quantity H is obtained from Eq. (4.50) as

$$N = e\sinh H - H \tag{4.58}$$

The inverse function, expressing ς in terms of H and written symbolically as $\varsigma = \text{gd}\, H$, is called the *Gudermannian* of H. Explicitly,

$$\varsigma = \text{gd}\, H = 2\arctan(e^H) - \tfrac{1}{2}\pi \tag{4.59}$$

This name was given by Arthur Cayley† in honor of the German mathematician Christof Gudermann (1798–1852) who was largely responsible for the introduction of the hyperbolic functions into modern analysis.

◇ Problem 4–17

The hyperbolic form of Kepler's equation can be obtained formally from Kepler's equation by writing

$$E = -iH \qquad \text{and} \qquad M = iN$$

where $i = \sqrt{-1}$.

Geometrical Representation of H

If A is the area swept out by the radius vector, then, from Prob. 2–16,

$$dA = \tfrac{1}{2}(x\,dy - y\,dx)$$

Hence, for the unit circle

$$x^2 + y^2 = 1 \qquad \text{or} \qquad x = \cos E, \quad y = \sin E$$

and for the unit equilateral hyperbola

$$x^2 - y^2 = 1 \qquad \text{or} \qquad x = \cosh H, \quad y = \sinh H$$

we have

$$dA = \tfrac{1}{2}dE \qquad \text{(unit circle)}$$
$$dA = \tfrac{1}{2}dH \qquad \text{(unit equilateral hyperbola)}$$

Furthermore, as shown in Fig. 4.13, with AQ an arc of the circle and the shaded area equal to $\frac{1}{2}E$, there obtains

$$CR = \cos E \qquad RQ = \sin E \qquad AD = \tan E$$

† Although Sir Arthur Cayley (1821–1895) contributed much to mathematics, he is is generally remembered as the creator of the theory of matrices. Logically, the idea of a *matrix* should precede that of a *determinant* but historically the order was the reverse. Cayley was the first to recognize the matrix as an entity in its own right and the first to publish a series of papers on the subject.

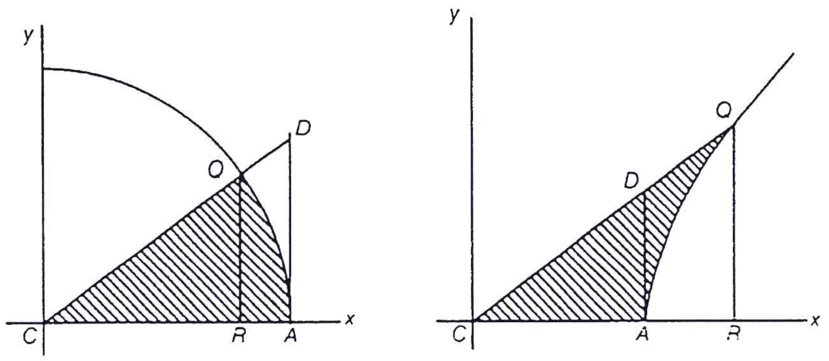

Fig. 4.13: Geometrical significance of E and H.

Similarly, with AQ an arc of the hyperbola and the shaded area equal to $\frac{1}{2}H$, then

$$CR = \cosh H \qquad RQ = \sinh H \qquad AD = \tanh H$$

Trigonometric functions are frequently called *circular functions* and this analogy between circular and hyperbolic functions is the reason for the designation of the latter as hyperbolic.

From this discussion, it is clear that the analog of the auxiliary circle, used in the analysis of the ellipse, should be the equilateral hyperbola having the same major axis as the hyperbolic orbit under consideration.

Refer to Fig. 4.14 where the points C and F are the center and focus of the hyperbola. The point A is the vertex or pericenter position. The axis through F and A is called the *transverse axis*. The other axis through the center, called the *conjugate axis*, does not intersect the curve. Let P be the position of a body on the hyperbola and let Q be the point where the perpendicular to the transverse axis through P cuts the auxiliary equilateral hyperbola. Then the area CAQ, bounded by the two straight lines CA, CQ, and the arc AQ, is

$$\text{Area } CAQ = \tfrac{1}{2}a^2 H \tag{4.60}$$

Problem 4–18

Derive the hyperbolic form of Kepler's equation geometrically, using the same pattern of argument as for elliptic orbits. Further, show that if a fictitious body starts from C when the real body is at A and moves along the asymptote of the equilateral hyperbola with a constant speed equal to the ultimate speed of the real body, then

$$N = \frac{2}{a^2}\text{ Area } F_0CP'$$

where F_0CP' is a triangle whose vertices are F_0, the focus of the equilateral hyperbola, C, the center, and P', the position of the fictitious body.

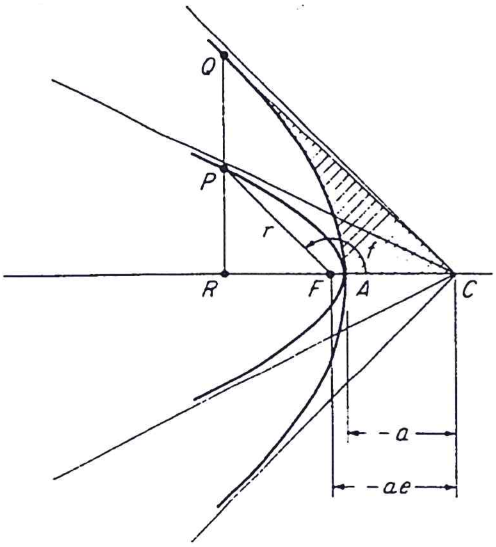

Fig. 4.14: Orbital relations for hyperbolic motion.

Lagrangian Coefficients

The position and velocity vectors in orbital-plane coordinates are readily obtained as

$$\begin{aligned} \mathbf{r} &= a(\cosh H - e)\,\mathbf{i}_e + \sqrt{-ap}\sinh H\,\mathbf{i}_p \\ \mathbf{v} &= -\frac{\sqrt{-\mu a}}{r}\sinh H\,\mathbf{i}_e + \frac{\sqrt{\mu p}}{r}\cosh H\,\mathbf{i}_p \end{aligned} \tag{4.61}$$

For the Lagrangian coefficients, we first establish

$$e\cosh H_0 = 1 - \frac{r_0}{a} \qquad e\sinh H_0 = \frac{\sigma_0}{\sqrt{-a}}$$

and then determine

$$\begin{aligned} F &= 1 - \frac{a}{r_0}[1 - \cosh(H - H_0)] \\ G &= \frac{a\sigma_0}{\sqrt{\mu}}[1 - \cosh(H - H_0)] + r_0\sqrt{\frac{-a}{\mu}}\sinh(H - H_0) \\ F_t &= -\frac{\sqrt{-\mu a}}{r r_0}\sinh(H - H_0) \\ G_t &= 1 - \frac{a}{r}[1 - \cosh(H - H_0)] \end{aligned} \tag{4.62}$$

where

$$r = -a + (r_0 + a)\cosh(H - H_0) + \sigma_0\sqrt{-a}\sinh(H - H_0) \tag{4.63}$$

with the quantity $H - H_0$ obtained as the solution of

$$N - N_0 = -(H - H_0) + \frac{\sigma_0}{\sqrt{-a}}[\cosh(H - H_0) - 1] + \left(1 - \frac{r_0}{a}\right)\sinh(H - H_0) \tag{4.64}$$

The Flight-Direction Angle

The angle γ between the position vector $\mathbf{r}$ and the velocity vector $\mathbf{v}$ will be referred to as the *flight-direction angle*. This name distinguishes it from the more traditional *flight-path angle* which is the complement of γ. Clearly, from the figure we have

$$\begin{aligned} \sin\gamma &= \frac{\mu}{hv}(1 + e\cos f) \\ \cos\gamma &= \frac{\mu}{hv} e \sin f \end{aligned} \tag{3.30}$$

which relate the flight-direction angle and the true anomaly.

◇ **Problem 3–23**

Derive the expressions

$$\sigma \equiv \frac{\mathbf{r}\cdot\mathbf{v}}{\sqrt{\mu}} = \frac{\sqrt{\mu}\, re\sin f}{h} = \frac{rv\cos\gamma}{\sqrt{\mu}} = \sqrt{p}\cot\gamma$$
$$h = rv\sin\gamma$$

◇ **Problem 3–24**

From the results of Prob. 3–23 and the vis-viva integral, derive the following expressions for the parameter p and the velocity vector $\mathbf{v}$ in terms of the flight-direction angle γ:

$$p = r\left(2 - \frac{r}{a}\right)\sin^2\gamma$$
$$\mathbf{v} = \frac{h}{r}(\cot\gamma\, \mathbf{i}_r + \mathbf{i}_\theta) = \frac{\mu e \sin f}{h}\mathbf{i}_r + \frac{h}{r}\mathbf{i}_\theta$$

◇ **Problem 3–25**

The quantity $Q \equiv \sigma = \sqrt{p}\cot\gamma$ is a solution of Eq. (3.8). Use the method of Sect. 3.2 to expand $\cot\gamma$ in a Taylor series.

3.6 The Lagrangian Coefficients

The components of the position and velocity vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ at a given instant of time t_0 serve to describe completely the motion of one body relative to another. In fact, these components can be used as orbital elements and, indeed, for some applications may be the most natural choice. When such is the case, we will require equations for $\mathbf{r}(t)$ and $\mathbf{v}(t)$ in terms of $\mathbf{r}_0$ and $\mathbf{v}_0$. For this purpose, we note that the position and velocity vectors may be expressed in terms of orbital plane coordinates as

$$\begin{aligned} \mathbf{r} &= r\cos f\, \mathbf{i}_e + r\sin f\, \mathbf{i}_p \\ \mathbf{v} &= -\frac{\mu}{h}\sin f\, \mathbf{i}_e + \frac{\mu}{h}(e + \cos f)\, \mathbf{i}_p \end{aligned} \tag{3.31}$$

$\vec{v} = \frac{\mu}{h^2} \vec{h} \times \left(\vec{e} + \frac{\vec{r}}{r} \right)$ segue dalla definizione di $\vec{e}$

(The equation for $\mathbf{v}$ follows at once from Eq. (3.28) with $\mathbf{e} = e\,\mathbf{i}_e$ and $\mathbf{h} = h\,\mathbf{i}_h$.) These equations, of course, are valid at the initial point for which the position and velocity are $\mathbf{r}_0$ and $\mathbf{v}_0$. When they are inverted, the coordinate unit vectors are obtained in terms of these initial vectors.

The inversion is readily accomplished by first observing that the determinant of the two-dimensional matrix of coefficients in Eqs. (3.31) is simply h. Hence,

$$\mathbf{i}_e = \frac{\mu}{h^2}(e + \cos f_0)\,\mathbf{r}_0 - \frac{r_0}{h}\sin f_0\,\mathbf{v}_0$$
$$\mathbf{i}_p = \frac{\mu}{h^2}\sin f_0\,\mathbf{r}_0 + \frac{r_0}{h}\cos f_0\,\mathbf{v}_0 \qquad (3.32)$$

and substitution into Eq. (3.31) gives the desired result in the form

$$\mathbf{r} = F\mathbf{r}_0 + G\mathbf{v}_0$$
$$\mathbf{v} = F_t\mathbf{r}_0 + G_t\mathbf{v}_0 \qquad (3.33)$$

The two-dimensional matrix of coefficients

$$\mathbf{\Phi} = \begin{bmatrix} F & G \\ F_t & G_t \end{bmatrix} \qquad (3.34)$$

acts as a *transition matrix* and the matrix elements are the *Lagrangian coefficients*. Clearly, the coefficients F_t and G_t are simply the respective time derivatives of F and G.

Two basic properties of $\mathbf{\Phi}$ are readily established:

1. The value of the determinant

$$|\mathbf{\Phi}| = FG_t - GF_t = 1 \qquad (3.35)$$

follows from the conservation of angular momentum

$$\mathbf{r} \times \mathbf{v} = (FG_t - GF_t)\mathbf{r}_0 \times \mathbf{v}_0 = \mathbf{r}_0 \times \mathbf{v}_0$$

The inverse of $\mathbf{\Phi}$ is simply

$$\mathbf{\Phi}^{-1} = \begin{bmatrix} G_t & -G \\ -F_t & F \end{bmatrix} \qquad (3.36)$$

so that $\mathbf{\Phi}$ is a symplectic matrix.†

2. For any three points on an orbit $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$

$$\mathbf{\Phi}_{2,0} = \mathbf{\Phi}_{2,1}\mathbf{\Phi}_{1,0} \qquad (3.37)$$

which is proved by successive applications of Eqs. (3.33).

Closed-form equations for the elements of $\mathbf{\Phi}$ do not generally exist in terms of the times t, t_0. However, they are readily obtained as functions

† The symplectic matrix is defined in Chapter 9. It is shown there that any two-dimensional matrix, whose determinant is equal to one, is symplectic.

of the true anomalies f, f_0 by multiplying the two matrices composed of the coefficients of Eqs. (3.31) and (3.32).

It is more convenient to express the Lagrangian coefficients in terms of the true anomaly difference

$$\theta = f - f_0 \tag{3.38}$$

For this purpose, write

$$\cos f = \cos(\theta + f_0) = \cos\theta\cos f_0 - \sin\theta\sin f_0$$

Then, obtain $\cos f_0$ from the equation of orbit and $\sin f_0$ by calculating the scalar product of the two equations in (3.31). Thus,

$$e\cos f_0 = \frac{p}{r_0} - 1 \qquad \text{and} \qquad e\sin f_0 = \frac{\sqrt{p}\,\sigma_0}{r_0} \tag{3.39}$$

where σ_0, which occurs frequently in other contexts, is defined by

$$\sigma_0 \equiv \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}} \tag{3.40}$$

Then the polar form of the equation of orbit, Eq. (3.20), may be written as

$$r = \frac{p r_0}{r_0 + (p - r_0)\cos\theta - \sqrt{p}\,\sigma_0 \sin\theta} \tag{3.41}$$

and the Lagrangian coefficients as

$$\begin{aligned} F &= 1 - \frac{r}{p}(1 - \cos\theta) & G &= \frac{r r_0}{\sqrt{\mu p}}\sin\theta \\ F_t &= \frac{\sqrt{\mu}}{r_0 p}[\sigma_0(1 - \cos\theta) - \sqrt{p}\sin\theta] & G_t &= 1 - \frac{r_0}{p}(1 - \cos\theta) \end{aligned} \tag{3.42}$$

Equations (3.41) and (3.42) are of major importance in our later work.

◇ **Problem 3–26**

Let the skew-symmetric matrix $\mathbf{H}$ be defined as

$$\mathbf{H} = \mathbf{v}\mathbf{r}^T - \mathbf{r}\mathbf{v}^T$$

where the individual matrices $\mathbf{v}\mathbf{r}^T$ and $\mathbf{r}\mathbf{v}^T$ are *dyadic products*. Then, the product $\mathbf{Hw}$ is the vector product of the angular momentum $\mathbf{h}$ and a vector $\mathbf{w}$.

◇ **Problem 3–27**

Derive the following expressions for the reference unit vectors of the orbital plane:

$$\begin{aligned} \mathbf{i}_e &= \frac{1}{e}\left[\left(\frac{v^2}{\mu} - \frac{1}{r}\right)\mathbf{r} - \frac{\sigma}{\sqrt{\mu}}\mathbf{v}\right] \\ \mathbf{i}_p &= \frac{1}{e\sqrt{p}}\left[\frac{\sigma}{r}\mathbf{r} + \frac{p - r}{\sqrt{\mu}}\mathbf{v}\right] \\ \mathbf{i}_h &= \frac{1}{\sqrt{\mu p}}\mathbf{r} \times \mathbf{v} \end{aligned}$$

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4.5 Universal Formulas for Conic Orbits

Thus far we have been obliged to use different formulations to describe the motion of a body in each of the various possible orbits. However, a generalization of the problem is possible using a new family of transcendental functions. With these functions, universally applicable formulas can be developed which are simultaneously valid for the parabola, the ellipse, and the hyperbola.

To motivate the development, the key differential relationships, derived in the previous three sections, can be summarized as

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$$df = \frac{1}{r}\begin{cases} p\,d(\tan\frac{1}{2}f) \\ b\,dE \\ b\,dH \end{cases} = \frac{h}{r^2}\,dt$$

Since, for the three kinds of orbits, we have, respectively,

$$\frac{h}{p} = \sqrt{\frac{\mu}{p}} \qquad \frac{h}{b} = \sqrt{\frac{\mu}{a}} \qquad \frac{h}{b} = \sqrt{\frac{\mu}{-a}}$$

then we may write

$$\sqrt{\mu}\,dt = r\begin{cases} d(\sqrt{p}\tan\frac{1}{2}f) \\ d(\sqrt{a}\,E) \\ d(\sqrt{-a}\,H) \end{cases} = r\,d\chi \tag{4.70}$$

where χ is to be regarded as a new independent variable—a kind of generalized anomaly. It is remarkable that when χ is used as the independent variable instead of the time t, then the nonlinear equations of motion can be converted into linear constant-coefficient differential equations.

The transformation defined by

$$\sqrt{\mu}\,\frac{dt}{d\chi} = r \tag{4.71}$$

is called a *Sundman transformation*† and we shall now demonstrate that $\mathbf{r}$, r, σ, and t can all be obtained as solutions of simple differential equations.

To begin, we differentiate the identity

$$r^2 = \mathbf{r}\cdot\mathbf{r}$$

and obtain

$$r\frac{dr}{d\chi} = \mathbf{r}\cdot\frac{d\mathbf{r}}{d\chi} = \frac{dt}{d\chi}\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = \frac{r}{\sqrt{\mu}}\mathbf{r}\cdot\mathbf{v} = r\sigma$$

† Karl Frithiof Sundman (1873–1949), professor of astronomy at the University of Helsinki and director of the Helsinki Observatory, introduced this transformation in his paper "Mémoire sur le Problème des Trois Corps" published in *Acta Mathematika*, Vol. 36, 1912.

Cancelling the factor r and differentiating a second time, we have

$$\frac{d^2r}{d\chi^2} = \frac{d\sigma}{d\chi} = \frac{r}{\mu}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{v}) = \frac{r}{\mu}\left(v^2 + \mathbf{r}\cdot\frac{d\mathbf{v}}{dt}\right) = \frac{r}{\mu}\left(\frac{2\mu}{r} - \frac{\mu}{a} - \frac{\mu}{r}\right) = 1 - \frac{r}{a}$$

It is convenient here and in the sequel to write α for the reciprocal of a so that α is defined as

$$\alpha \equiv \frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu} \tag{4.72}$$

and may be positive, negative, or zero.

In summary, then

$$\frac{dr}{d\chi} = \sigma = \sqrt{\mu}\,\frac{d^2t}{d\chi^2}$$

$$\frac{d^2r}{d\chi^2} = \frac{d\sigma}{d\chi} = \sqrt{\mu}\,\frac{d^3t}{d\chi^3} = 1 - \alpha r$$

$$\frac{d^3r}{d\chi^3} = \frac{d^2\sigma}{d\chi^2} = \sqrt{\mu}\,\frac{d^4t}{d\chi^4} = -\alpha\frac{dr}{d\chi} = -\alpha\sigma = -\alpha\sqrt{\mu}\,\frac{d^2t}{d\chi^2}$$

so that σ, r, and t are solutions of the equations

$$\frac{d^2\sigma}{d\chi^2} + \alpha\sigma = 0 \qquad \frac{d^3r}{d\chi^3} + \alpha\frac{dr}{d\chi} = 0 \qquad \frac{d^4t}{d\chi^4} + \alpha\frac{d^2t}{d\chi^2} = 0 \tag{4.73}$$

The derivatives of the position vector $\mathbf{r}$

$$\frac{d\mathbf{r}}{d\chi} = \frac{r}{\sqrt{\mu}}\mathbf{v} \qquad \frac{d^2\mathbf{r}}{d\chi^2} = \frac{\sigma}{\sqrt{\mu}}\mathbf{v} - \frac{1}{r}\mathbf{r}$$

lead to

$$\frac{d^3\mathbf{r}}{d\chi^3} + \alpha\frac{d\mathbf{r}}{d\chi} = \mathbf{0} \tag{4.74}$$

in a similar manner.

Linear differential equations with constant coefficients present no particular difficulty in their solution. Nevertheless, it is advantageous in this case to develop the solutions in a form utilizing a family of special functions defined solely for this purpose.

The Universal Functions $U_n(\chi;\alpha)$

To construct the family of special functions, we begin by determining the power series solution of

$$\frac{d^2\sigma}{d\chi^2} + \alpha\sigma = 0$$

by substituting

$$\sigma = \sum_{k=0}^{\infty} a_k\chi^k$$

and equating coefficients of like powers of χ. We are led to

$$a_{k+2} = -\frac{\alpha}{(k+1)(k+2)}a_k \qquad \text{for} \qquad k = 0, 1, \ldots$$

as a recursion formula for the coefficients. Hence

$$\sigma = a_0\left[1 - \frac{\alpha\chi^2}{2!} + \frac{(\alpha\chi^2)^2}{4!} - \cdots\right] + a_1\chi\left[1 - \frac{\alpha\chi^2}{3!} + \frac{(\alpha\chi^2)^2}{5!} - \cdots\right]$$

where a_0 and a_1 are two arbitrary constants. We shall designate the two series expansions by $U_0(\chi;\alpha)$ and $U_1(\chi;\alpha)$ so that

$$\sigma = a_0U_0(\chi;\alpha) + a_1U_1(\chi;\alpha)$$

The function U_1 is simply the integral of U_0 so that we are motivated to define a sequence of functions

$$U_1 = \int_0^\chi U_0\,d\chi \qquad U_2 = \int_0^\chi U_1\,d\chi \qquad U_3 = \int_0^\chi U_2\,d\chi \qquad \text{etc.}$$

The n^{th} function of such a sequence is easily seen to be

$$U_n(\chi;\alpha) = \chi^n\left[\frac{1}{n!} - \frac{\alpha\chi^2}{(n+2)!} + \frac{(\alpha\chi^2)^2}{(n+4)!} - \cdots\right] \tag{4.75}$$

A basic identity for the U functions is at once apparent from the series definition of $U_n(\chi;\alpha)$. Since Eq. (4.75) may be written as

$$U_n(\chi;\alpha) = \frac{\chi^n}{n!} - \alpha\chi^{n+2}\left[\frac{1}{(n+2)!} - \frac{\alpha\chi^2}{(n+4)!} + \frac{(\alpha\chi^2)^2}{(n+6)!} - \cdots\right]$$

we have

$$U_n(\chi;\alpha) + \alpha U_{n+2}(\chi;\alpha) = \frac{\chi^n}{n!} \tag{4.76}$$

It is clear, from the manner in which the family of functions was constructed, that

$$\frac{dU_n}{d\chi} = U_{n-1} \qquad \text{for} \qquad n = 1, 2, \ldots \tag{4.77}$$

and, by differentiating the series for U_0, we can easily show that

$$\frac{dU_0}{d\chi} = -\alpha U_1 \tag{4.78}$$

Now, if we differentiate the identity (4.76) $m+1$ times, where $m \geqslant n$, and use Eq. (4.77), we obtain

$$\frac{d^{m+1}U_n}{d\chi^{m+1}} + \alpha\frac{d^{m-1}U_n}{d\chi^{m-1}} = 0 \qquad \text{for} \qquad n = 0, 1, \ldots, m \tag{4.79}$$

It follows that U_0 and U_1 are each solutions of the second-order differential equation satisfied by σ, and we recall that σ was, indeed, found to be a linear combination of U_0 and U_1.

Finally, by applying the identity (4.79) to the other two differential equations in (4.73), we conclude that r is a linear combination of U_0, U_1, U_2 while t is a linear combination of U_0, U_1, U_2, U_3. These will be the general solutions provided, of course, that the U functions are linearly independent.

Linear Independence of $U_n(\chi;\alpha)$

The functions $U_0, U_1, \ldots, U_n$ will be *linearly independent* if no one of the functions can be expressed as a linear combination of the others, or, equivalently, if no linear combination of the functions is identically zero over any interval of χ under consideration.

It is known that the functions will be linearly independent if the associated *Wronskian determinant*† is not identically zero. The elements of the first row of this determinant are the functions $U_0, U_1, \ldots, U_n$. The second row consists of the first derivatives of these functions, the third row, the second derivatives, and so forth with the last or $(n+1)^{\text{th}}$ row containing the n^{th} derivatives.

For example, if $n = 3$, the Wronskian is

$$W = \begin{vmatrix} U_0 & U_1 & U_2 & U_3 \\ -\alpha U_1 & U_0 & U_1 & U_2 \\ -\alpha U_0 & -\alpha U_1 & U_0 & U_1 \\ \alpha^2 U_1 & -\alpha U_0 & -\alpha U_1 & U_0 \end{vmatrix}$$

where we have used the identities (4.77) and (4.78) to replace the derivatives by the appropriate U functions.

To evaluate the determinant, we multiply the first row by α and add to the third row. Then, the second row is multiplied by α and added to the fourth row. Where appropriate, we utilize the identity (4.76) and obtain

$$W = \begin{vmatrix} U_0 & U_1 & U_2 & U_3 \\ -\alpha U_1 & U_0 & U_1 & U_2 \\ 0 & 0 & 1 & \chi \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Hence, the value of W is simply $U_0^2 + \alpha U_1^2$. Indeed, it is easy to see that W will have this value for any $n > 0$. Therefore, the question of linear independence will be resolved when we show that

$$U_0^2 + \alpha U_1^2 = 1 \tag{4.80}$$

for all values of χ.

† The name was given by Thomas Muir in 1882 to honor the Polish mathematician and philosopher Józef Maria Höené-Wroński (1776–1853) who first used this determinant in his studies of differential equations.

To this end, we multiply the identity [Eq. (4.76) with $n = 0$]

$$U_0 + \alpha U_2 = 1$$

by U_1 and integrate with respect to χ. We have

$$U_1^2 + \alpha U_2^2 = 2U_2$$

or

$$U_1^2 + U_2(1 - U_0) = 2U_2$$

Hence

$$U_2 = U_1^2 - U_0 U_2$$

Substituting this, for U_2 in the equation $U_0 + \alpha U_2 = 1$, yields

$$\begin{aligned} U_0 + \alpha U_2 = 1 &= U_0 + \alpha(U_1^2 - U_0 U_2) \\ &= U_0 + \alpha U_1^2 - U_0(1 - U_0) \\ &= U_0^2 + \alpha U_1^2 \end{aligned}$$

and the identity (4.80) is established.

Lagrangian Coefficients and Other Orbital Quantities

Since the U functions are linearly independent, the general solution of the differential equation for t may be written as

$$\sqrt{\mu}(t - t_0) = a_0 U_0 + a_1 U_1 + a_2 U_2 + a_3 U_3$$

If we require $t = t_0$ when $\chi = 0$, then we find that a_0 must be zero. The derivative of this expression, according to Eq. (4.71), yields

$$r = a_1 U_0 + a_2 U_1 + a_3 U_2$$

Setting $\chi = 0$, gives $a_1 = r_0$. Differentiating again produces

$$\sigma = -\alpha r_0 U_1 + a_2 U_0 + a_3 U_1$$

so that $a_2 = \sigma_0$. Finally, calculating one more derivative, we have

$$1 - \alpha r = -\alpha r_0 U_0 - \alpha \sigma_0 U_1 + a_3 U_0$$

from which $a_3 = 1$.

In this manner, we obtain the generalized form of Kepler's equation

$$\sqrt{\mu}(t - t_0) = r_0 U_1(\chi; \alpha) + \sigma_0 U_2(\chi; \alpha) + U_3(\chi; \alpha) \tag{4.81}$$

together with

$$r = r_0 U_0(\chi; \alpha) + \sigma_0 U_1(\chi; \alpha) + U_2(\chi; \alpha) \tag{4.82}$$

$$\sigma = \sigma_0 U_0(\chi; \alpha) + (1 - \alpha r_0) U_1(\chi; \alpha) \tag{4.83}$$

In a similar fashion, we write

$$\mathbf{r} = U_0\mathbf{a}_0 + U_1\mathbf{a}_1 + U_2\mathbf{a}_2$$
$$\frac{r}{\sqrt{\mu}}\mathbf{v} = -\alpha U_1\mathbf{a}_0 + U_0\mathbf{a}_1 + U_1\mathbf{a}_2$$
$$\frac{\sigma}{\sqrt{\mu}}\mathbf{v} - \frac{1}{r}\mathbf{r} = -\alpha U_0\mathbf{a}_0 - \alpha U_1\mathbf{a}_1 + U_0\mathbf{a}_2$$

and determine the vectors $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ by setting $\chi = 0$. Thus, we obtain the following expressions for the Lagrangian coefficients

$$\begin{aligned} F &= 1 - \frac{1}{r_0}U_2(\chi;\alpha) & G &= \frac{r_0}{\sqrt{\mu}}U_1(\chi;\alpha) + \frac{\sigma_0}{\sqrt{\mu}}U_2(\chi;\alpha) \\ F_t &= -\frac{\sqrt{\mu}}{rr_0}U_1(\chi;\alpha) & G_t &= 1 - \frac{1}{r}U_2(\chi;\alpha) \end{aligned} \tag{4.84}$$

These equations are "universal" in the sense that they are valid for all conic orbits† and are void of singularities. For this reason the U functions are referred to as *universal functions*. As we indicated at the beginning of this section, χ is a generalized anomaly and is related to the classical ones by

$$\chi = \begin{cases} \sqrt{p}\,(\tan\frac{1}{2}f - \tan\frac{1}{2}f_0) = \sigma - \sigma_0 \\ \sqrt{a}\,(E - E_0) \\ \sqrt{-a}\,(H - H_0) \end{cases} \tag{4.85}$$

Finally, an important relation for χ can be derived. If we multiply Eq. (4.81) by α and add Eq. (4.83), we have

$$\alpha\sqrt{\mu}(t - t_0) + \sigma = U_1 + \alpha U_3 + \sigma_0(U_0 + \alpha U_2)$$

Hence, using Eq. (4.76),

$$\chi = \alpha\sqrt{\mu}(t - t_0) + \sigma - \sigma_0 \tag{4.86}$$

is obtained as an explicit expression for χ which does not involve any of the U functions.‡

† The case of the parabola was considered separately in Sect. 4.2.

‡ Equation (4.86) was discovered in August of 1967 by Charles M. Newman—a staff member of the MIT Instrumentation Laboratory during the era of Apollo. His derivation was more involved than the one presented here.

◇ Problem 4–21

The U functions are, of course, related to the elementary functions but the particular relations depend on whether the orbit is a parabola $\alpha = 0$, an ellipse $\alpha > 0$, or a hyperbola $\alpha < 0$. The first four of the U functions are given by

$$U_0(\chi;\alpha) = \begin{cases} 1 \\ \cos(\sqrt{\alpha}\,\chi) \\ \cosh(\sqrt{-\alpha}\,\chi) \end{cases} \qquad U_1(\chi;\alpha) = \begin{cases} \chi \\ \dfrac{\sin(\sqrt{\alpha}\,\chi)}{\sqrt{\alpha}} \\ \dfrac{\sinh(\sqrt{-\alpha}\,\chi)}{\sqrt{-\alpha}} \end{cases}$$

$$U_2(\chi;\alpha) = \begin{cases} \dfrac{\chi^2}{2} \\ \dfrac{1-\cos(\sqrt{\alpha}\,\chi)}{\alpha} \\ \dfrac{\cosh(\sqrt{-\alpha}\,\chi)-1}{-\alpha} \end{cases} \qquad U_3(\chi;\alpha) = \begin{cases} \dfrac{\chi^3}{6} \\ \dfrac{\sqrt{\alpha}\,\chi-\sin(\sqrt{\alpha}\,\chi)}{\alpha\sqrt{\alpha}} \\ \dfrac{\sinh(\sqrt{-\alpha}\,\chi)-\sqrt{-\alpha}\,\chi}{-\alpha\sqrt{-\alpha}} \end{cases}$$

Problem 4–22

If we define a new universal anomaly ψ as

$$\psi = \frac{r_0\chi}{\sqrt{\mu}(t-t_0)}$$

then the universal form of Kepler's equation may be written either as

$$1 = U_1(\psi;\varsigma) + \xi U_2(\psi;\varsigma) + \eta U_3(\psi;\varsigma)$$

where

$$\xi = \frac{\sqrt{\mu}(t-t_0)\sigma_0}{r_0^2} \qquad \eta = \frac{\mu(t-t_0)^2}{r_0^3} \qquad \varsigma = \frac{\alpha\mu(t-t_0)^2}{r_0^2}$$

or as

$$1 = \psi + \xi U_2(\psi;\varsigma) + \lambda U_3(\psi;\varsigma)$$

where

$$\lambda = \eta - \varsigma$$

Observe that for parabolic orbits the second form of Kepler's equation becomes

$$1 = \psi + \tfrac{1}{2}\xi\psi^2 + \tfrac{1}{6}\lambda\psi^3$$

the solution of which provides a good initial approximation for the near parabolic case.

Also, for circular orbits, the solution is simply $\psi = 1$, providing a good approximation for near circular orbits.

Karl Stumpff 1958

◇ **Problem 4–23**

Introduce the quantity

$$\psi = \alpha\chi^2$$

so that a family of functions $c_n(\psi)$ can be defined in terms of the U functions by

$$\chi^n c_n(\psi) = U_n(\chi;\alpha)$$

Indeed, the entire subject of universal functions can be developed in terms of these alternate functions $c_n(\psi)$.

(a) Derive the series representation

$$c_n(\psi) = \frac{1}{n!} - \frac{\psi}{(n+2)!} + \frac{\psi^2}{(n+4)!} - \cdots$$

together with the recursion formula

$$c_n + \psi c_{n+2} = \frac{1}{n!}$$

and the identity

$$c_0^2 + \psi c_1^2 = 1$$

(b) Derive the following derivative formulas

$$\frac{dc_0}{d\psi} = -\frac{c_1}{2}$$

$$\frac{dc_n}{d\psi} = \frac{1}{2\psi}(c_{n-1} - nc_n) \qquad \text{for} \qquad n = 1, 2, \ldots$$

$$= \tfrac{1}{2}(nc_{n+2} - c_{n+1}) \qquad \text{for} \qquad n = 0, 1, \ldots$$

(c) The first four c functions† are related to the elementary functions as follows

$$c_0(\psi) = \begin{cases} 1 \\ \cos\sqrt{\psi} \\ \cosh\sqrt{-\psi} \end{cases} \qquad c_1(\psi) = \begin{cases} 1 \\ \dfrac{\sin\sqrt{\psi}}{\sqrt{\psi}} \\ \dfrac{\sinh\sqrt{-\psi}}{\sqrt{-\psi}} \end{cases}$$

$$c_2(\psi) = \begin{cases} \dfrac{1}{2} \\ \dfrac{1-\cos\sqrt{\psi}}{\psi} \\ \dfrac{\cosh\sqrt{-\psi}-1}{-\psi} \end{cases} \qquad c_3(\psi) = \begin{cases} \dfrac{1}{6} \\ \dfrac{\sqrt{\psi}-\sin\sqrt{\psi}}{\psi\sqrt{\psi}} \\ \dfrac{\sinh\sqrt{-\psi}-\sqrt{-\psi}}{-\psi\sqrt{-\psi}} \end{cases}$$

where the alternate representations depend upon the sign of ψ.

† The functions $c_2(\psi)$ and $c_3(\psi)$ are identical with the functions $C(x)$ and $S(x)$ originally defined by the author in his book *Astronautical Guidance*. Their use in the Apollo program is documented in the Epilogue of this book.

◇ Problem 4–24

Consider another form of the Sundman transformation

$$\frac{dt}{d\chi} = r$$

(a) If $\sqrt{\mu}\,\sigma$ is replaced by σ (that is, σ is defined as $\sigma = \mathbf{r}\cdot\mathbf{v}$), then t, r, and σ are given by

$$\begin{aligned} t - t_0 &= r_0 U_1(\chi;\mu\alpha) + \sigma_0 U_2(\chi;\mu\alpha) + \mu U_3(\chi;\mu\alpha) \\ r &= r_0 U_0(\chi;\mu\alpha) + \sigma_0 U_1(\chi;\mu\alpha) + \mu U_2(\chi;\mu\alpha) \\ \sigma &= \sigma_0 U_0(\chi;\mu\alpha) + \mu(1 - \alpha r_0) U_1(\chi;\mu\alpha) \end{aligned}$$

(b) Further obtain the Lagrangian coefficients

$$F = 1 - \frac{\mu}{r_0} U_2(\chi;\mu\alpha) \qquad G = r_0 U_1(\chi;\mu\alpha) + \sigma_0 U_2(\chi;\mu\alpha)$$

$$F_t = -\frac{\mu}{r r_0} U_1(\chi;\mu\alpha) \qquad G_t = 1 - \frac{\mu}{r} U_2(\chi;\mu\alpha)$$

NOTE: In this form, the solutions of the two-body equations of motion do not require that μ be positive so that they are equally valid for repulsive as well as attractive forces.

William H. Goodyear† 1965

Problem 4–25

Parabolic coordinates ξ, η are defined by the transformation

$$x = \xi^2 - \eta^2 \qquad y = 2\xi\eta$$

which provides a mapping of the ξ, η plane onto the x, y plane. The inverse transformation is most conveniently expressed in terms of polar coordinates r, θ in the x, y plane.

(a) Show that

$$\xi = \sqrt{r}\cos\tfrac{1}{2}\theta \qquad \eta = \sqrt{r}\sin\tfrac{1}{2}\theta$$

is the appropriate mapping of the x, y plane onto the ξ, η plane.

(b) The two-body equations of motion in the x, y plane are transformed into

$$\frac{d^2\xi}{d\chi^2} + \frac{\alpha}{4}\xi = 0 \qquad \frac{d^2\eta}{d\chi^2} + \frac{\alpha}{4}\eta = 0$$

in the ξ, η plane, where $\alpha = 1/a$ is the reciprocal of the semimajor axis and χ is defined by the Sundman or *regularization transformation*

$$\sqrt{\mu}\frac{dt}{d\chi} = r$$

Thus, we see that the two-body motion in parabolic coordinates consists of two independent harmonic oscillators of the same frequency.

André Deprit 1968

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† "Completely General Closed-Form Solution for Coordinates and Partial Derivatives of the Two-Body Problem," *The Astronomical Journal*, Vol. 70, April 1965, pp. 189–192.

4.6 Identities for the Universal Functions

There are a variety of identities involving the functions $U_n(\chi;\alpha)$, many of which will be required in further applications. These will be developed and collected in this section to serve as a ready reference when needed.

Because of the direct relationship between U_0, U_1 and the circular and hyperbolic functions, as seen in Prob. 4–21, we can immediately recognize

$$U_0^2 + \alpha U_1^2 = 1 \tag{4.87}$$

as the best known identity between sines and cosines or hyperbolic sines and cosines. Similarly, we can write

$$\begin{aligned} U_0(\chi \pm \psi) &= U_0(\chi)U_0(\psi) \mp \alpha U_1(\chi)U_1(\psi) \\ U_1(\chi \pm \psi) &= U_1(\chi)U_0(\psi) \pm U_0(\chi)U_1(\psi) \end{aligned} \tag{4.88}$$

and

$$\begin{aligned} U_0(2\chi) &= U_0^2(\chi) - \alpha U_1^2(\chi) = 2U_0^2(\chi) - 1 = 1 - 2\alpha U_1^2(\chi) \\ U_1(2\chi) &= 2U_0(\chi)U_1(\chi) \end{aligned} \tag{4.89}$$

as counterparts of other familiar identities. Just as Eq. (4.87) was derived earlier, without resort to its relation with the elementary functions, so also could these and all identities involving just U_0 and U_1.

For the higher order U functions, the analogy with the elementary functions is not convenient to exploit and other techniques will have to be employed.

Identities Involving Compound Arguments

The basic equation, from which all the identities will evolve, is

$$U_n + \alpha U_{n+2} = \frac{\chi^n}{n!} \tag{4.90}$$

For $n = 0$, we have

$$\alpha U_2(\chi \pm \psi) = 1 - U_0(\chi \pm \psi) = 1 - U_0(\chi)U_0(\psi) \pm \alpha U_1(\chi)U_1(\psi)$$

but this equation is not useful to calculate $U_2(\chi \pm \psi)$ since division by α would be required. (It will be a cardinal rule that we must *never* divide by α in any calculation involving universal functions.)

To obtain a proper identity, we write

$$\alpha U_2(\chi \pm \psi) = 1 - [1 - \alpha U_2(\chi)][1 - \alpha U_2(\psi)] \pm \alpha U_1(\chi)U_1(\psi)$$

so that α may be cancelled as a common factor. There results

$$U_2(\chi \pm \psi) = U_2(\chi)[1 - \alpha U_2(\psi)] + U_2(\psi) \pm U_1(\chi)U_1(\psi)$$

Hence, finally,

$$U_2(\chi \pm \psi) = U_2(\chi)U_0(\psi) + U_2(\psi) \pm U_1(\chi)U_1(\psi) \tag{4.91}$$

◇ Problem 4–26

A generalization of the well-known Euler identity for trigonometric functions is

$$e^{i\sqrt{\alpha}\,\chi} = U_0(\chi;\alpha) + i\sqrt{\alpha}\,U_1(\chi;\alpha)$$

where $i = \sqrt{-1}$. Use this relation to derive Eqs. (4.88).

◇ Problem 4–27

Derive the following identities for the universal functions of the sum and difference of two arguments:

$$U_3(\chi \pm \psi) = U_3(\chi) \pm U_3(\psi) + U_1(\chi)U_2(\psi) \pm U_2(\chi)U_1(\psi)$$
$$U_4(\chi \pm \psi) = U_2(\chi)U_2(\psi) + U_4(\chi) + U_4(\psi) \pm \psi U_3(\chi) \pm U_1(\chi)U_3(\psi)$$

Identities for $U_n^2(\chi;\alpha)$

The method used to establish the identity

$$U_1^2 = U_2(1 + U_0)$$

which was derived in the previous section as a part of the calculation of the Wronskian of the U functions, can be generalized to produce a sequence of identities. For this purpose, multiply Eq. (4.90) by U_{n+1} and rewrite as

$$\frac{d}{d\chi}(U_{n+1}^2 + \alpha U_{n+2}^2) = 2\frac{\chi^n}{n!}U_{n+1}$$

Hence

$$U_{n+1}^2 + \alpha U_{n+2}^2 = 2\left[\frac{\chi^n}{n!}U_{n+2} - \frac{\chi^{n-1}}{(n-1)!}U_{n+3} + \cdots \pm U_{2n+2}\right]$$

is obtained by integrating the right-hand side by parts. Then, using Eq. (4.90) again, we have

$$U_{n+1}^2 = U_{n+2}\left(\frac{\chi^n}{n!} + U_n\right) - 2\left[\frac{\chi^{n-1}}{(n-1)!}U_{n+3} - \frac{\chi^{n-2}}{(n-2)!}U_{n+4} + \cdots\right] \quad (4.92)$$

Therefore, by setting $n = 0,\ 1,\ 2,\ \ldots,$ we may establish successively

$$\begin{aligned} U_1^2 &= U_2(1 + U_0) \\ U_2^2 &= U_3(\chi + U_1) - 2U_4 \\ U_3^2 &= U_4(\tfrac{1}{2}\chi^2 + U_2) - 2(\chi U_5 - U_6) \qquad \text{etc.} \end{aligned} \quad (4.93)$$

These equations are particularly useful to calculate $U_4, U_6, U_8, \ldots$ in terms of the U functions with lower subscripts. Similar explicit relations for the odd-ordered functions do not seem to exist. Of course, Eq. (4.90) permits a simple solution to the reverse problem, i.e., calculating lower-order functions from higher-order ones.

Identities for $U_{n+1}U_{n+1-m} - U_{n+2}U_{n-m}$

For any integer $m \leq n$, the identity (4.90) may be written as

$$U_{n-m} + \alpha U_{n+2-m} = \frac{\chi^{n-m}}{(n-m)!}$$

Now multiply this by U_{n+1} and multiply Eq. (4.90) by U_{n+1-m}. Adding the resulting two equations gives

$$\frac{d}{d\chi}(U_{n+1}U_{n+1-m} + \alpha U_{n+2}U_{n+2-m}) = \frac{\chi^{n-m}}{(n-m)!}U_{n+1} + \frac{\chi^n}{n!}U_{n+1-m}$$

Hence

$$\begin{aligned} U_{n+1}U_{n+1-m} &+ \alpha U_{n+2}U_{n+2-m} = \\ &\frac{\chi^{n-m}}{(n-m)!}U_{n+2} - \frac{\chi^{n-m-1}}{(n-m-1)!}U_{n+3} + \cdots \pm U_{2n+2-m} \\ &\quad + \frac{\chi^n}{n!}U_{n+2-m} - \frac{\chi^{n-1}}{(n-1)!}U_{n+3-m} + \cdots \pm U_{2n+2-m} \end{aligned}$$

or

$$\begin{aligned} U_{n+1}U_{n+1-m} &- U_{n+2}U_{n-m} = \\ &\frac{\chi^n}{n!}U_{n+2-m} - \frac{\chi^{n-1}}{(n-1)!}U_{n+3-m} + \cdots \pm U_{2n+2-m} \\ &- \frac{\chi^{n-m-1}}{(n-m-1)!}U_{n+3} + \frac{\chi^{n-m-2}}{(n-m-2)!}U_{n+4} - \cdots \mp U_{2n+2-m} \end{aligned} \quad (4.94)$$

which agrees with Eq. (4.92) for $m = 0$. The following identities result for $(m, n) = (1, 1)$, $(1, 2)$, $(2, 2)$:

$$\begin{aligned} U_2U_1 - U_3U_0 &= \chi U_2 - U_3 \\ U_3U_2 - U_4U_1 &= \tfrac{1}{2}\chi^2U_3 - \chi U_4 \\ U_3U_1 - U_4U_0 &= \tfrac{1}{2}\chi^2U_2 - \chi U_3 + U_4 \end{aligned} \quad (4.95)$$

◇ **Problem 4–28**

Derive the identity

$$U_n(m\chi) + \alpha U_{n+2}(m\chi) = m^n[U_n(\chi) + \alpha U_{n+2}(\chi)]$$

where m is an integer.

◇ **Problem 4–29**

Show that

$$U_n(k\chi; \alpha) = k^n U_n(\chi; k^2\alpha)$$

obtains for any value of the parameter k.

Problem 4–30
Derive the identity

$$U_3^2 = \tfrac{1}{6}\chi^3 U_3 + U_5(U_1 - \chi)$$

◇ **Problem 4–31**
Derive the following double argument identities for the universal functions:

$$U_2(2\chi) = 2U_1^2(\chi) \qquad U_4(2\chi) = 2U_2^2(\chi) + 4U_4(\chi)$$
$$U_3(2\chi) = 2U_3(\chi) + 2U_1(\chi)U_2(\chi) \qquad = 2U_3(\chi)[\chi + U_1(\chi)]$$
$$= 2U_0(\chi)U_3(\chi) + 2\chi U_2(\chi) \qquad U_5(2\chi) = 2U_1(\chi)U_4(\chi) + \chi^2 U_3(\chi) + 2U_5(\chi)$$

Identities Involving the True Anomaly Difference

Important relationships between the functions $U_n(\chi;\alpha)$ and trigonometric functions of the true anomaly difference $\theta = f - f_0$ can be obtained by comparing Eqs. (3.42) and (4.84). Thus,

$$\begin{aligned} U_1(\chi;\alpha) &= \frac{r}{p}[\sqrt{p}\sin\theta - \sigma_0(1-\cos\theta)] \\ U_2(\chi;\alpha) &= \frac{rr_0}{p}(1-\cos\theta) \end{aligned} \tag{4.96}$$

Also, by using the identities

$$U_1(\chi) = 2U_0(\tfrac{1}{2}\chi)U_1(\tfrac{1}{2}\chi) \qquad \text{and} \qquad U_2(\chi) = 2U_1^2(\tfrac{1}{2}\chi)$$

we find that

$$\begin{aligned} U_0(\tfrac{1}{2}\chi;\alpha) &= \sqrt{\frac{r}{r_0 p}}(\sqrt{p}\cos\tfrac{1}{2}\theta - \sigma_0\sin\tfrac{1}{2}\theta) \\ U_1(\tfrac{1}{2}\chi;\alpha) &= \sqrt{\frac{rr_0}{p}}\sin\tfrac{1}{2}\theta \end{aligned} \tag{4.97}$$

which may be written alternately as

$$\begin{aligned} \sin\tfrac{1}{2}\theta &= \sqrt{\frac{p}{rr_0}}U_1(\tfrac{1}{2}\chi;\alpha) \\ \cos\tfrac{1}{2}\theta &= \frac{1}{\sqrt{rr_0}}[r_0U_0(\tfrac{1}{2}\chi;\alpha) + \sigma_0U_1(\tfrac{1}{2}\chi;\alpha)] \end{aligned} \tag{4.98}$$

In particular, we obtain

$$\tan\tfrac{1}{2}\theta = \frac{\sqrt{p}\,U_1(\tfrac{1}{2}\chi;\alpha)}{r_0U_0(\tfrac{1}{2}\chi;\alpha) + \sigma_0U_1(\tfrac{1}{2}\chi;\alpha)} \tag{4.99}$$

as a convenient formula for determining θ from χ.

1) Spiegazione della relazione

$$df = \frac{1}{r} p \, d\left(\tan \frac{f}{2}\right)$$

→ verifica diretta

$$df = (1+\cos f) \frac{1}{2\cos^2 \frac{f}{2}} df \qquad 1+\cos f = 2\cos^2 \frac{f}{2}$$

2) Spiegazione della relazione

$$df = \frac{1}{r} b \, dE$$

→

da $\cos f = \dfrac{\cos E - e}{1 - e\cos E}$ si ha $2\cos^2 \dfrac{f}{2} = 1 + \dfrac{\cos E - e}{1 - e\cos E}$

$$\cos^2 \frac{f}{2} = \frac{a}{r}(1-e)\cos^2 \frac{E}{2} \quad *$$

calcoliamo il differenziale di

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$

$$\frac{1}{2} \frac{1}{\cos^2 \frac{f}{2}} df = \frac{1}{2} \sqrt{\frac{1+e}{1-e}} \frac{1}{\cos^2 \frac{E}{2}} dE$$

e usando * si ottiene

$$\frac{r}{a(1-e)} df = \sqrt{\frac{1+e}{1-e}} dE$$

$$df = \frac{a}{r}\sqrt{1-e^2}\, dE = \frac{b}{r} dE$$

3) Spiegazione della relazione

$$df = \frac{1}{r} b \, dH$$

→ trattazione caso iperbolico

e poi si procede come nel caso dell'anomalia eccentrica usando le relazioni

$$\cos f = \frac{e - \cosh H}{e \cosh H - 1}$$

$$\tan \frac{f}{2} = \sqrt{\frac{e+1}{e-1}} \tanh \frac{H}{2}$$