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« Theory of orbits »

Chapter 3

Regularization

3.1 Introduction

A significant difference between the motion of natural (celestial) bodies and that of artificial bodies is that close approaches are common occurrences in the latter case while in the former they happen but seldom.

The consequences of this fact can be understood when the property of the Newtonian gravitational force field is recalled, according to which the forces acting between particles approach infinity when the distance between the bodies approaches zero. Therefore at collision (when r_1 or r_2 is zero) the equations show singularities.

Space probes often require orbits which connect two celestial bodies. The actual orbits, of course, do not go through the points of singularity since before this happens the trajectory ends at the point of impact between the probe and the surface of the celestial body. In the framework of the problem of three bodies this impact can take place only at the singularity since the participating bodies are point masses. Therefore, in the physical sense singularities are never reached by means of collision in celestial mechanics. In numerical (computational) and conceptual respects the singularity aspects of the problem of three bodies are of utmost importance.

Both the force acting on the third body and its velocity increase as the body approaches the vicinity of one of the primaries. The step size of a numerical integration must be decreased significantly in this region in order to obtain reliable results. The physical aspects of space

trajectories also require an increased accuracy in these critical regions near the singularities.

The conceptual aspects of the singularities in the field are connected with the existence of the solution of differential equations.

Since the singularities occurring at collisions are not of essential character, they can be eliminated by the proper choice of the independent variable. Once this has been done the following have been effected:

(i) existence of solutions has been established for an arbitrary selection of the initial conditions;
(ii) solutions going through singularities can be traced analytically;
(iii) solutions may be established numerically before, at, and after collision;
(iv) close approaches may be treated with analytical and numerical precision.

In this chapter we proceed from discussing the simplest straightline collision-orbit without regularization, to the complete regularization of the restricted problem. Accordingly, the first subject is the problem of two bodies. The corresponding equations of motion are regularized in two steps. First we treat collision orbits (Section 3.2), then the general problem is regularized (Section 3.3). This is followed by the regularization of the equations of motion of the restricted problem. Here first we solve the problem "locally," by which in this context we mean that we regularize the equations of motion only at one of the two singularities (Section 3.4). Following Birkhoff's terminology, we then affect "global" regularizations; we eliminate both singularities simultaneously. Both the "global" and the "local" regularizations may be, according to mathematical usage, considered local operations; nevertheless, we accept Birkhoff's terminology at this point and defer additional remarks to Section 3.10. The global regularization may be performed with several transformations, and we describe three methods in Sections 3.5–3.7. Generalizations and comparisons of global regularization techniques are offered then in Sections 3.8 and 3.9. Section 3.10 contains the theorem of existence of solutions of the restricted problem for finite intervals of time. The chapter is concluded with two major areas of application: space dynamics and stellar dynamics,

The following two additional remarks are in order at this point. The *regularization of the differential equations* of motion is the principal problem and the main subject of this chapter. The *regularization of the solution* at collision can always be accomplished by introducing the eccentric anomaly since the collision of two bodies in any problem can be regularized in this way.

The second remark is that the inclusion of a chapter on regularization in a book on celestial mechanics might serve two purposes. It helps to establish the existence of solutions from the point of view of analysis and it extends the applicability of celestial mechanics to collision orbits. Books on classical celestial mechanics often omit entirely the subject of regularization since the analyticity of solutions is not of central interest to astronomers and because collisions do not occur in classical celestial mechanics.

For our purposes inclusion of the chapter on regularization is a necessity in order to satisfy our interest in applications to space dynamics as well as to establish the existence of the solutions of the restricted problem. To these solutions, after all, our whole volume is dedicated.

The questions of existence are treated at the end of the chapter, only after a relatively leisurely journey through the methods of regularization. The first three sections after the introduction (Sections 3.2, 3.3, and 3.4) the reader will find comfortable, mostly easy going, and at some places even repetitious. Teaching experience suggests such a treatment to lay a sound foundation for the more complex aspects. Since regularization is essential in space dynamics, the presentation of Sections 3.2, 3.3, and 3.4 is directed to workers in this field, leaving some of the analytical aspects to Section 3.10.

3.2 Regularization of collision orbits in the problem of two bodies

Part (A) of this section describes the dynamics of a simple collision orbit without the use of regularizing variables. In Part (B) regularizing variables are introduced in a general form, and in Part (C) a special transformation concerning only the independent variable is treated. Part (D) offers an explanation of the mathematical result according to which the particle is reflected after collision, shows the regularizing role of the eccentric anomaly, and proves that the distance to collision varies as the 2/3 power of the time. In Part (E) regularization is affected by transforming the independent as well as the dependent variables.

(A) Consider the equations of motion [Section 1.5, Eq. (52)] of the restricted problem and let $\mu_1 = 1$, $\mu_2 = 0$. The corresponding function Ω is

$$\Omega = \frac{1}{r} + \frac{1}{2}r^2 \tag{1}$$

since $r^2 = r_1^2 = x^2 + y^2$ [see Fig. 3.1(a)]. The equations of motion are

$$\ddot{x} - 2\dot{y} = x(1 - 1/r^3), \qquad \ddot{y} + 2\dot{x} = y(1 - 1/r^3), \tag{2}$$

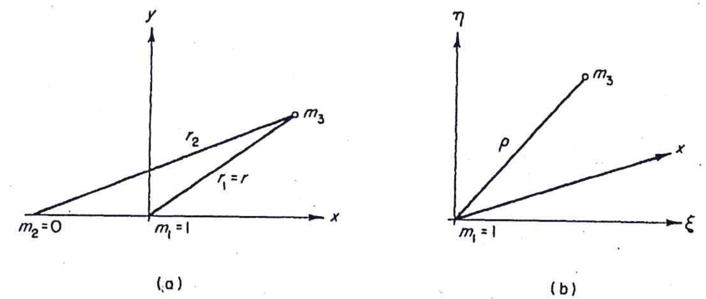

FIG. 3.1. Problem of two bodies.

and the Jacobian integral is

$$\dot{x}^2 + \dot{y}^2 = r^2 + 2/r - C. \tag{3}$$

Equations (2) describe the problem of two bodies; in fact they refer to a simplified restricted problem of three bodies in which the mass of one of the two primaries is zero. This description, however, is in a synodic coordinate system which renders the equations rather complicated. The equations of motion in a corresponding fixed system are

$$\ddot{\xi} = -\xi/\rho^3 \quad \text{and} \quad \ddot{\eta} = -\eta/\rho^3, \tag{4}$$

where $\rho^2 = r^2 = x^2 + y^2 = \xi^2 + \eta^2$ [see Fig. 3.1(b)]. The energy integral of Eqs. (4) is

$$\dot{\xi}^2 + \dot{\eta}^2 = 2/\rho - C. \tag{5}$$

In the first instance the simplest possible close approach, a collision, will be considered. Since a two-body collision orbit in a fixed system of coordinates is a straight line, we might specify the following initial conditions: at $t = 0$, $\xi = \xi_0$, $\dot{\xi} = \dot{\xi}_0$, and $\eta \equiv 0$, $\dot{\eta} \equiv 0$, for any t. Noting that $\rho = |\xi|$ we have

$$\ddot{\xi} = \mp 1/\xi^2 \tag{6}$$

for $\xi \gtrless 0$. The energy integral gives

$$\dot{\xi}^2 = 2/|\xi| - C = \pm 2/\xi - C, \tag{7}$$

again for $\xi \gtrless 0$, and in order to evaluate C we can substitute the initial conditions, obtaining

$$C = \pm 2/\xi_0 - \dot{\xi}_0^2. \tag{8}$$

In order to discuss a specific problem, let $\dot{\xi}_0 = 0$ and $\xi_0 > 0$. Thus $C = 2/\xi_0 > 0$, and Eq. (7) gives

$$\pm \int_{\xi_0}^{\xi} \left(\frac{\xi}{2 - C\xi}\right)^{1/2} d\xi = t. \tag{9}$$

Note that, since $2/|\xi| - 2/\xi_0 \geqslant 0$, $|\xi| \leqslant \xi_0$; the particle released at $t = 0$ from $\xi = \xi_0 > 0$ will never depart farther from the origin than its initial position. The velocity of the particle is directed toward the origin and is negative during the time interval $0 < t < t_c$, where t_c is the time of collision; that is,

$$\dot{\xi} = -\left(\frac{2}{|\xi|} - C\right)^{1/2}, \tag{10}$$

for $0 < \xi < \xi_0$. Equation (6) shows that this negative velocity increases in absolute value as the particle approaches the origin (point of collision), since $\ddot{\xi} < 0$ for $\xi > 0$. As $\xi \to 0$, $|\dot{\xi}| \to \infty$, and therefore Eq. (10) can be integrated from $t = 0$ to $t_c - \delta$, where the time $t_c - \delta$ corresponds to $\xi = \epsilon > 0$ and the time $t = 0$ to $\xi = \xi_0$. The limit is to be evaluated as $\epsilon \to 0$ and $\delta \to 0$. Now Eq. (9) may be evaluated with the negative sign, giving

$$t = \frac{1}{C}[\xi(2 - C\xi)]^{1/2} + \frac{2}{C^{3/2}}\arctan\left(\frac{2 - C\xi}{C\xi}\right)^{1/2}. \tag{11}$$

As $\xi \to 0$, from Eq. (11)

$$t \to t_c = \pi/C^{3/2} \tag{12}$$

which connects the initial conditions with the instant of collision since $C = 2/\xi_0$ and so $\xi_0 = 2(t_c/\pi)^{2/3}$.

The monotonically decreasing function $\xi = \xi(t)$ as obtained from Eq. (11) is shown in Fig. 3.2. From Eq. (10), $|\dot{\xi}| \to \infty$ as $\xi \to 0$ (i.e., as $t \to t_c$), so the curve intersects the t axis perpendicularly. The initial conditions show that the curve starts perpendicularly to the ξ axis; $t = 0$, $\xi = \xi_0$, $\dot{\xi} = \dot{\xi}_0 = 0$. Note that the second derivative $\ddot{\xi}$ is negative in the region $0 \leqslant t < t_c$ so the curve is concave.

What happens to the particle following the instant t_c for $t \geqslant t_c$ is, of course, not clear from the preceding results, which cease to be meaningful at $t = t_c$. The continuation of the orbit after collision is not feasible since the solution encounters the singularity present in the problem.

(B) To eliminate the singularity, new dependent and independent variables are introduced:

$$\xi = f(u) \tag{13}$$

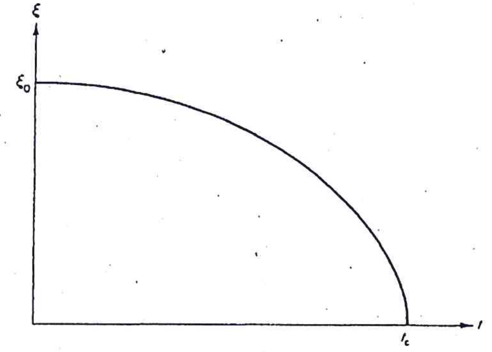

FIG. 3.2. One-dimensional collision orbit. Displacement ξ as a function of time t.

and

$$\tau = \int_{t_0}^{t} \frac{dt}{g(u)}. \tag{14}$$

The motion in the new system is described by the function $u = u(\tau)$; the new variable u depends on its own independent variable τ. When the functions $f(u)$ and $g(u)$ are known, Eq. (13) also gives u as a function of t since $\xi = \xi(t)$. Consequently Eq. (14) gives the relation between the old t and new τ time variables.

Another form of (14), which is frequently used in the literature, is

$$dt/d\tau = g(u). \tag{15}$$

This gives the ratio of the differentials of the old and new times as a function of the new, or, by Eq. (13), of the old dependent variable.

To understand what happens at and close to collision, the phenomenon must be slowed down by stretching the time scale so that the approach of the actual velocity to infinity can be handled.

The considerations leading to the selection of the functions f and g are presented in the following paragraphs.

The new velocity $u' = du/d\tau$ is related to the actual (physical) velocity $\dot{\xi} = d\xi/dt$ by

$$\frac{d\xi(t)}{dt} = \frac{df(u)}{du}\frac{du(\tau)}{d\tau}\frac{d\tau}{dt} \tag{16}$$

or

$$\dot{\xi} = u'f'/g, \tag{17}$$

if the notation $f' = df/du$ is introduced. Equation (16) follows from (13) using (15) and gives the new velocity:

$$u' = g\dot{\xi}/f'. \tag{18}$$

In order to have a finite value of this new velocity at collision, it is necessary that the ratio g/f' approach zero as $\dot{\xi} \to \infty$.

The energy integral (7) may be written as

$$\dot{\xi}^2 = 2/\xi - C = 2U, \tag{19}$$

where the $\pm$ sign is omitted; it should be remembered that $\xi > 0$. The energy integral in terms of the new variables becomes

$$(u')^2 = \frac{g^2}{(f')^2}\left(\frac{2}{f} - C\right) = \frac{g^2}{(f')^2}2U. \tag{20}$$

As $\xi \to 0$, $f(u) \to 0$ and $2U \to \infty$. In order to have a finite velocity u' at collision, we must have $[g^2/(f')^2]U$ finite as $\xi \to 0$.

Since $2U = (2/\xi) - C$, $U \to \infty$ as $\xi \to 0$, and close to collision $U = 1/\xi = 1/f$. So the requirement for finite velocity in the (u, τ) system is that

$$\frac{g^2}{(f')^2 f}$$

remain finite as $\xi \to 0$ or that

$$\frac{g}{f'}\frac{1}{f^{1/2}}$$

be finite as $f \to 0$.

If g/f' is represented by a power series in $f^{1/2}$, the lowest term in this series must be $(\text{const})f^{1/2}$. This follows from writing $g/f' = (f^{1/2})^n$ which gives

$$\frac{g}{f'}\frac{1}{f^{1/2}} = (f^{1/2})^{n-1}.$$

Therefore the above limit requirement is satisfied if $n - 1 \geqslant 0$, and the lowest allowable power of $f^{1/2}$ in the series must be 1. The series is

$$\frac{g}{f'} = A_1 f^{1/2} + A_2 f + A_3 f^{3/2} + A_4 f^2 + \cdots. \tag{21}$$

Consequently

$$\frac{g}{f' f^{1/2}} = A_1 + A_2 f^{1/2} + \cdots, \tag{22}$$

and, as $f \to 0$, $(g/f' f^{1/2}) \to A_1$. Only if $A_1 = 0$ does the limit process lead to $g/(f' f^{1/2}) = 0$. Therefore, the velocity in the system u, τ is finite at the singularity, $u' = 2^{1/2} A_1$, provided g and f are selected satisfying Eq. (22).

For instance, if $\xi = f(u) = u^n$, we have

$$g = A_1 f' f^{1/2} = n A_1 u^{(3/2)n-1}. \tag{23}$$

The next step is to investigate the equation of motion regarding singularities. Similarly to Eq. (16), we have

$$\ddot{\xi} = f'u'\frac{d^2\tau}{dt^2} + (f'u'' + f''u'^2)\left(\frac{d\tau}{dt}\right)^2$$

or, since

$$\frac{d^2\tau}{dt^2} = \frac{d}{dt}\frac{1}{g(u)} = -\frac{g'u'}{g^3},$$

we have

$$\ddot{\xi} = -u'^2\frac{f'g'}{g^3} + \frac{f'u'' + f''u'^2}{g^2}. \tag{24}$$

The equation of motion in the system u, τ therefore becomes, from Eq. (6) using only the region $\xi > 0$,

$$u''\frac{f'}{g^2} + u'^2\left(\frac{f''}{g^2} - \frac{f'g'}{g^3}\right) = \frac{1}{f'}\frac{dU}{du} \tag{25}$$

or

$$u'' + u'^2\frac{gf'' - f'g'}{f'g} = \frac{g^2}{f'^2}\frac{dU}{du}. \tag{26}$$

In connection with the regularization of the velocity in the (u, τ) system, Eq. (20) leads to the requirement that $(g^2/f'^2)2U$ must be finite. In order to utilize this requirement we compute

$$\frac{d}{du}U\frac{g^2}{f'^2} = \frac{g^2}{f'^2}\frac{dU}{du} + \frac{u'^2}{f'g}(g'f' - gf''). \tag{27}$$

Solving this equation for the required term on the right side and substituting it into (26) we have

$$u'' = \frac{d}{du}\frac{g^2}{f'^2}U, \tag{28}$$

which also follows, of course, from (20) by differentiation since $du'/du = u''/u'$.

(C) Regarding the selection of the functions f and g we first recall Sundman's and Levi-Civita's idea according to which the essential part of the regularization is the time transformation, the selection of the function g. The basic idea follows from the previously mentioned process of slowing down the phenomenon, and the equations

$$dt/d\tau = g(\xi) \tag{29}$$

and

$$\xi = u \tag{30}$$

take the place of Eqs. (13) and (14). The new velocity $u' = \xi' = d\xi/d\tau$ becomes, from Eq. (20),

$$\xi'^2 = g^2(2/\xi - C), \tag{31}$$

and in order to maintain a finite ξ' as $\xi \to 0$ we must have

$$g^2 = A_1\xi + B_1\xi^2 + C_1\xi^3 + \cdots. \tag{32}$$

This gives

$$\xi'^2 = 2A_1 + \xi(2B_1 - CA_1) + \xi^2(2C_1 - CB_1) + \cdots, \tag{33}$$

and so the new velocity is $(2A_1)^{1/2}$ at collision. If $A_1 = C_1 = \cdots = 0$, but $B_1 = 1$, we have $g = \xi$, $\xi'^2 = 2\xi - C\xi^2$, and from Eq. (29)

$$dt/d\tau = \xi, \tag{34}$$

or

$$d\tau = dt/\xi = \Omega(\xi)\, dt. \tag{35}$$

The new time variable τ is therefore directly related to the potential $\Omega(\xi) = 1/\xi$ of the dynamical problem. Since ξ in the present problem is the distance of the moving particle from the singularity, we have $\Omega(\xi) = 1/r$ or

$$\tau = \int_{t_0}^{t} \frac{dt}{r}, \tag{36}$$

which is a form popular in the literature.

The actual solution of the problem in the (ξ, τ) system follows from integrating Eq. (31). Instead of using the general form, Eq. (32), for $g(\xi)$, the special case $B_1 = 1$, $A_1 = C_1 = \cdots = 0$ is continued:

$$\xi'^2 = \xi(2 - C\xi). \tag{37}$$

The initial conditions are $t = 0$, $\tau = 0$, $\xi = 2/C$ and the solution is obtained from Eq. (37) in the form

$$\xi = \frac{1}{C}(1 + \cos C^{1/2}\tau) \tag{38}$$

with

$$t = \frac{1}{C}[\tau + (1/C^{1/2}) \sin C^{1/2}\tau]. \tag{39}$$

The functions $\xi(\tau)$ and $t(\tau)$ are shown in Figs. 3.3 and 3.4. Equations (38) may be obtained either from Eq. (37) or from

$$\xi'' + C\xi - 1 = 0, \tag{40}$$

which is a consequence of (26) or (28). Equation (39) follows from

$$= \int_0^\tau \xi\, d\tau. \tag{41}$$

At collision, $t = t_c = \pi/C^{3/2}$, $\tau = \tau_c = \pi/C^{1/2}$, and at the beginning of the motion $t = \tau = 0$.

The motion of the particle, according to Eqs. (38) and (39) and as shown in the corresponding Figs. 3.3 and 3.4, is oscillatory. At the beginning of the motion $t = \tau = 0$, $\xi = 2/C > 0$, $\dot{\xi} = 0$, and $\xi' = 0$. The particle moves toward the origin ($\xi = 0$) with $\dot{\xi} < 0$ and $\xi' < 0$ after the beginning of the motion. At $\tau = \tau_c/2$, $t = [(2 + \pi)/2\pi]\, t_c$, $\xi = 1/C$, $\dot{\xi} < 0$, $\xi' < 0$. As the particle approaches the origin, $|\dot{\xi}|$

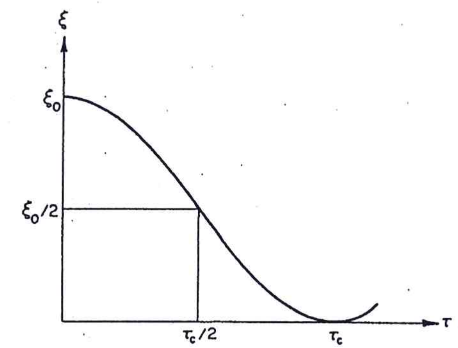

FIG. 3.3. Displacement ξ as a function of the regularized time τ for a one-dimensional collision orbit.

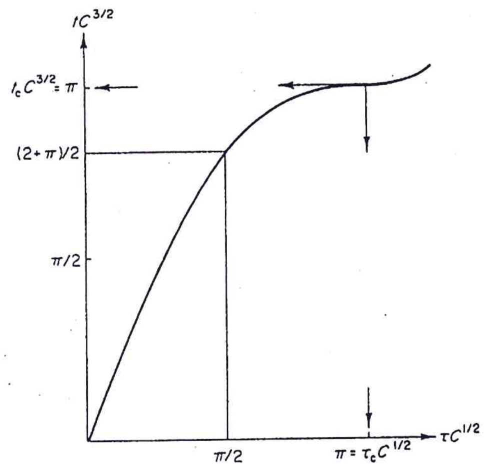

FIG. 3.4. Relation between physical t and pseudo time τ.

increases as $1/\xi^{1/2}$ and ξ' approaches zero. At collision $\xi = 0$, $t = t_c$, $\tau = \tau_c$, $\xi' = 0$, and $|\dot{\xi}| = +\infty$. Shortly after collision $t > t_c$, $\tau > \tau_c$, $\dot{\xi} > 0$, and $\xi' > 0$. At $t = 2t_c$, $\tau = 2\tau_c$, the particle is back at $\xi = 2/C$ with $\dot{\xi} = \xi' = 0$ and the cycle repeats itself. The function $\xi(\tau)$ is regular everywhere and so is the new velocity,

$$\xi'(\tau) = -\frac{1}{C^{1/2}} \sin C^{1/2}\tau.$$

The oscillation takes place along the positive ξ axis between $\xi_0 = 2/C$ and $\xi = 0$ with period $2t_c = 2\pi/C^{3/2}$. Therefore, for unit mean motion $n = 2\pi/2t_c = 1$, we have $t_c = \pi$, $C = 1$, and $\xi_0 = 2$.

(D) The regularized velocity ξ' changes sign at collision; $\xi' < 0$ before collision and $\xi' > 0$ after collision. Since the actual velocity $|\dot{\xi}| = \infty$ at collision, the incoming particle in this system arrives with a velocity $\dot{\xi} \to -\infty$ and leaves with a velocity $\dot{\xi} \to \infty$. Visualization of this is helped if the limiting degeneration of an elliptical two-dimensional orbit is considered.

As in Fig. 3.5, let $\dot{\eta}_0 > 0$ be a small vertical velocity at point $\xi = \xi_0$, $\eta = 0$ so that

$$2/\xi_0 > \dot{\eta}_0^2.$$

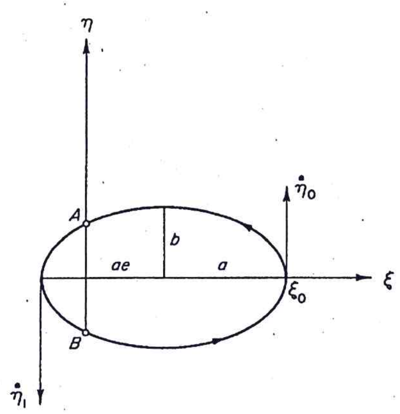

FIG. 3.5. Close approach as $\dot{\eta}_0 \to 0$.

The particle describes an ellipse with focus at the origin of the ξ, η coordinate system. The semimajor axis of this ellipse is related to the initial conditions by the energy integral $(2/r) - (1/a) = v^2$ or

$$\frac{1}{a} = \frac{2}{\xi_0} - \dot{\eta}_0^2, \tag{42}$$

and the eccentricity of the elliptic orbit is obtained from $\xi_0 = a(1 + e)$ or

$$e = \frac{\xi_0 - a}{a}. \tag{43}$$

When $\xi_0 = 2a$, then $e = 1$, $\dot{\eta}_0 = 0$, and with $v^2 = \dot{\xi}_0^2$ we have

$$\frac{2}{\xi_0} - \frac{1}{a} = \dot{\xi}_0^2 \tag{44}$$

from which $C = 1/a$.

The velocity at the pericenter is

$$\dot{\eta}_1 = \frac{1}{a^{1/2}}\left(\frac{1+e}{1-e}\right)^{1/2},$$

which goes to infinity as $e \to 1$. The velocity at the apocenter is

$$\dot{\eta}_0 = \frac{1}{a^{1/2}}\left(\frac{1-e}{1+e}\right)^{1/2}, \tag{45}$$

and when $e \to 1$, $\dot{\eta}_0 \to 0$.

The speed at point A with coordinates $\xi = 0$, $\eta = \eta_0$ is

$$v_A = \frac{1}{a^{1/2}}\left(\frac{1+e^2}{1-e^2}\right)^{1/2} \tag{46}$$

and the velocity components at points A and B become

$$\dot{\xi}_A = -\frac{1}{[a(1-e^2)]^{1/2}} = -\dot{\xi}_B,$$
$$\dot{\eta}_A = -\frac{e}{[a(1-e^2)]^{1/2}} = \dot{\eta}_B. \tag{47}$$

As $e \to 1$, the ellipse flattens out ($b = a(1-e^2)^{1/2} \to 0$) and the above components tend toward ∞. The ξ component of the velocity changes sign as the particle goes around the singularity, $\dot{\xi}_A = -\dot{\xi}_B$.

The period of the motion on the ellipse is $T = 2\pi/n$, with $n = a^{-3/2}$. When $\eta_0 \to 0$, $a^{-1} \to 2/\xi_0$ and so $T \to 2\pi(2/\xi_0)^{-3/2}$. Since $C = 2/\xi_0$ and $t_c = T/2$, the limiting condition of the elliptic motion furnishes the previously obtained time for collision, t_c.

It is not expected that the motion can be followed completely in the case $e = 1$; nevertheless, the preceding discussion suggests that the results obtained with the regularization process are reasonable not only from a mathematical but also from a physical point of view.

Prior to leaving this simplest case of regularization, attention is called to Eq. (38) connecting the distance $\xi = r$ with the new time variable τ. If the substitutions $a = 1/C$ and $na = aa^{-3/2} = C^{1/2}$ are made, we obtain

$$r = a(1 + \cos na\tau),$$

where $na\tau = u$ is the eccentric anomaly, and consequently Kepler's equation

$$nt = u + e \sin u$$

becomes Eq. (39) with $e = 1$. Note that in the last two equations the plus signs become minus signs if $u = \tau = 0$ corresponds to perihelion instead of to aphelion as in Fig. 3.5.

The conclusion is that the eccentric anomaly is a regularizing variable for the problem of two bodies. In fact, a comparison of Eq. (36),

$$d\tau = dt/r,$$

with

$$du = \frac{na}{r}\,dt$$

shows that our "new time" τ is essentially the eccentric anomaly.

If the time is measured from collision by $t^* = t - t_c$ or by $\tau^* = \tau - \tau_c$ with $Ct_c = \tau_c = \pi/C^{1/2}$ then Eqs. (38) and (39) become

$$\xi = \frac{1}{C}(1 - \cos C^{1/2}\tau^*)$$

and

$$t^* = \frac{1}{C}\left(\tau^* - \frac{1}{C^{1/2}}\sin C^{1/2}\tau^*\right).$$

Power series expansions for these functions may be written as

$$\xi = \frac{\tau^{*2}}{2!} - \frac{C\tau^{*4}}{4!} \pm \cdots = \tau^{*2}\,\Xi(\tau^*)$$

and

$$t^* = \frac{\tau^{*3}}{3!} - \frac{C\tau^{*5}}{5!} \pm \cdots = \tau^{*3}\,T(\tau^*),$$

where the series $\Xi(\tau^*)$ and $T(\tau^*)$ are convergent for any τ^* since they are essentially Taylor series for the functions sine and cosine. The constant terms in the series Ξ and T are $(2!)^{-1}$ and $(3!)^{-1}$, respectively. Consequently for sufficiently small t^* we have

$$\xi = (t^*)^{2/3}\,X(t^{*1/3}),$$

where once again the function $X(t^{*1/3})$ is a power series with the constant term $(9/2)^{1/3}$. The original solution therefore has at t^* a branch point of order 2 and the synodic path possesses a cusp at collision.

(E) Equations (13) and (14) propose the performance of the regularization by introducing two transformations,

$$\xi = f(u) \quad \text{and} \quad dt/d\tau = g(u).$$

The selection for $f(u)$ in the previous example was simply $f(u) = \xi = u$, and so the dependent variable was *not* transformed. The transformation of the independent variable followed the formula $dt/d\tau = g(u) = g(\xi) = \xi$. Regularization was performed, therefore, by transforming only the independent variable, as was expected since this was the essential step in the regularization process.

If the dependent variable is also transformed the situation is different because Eq. (23) furnishes a possible function $g(u)$, once $f(u)$ is selected. The selection of

$$\xi = f(u) = u^2 \tag{48}$$

is the next natural step. Note that, when n is even, Eq. (23) gives rational functions for g. In the present case

$$g(u) = Bu^2, \tag{49}$$

where $B = \text{const}$. The new velocity, from Eq. (20), becomes

$$(u')^2 = \left(\frac{1}{2} - \frac{Cu^2}{4}\right) B^2$$

and the selection of $B = 4$ gives

$$u' = \pm\, 2(2 - Cu^2)^{1/2}. \tag{50}$$

The selection of the constant B is quite arbitrary. If $B = 4$ we have $g = 4u^2 = (f')^2$, which will be of interest later.

Regarding the sign in Eq. (50), note that, since $u = \pm\xi^{1/2}$, from Eq. (48), and

$$\dot{\xi} = \frac{u'f'}{g} = \frac{u'}{2u}, \tag{51}$$

from Eq. (17), we have the result that, when $\dot{\xi} > 0$, either both $u > 0$, $u' > 0$ or both $u < 0$, $u' < 0$. When $\dot{\xi} < 0$, the signs of u and u' must be opposite. It is to be realized that there are two distinct values of u corresponding to any value of ξ except to $\xi = 0$.

The equation of motion is Eq. (50). The second-order equation can be established either by differentiating Eq. (50) or by using Eq. (28). The result is

$$u'' + 4Cu = 0. \tag{52}$$

The solution of Eq. (50) or (52) is

$$u = (2/C)^{1/2} \cos 2C^{1/2}\tau, \tag{53}$$

where the initial conditions $\tau = 0$, $u = u_0 = \xi_0^{1/2} = (2/C)^{1/2}$, and $u_0' = 0$ are used. The last condition follows from Eq. (51), since

$$u_0' = 2u_0\dot{\xi}_0 = 0.$$

Collision corresponds to $\xi = u = 0$, i.e., to $\tau_c = \pi/4C^{1/2}$. The new velocity, from Eq. (53), becomes

$$u' = -\,2(2)^{1/2} \sin 2C^{1/2}\tau \tag{54}$$

and its value at collision is

$$u_c' = -2(2)^{1/2};$$

the same results can also be obtained from Eq. (50) with $u = u_c = 0$.

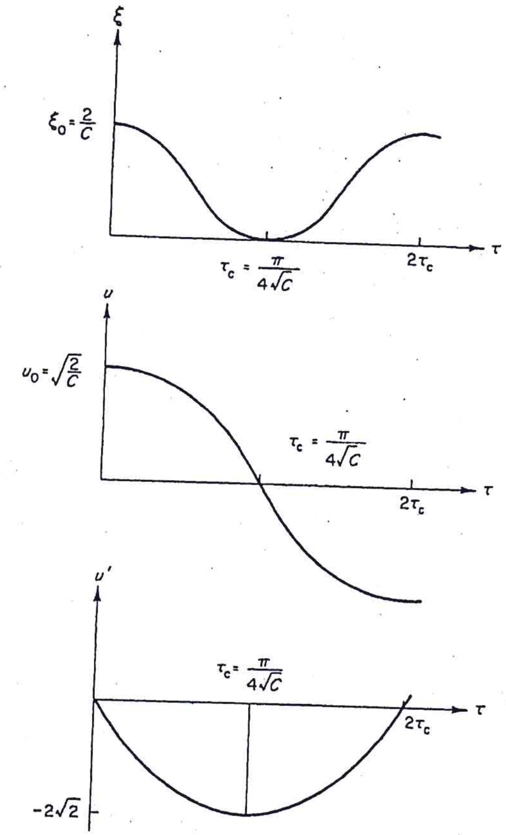

Fig. 3.6. Collision of two bodies with transformed time and coordinate.

In Fig. 3.6 are shown the relations $\xi(\tau)$, $u(\tau)$, and $u'(\tau)$. Only the positive value of the relation $u = \xi^{1/2}$ is shown. Between the initial position and collision, $0 \leqslant \tau \leqslant \tau_c$, $u_0 \geqslant u \geqslant 0$, $\xi_0 \geqslant \xi \geqslant 0$, and $0 \geqslant u' \geqslant -2(2)^{1/2}$. Observe that in this region $0 \geqslant \dot{\xi} \geqslant -\infty$, i.e., $\dot{\xi} < 0$ and sign $u = -$sign u'. After the collision in the time domain $\tau_c \leqslant \tau \leqslant 2\tau_c$ the particle returns from the origin to the position $\xi = \xi_0$, which it occupied at $\tau = 0$. During this time $0 \geqslant u \geqslant -u_0$, $0 \leqslant \xi \leqslant \xi_0$, and $-2(2)^{1/2} \leqslant u' \leqslant 0$. We have sign $u =$ sign u' in this region and $\dot{\xi} > 0$.

To find the relation between the new and old times τ and t we have $dt/d\tau = 4u^2$ from Eq. (15). Using (53) we obtain

$$t = \int_0^\tau \frac{8}{C}\cos^2 2C^{1/2}\tau\, d\tau,$$

or

$$t = \frac{4}{C}\left(\tau + \frac{\sin 4C^{1/2}\tau}{4C^{1/2}}\right), \tag{55}$$

the general shape of which is also shown in Fig. 3.4, since in this case $\tau_c = \pi/4C^{1/2}$.

There is no essential difference between the double transformation when both the time and the dependent variable are transformed, and the previously shown single time transformation. Both methods regularize the equation of motion and represent the solution. The first regularization results in zero velocity at collision, the second gives $2(2)^{1/2}$.

The transformation of the independent variable regularized the equation of motion which became

$$\xi'' + C\xi - 1 = 0$$

and the double transformation gave

$$u'' + 4Cu = 0.$$

Both these results are of the same mathematical simplicity.

The two-dimensional case to be discussed in the next section differs from the straight-line motion (from the point of view of regularization) in that the integral of energy does not furnish the equations of motion in the two-dimensional case. The introduction of the energy relation (20) into the equation of motion (26) eliminates the quadratic term in the velocity in Eq. (26), resulting in Eq. (28). It will be shown that in the two-dimensional case the introduction of a geometrical (coordinate) transformation, in addition to the time transformation, is necessary to eliminate the quadratic velocity term from the equations of motion.

Inasmuch as the regularization of the restricted problem is performed by double transformations, the above considerations [Eqs. (48) and (55)] might be looked upon as preparatory exercises.

3.3 Regularization of the general problem of two bodies

We now return to Eqs. (4) and study their general form. Introducing $\zeta = \xi + i\eta$ we have $\rho = |\zeta|$ and

$$\ddot{\zeta} = -\zeta/|\zeta|^3. \tag{56}$$

It is recognized that the singularity is at $\zeta = 0$, i.e., at collision; therefore, regularization from a mathematical point of view is of interest only when the conic-section orbit degenerates into a straight-line solution. From a practical point of view so-called close approach orbits also require regularization since, when an actual orbit is to be computed under these circumstances, problems of accuracy arise. For this reason we will regularize Eq. (56) by introducing a coordinate transformation,

$$\zeta = f(w), \tag{57}$$

and a time transformation,

$$dt/d\tau = g(w), \tag{58}$$

quite similar to Eqs. (13) and (14). Here $w = u + iv$, so while in the case of the straight-line motion the transformation established a relation between the points of the physical line ξ and the transformed line u, now the relation is between the physical plane ζ and the transformed plane w. The function $g(w)$ is a real function of the complex variable w, introducing in this way the new time τ as a real quantity.

The first transformation represents a conformal mapping; it contains the geometric information and it controls the accuracy of the shape of the orbit. The second transformation is the essential one, as was shown before, since it controls the kinematic aspects and it performs the regularization. The introduction of two transformations gives greater freedom and it will be shown in this section that significant simplifications of the transformed equations of motion are obtained by properly (and not independently) selecting the functions f and g.

The energy integral in the physical plane is

$$|\dot{\zeta}|^2 = 2/|\zeta| - C, \tag{59}$$

which is identical with the previously given Eq. (5).

Computation of $\dot{\zeta}$ follows the pattern established for the one-dimensional case:

$$\dot{\zeta} = \frac{d\zeta}{dw}\frac{dw}{d\tau}\frac{d\tau}{dt}, \tag{60}$$

or

$$|\dot{\zeta}|^2 = \frac{|f'(w)|^2\,|w'|^2}{g^2}. \tag{61}$$

Combining this with Eq. (59) gives the energy integral in the new system of variables:

$$|w'|^2 = \left(\frac{2}{|f|} - C\right)\frac{g^2}{|f'|^2}, \tag{62}$$

similar to Eq. (20). On the left side of Eq. (62) is the square of the new velocity and

$$|w'|^2 = (du/d\tau)^2 + (dv/d\tau)^2. \tag{63}$$

Prior to the treatment of the general situation and in complete analogy to the previously discussed case let $\zeta = w^2$, corresponding to $\xi = u^2$, as given by Eq. (48). Equation (36) suggests for $d\tau/dt$ the reciprocal of the distance; i.e., in the present case $d\tau/dt = r^{-1} = |\zeta|^{-1} = |w|^{-2}$. [The factor 4 which will be applied here as well as in Eq. (49) is of no consequence.] Therefore the transformation equations are

$$\zeta = f(w) = w^2 \tag{64}$$

and

$$\frac{dt}{d\tau} = g(w) = 4|w|^2. \tag{65}$$

The energy integral becomes

$$|w'|^2 = 8 - 4|w|^2 C \tag{66}$$

and the relation between the physical and the transformed velocities is

$$|\dot{\zeta}| = \frac{|w'|}{2|w|}, \tag{67}$$

which generalizes Eq. (51) in a straightforward manner.

The equation of motion using the transformed variables is

$$w'' + 4Cw = 0, \tag{68}$$

which equation can be obtained by transforming the original equation of motion, Eq. (56), and using Eq. (66).

Considering now the equation of motion (56), we write it as

$$\ddot{\zeta} = \operatorname{grad}_\zeta \frac{1}{|\zeta|}, \tag{69}$$

where for the real function F of a complex variable ζ we define

$$\operatorname{grad}_\zeta F(\zeta) = \frac{\partial F}{\partial \xi} + i\frac{\partial F}{\partial \eta}. \tag{70}$$

Using the notation

$$U = \frac{1}{|\zeta|} - \frac{C}{2} \tag{71}$$

as before, we have

$$\ddot{\zeta} = \operatorname{grad}_\zeta U. \tag{72}$$

The energy integral, Eq. (59), is

$$|\dot{\zeta}|^2 = 2U. \tag{73}$$

In order to transform Eq. (72) the expression given by Eq. (60) for $\dot{\zeta}$ is used:

$$\dot{\zeta} = \frac{df}{dw}\frac{dw}{d\tau}\frac{d\tau}{dt},$$

which, with

$$\frac{d\tau}{dt} = \dot{\tau}, \qquad \frac{df}{dw} = f'(w), \qquad \frac{dw}{d\tau} = w'(\tau),$$

becomes

$$\dot{\zeta} = f'w'\dot{\tau}. \tag{74}$$

The second derivative is

$$\ddot{\zeta} = f'w'\ddot{\tau} + (f''w'^2 + f'w'')\dot{\tau}^2. \tag{75}$$

The gradient operator will have to be transformed also and it will be shown that

$$\bar{f}' \operatorname{grad}_\zeta U = \operatorname{grad}_w U, \tag{76}$$

where the bar denotes the conjugate, i.e., $\bar{f}' = (\overline{df/dw})$.

Therefore Eq. (72) becomes

$$w'' + w'\frac{\ddot{\tau}}{\dot{\tau}^2} + w'^2\frac{f''}{f'} = \frac{\operatorname{grad}_w U}{\dot{\tau}^2|f'|^2}. \tag{77}$$

Computation of $\ddot{\tau}$ requires some care. Since $\dot{\tau} = 1/g$, we have

$$\ddot{\tau}/\dot{\tau}^2 = -\dot{g}. \tag{78}$$

Also, $g = g(w)$ is a real function of a complex variable, and so we may write it as $g = h(w)\bar{h}(w) = |h|^2$, where we require differentiability of $h(w)$. The derivative of g becomes

$$\frac{dg}{dt} = \left(h\frac{\overline{dh}}{dw}\frac{\overline{dw}}{d\tau} + \bar{h}\frac{dh}{dw}\frac{dw}{d\tau}\right)\dot{\tau}. \tag{79}$$

In computing the derivative of $\bar{h}$ we use the fact that

$$(\bar{h})' = \frac{d\bar{h}}{dw} = \overline{\frac{dh}{dw}} = (\overline{h'}).$$

Consequently,

$$\frac{d\bar{h}}{dt} = \overline{\frac{dh}{dw}}\,\overline{\frac{dw}{d\tau}}\,\frac{d\tau}{dt} = \overline{\frac{dh}{dw}}\,\overline{\frac{dw}{d\tau}}\,\dot{\tau} = \frac{\bar{h}'\bar{w}'}{hh}$$

and

$$\frac{\ddot{\tau}}{\dot{\tau}^2} = -\dot{g} = -\left(\frac{\bar{h}'\bar{w}'}{\bar{h}} + \frac{h'w'}{h}\right).$$

Substitution in Eq. (77) gives

$$w'' - \frac{|w'|^2}{\bar{h}}\frac{d\bar{h}}{dw} + w'^2\left(\frac{f''}{f'} - \frac{h'}{h}\right) = \frac{|h|^4}{|f'|^2}\operatorname{grad}_w U. \tag{80}$$

The energy integral can be used to change the second term on the left side since

$$|w'|^2 = \frac{|h|^4}{|f'|^2}2\tilde{U}. \tag{81}$$

The equation of motion becomes

$$w'' + (w')^2\frac{d}{dw}\left(\ln\frac{f'}{h}\right) = \frac{|h|^4}{|f'|^2}\left(2U\overline{\frac{d\ln h}{dw}} + \operatorname{grad}_w U\right), \tag{82}$$

which may also be written as

$$w'' = \operatorname{grad}_w\left|\frac{h^2}{f'}\right|^2 U - 2iw'\operatorname{Im}\left(w'\frac{d}{dw}\ln\frac{f'}{h}\right), \tag{83}$$

where the symbol Im stands for the imaginary part of the complex expression following it.

Before filling in a few steps in the derivation between Eqs. (82) and (83) we note that, if $f' = h$, we have

$$w'' = \operatorname{grad}_w |f'|^2 U, \tag{84}$$

and with $\zeta = f(w) = w^2$ as proposed by Eq. (64) we have

$$w'' = \operatorname{grad}_w 4|w|^2\left(\frac{1}{|w|^2} - \frac{C}{2}\right) = -4Cw, \tag{85}$$

in complete agreement with Eq. (68).

The reader can readily verify Eq. (83) by considering that

(i) $\operatorname{grad}_w g_1(w)g_2(w) = g_1\operatorname{grad}_w g_2 + g_2\operatorname{grad}_w g_1$,

where g_1 and g_2 are real functions of the complex variable w, and that

(ii) $\operatorname{grad}_w |G(w)|^2 = 2G\overline{\dfrac{dG}{dw}}$,

where G is a complex function of the complex variable w.

The proof of Statement (i) is left to the reader, while that of (ii) is shown as follows.

Since $G(w)$ is analytic, we have

$$G_u = \frac{\partial G}{\partial u} = \frac{dG}{dw} = -iG_v \qquad\text{and}\qquad \bar{G}_u = \overline{\frac{dG}{dw}} = i\bar{G}_v.$$

Furthermore,

$$\operatorname{grad}_w |G|^2 = G\operatorname{grad}_w \bar{G} + \bar{G}\operatorname{grad}_w G,$$

where

$$\operatorname{grad}_w G = G_u + iG_v = 0$$

and

$$\operatorname{grad}_w \bar{G} = \bar{G}_u + i\bar{G}_v = 2\overline{\frac{dG}{dw}}.$$

Q.E.D.

At this point Eq. (76) is also proved as follows. Consider the expanded form of $\operatorname{grad}_w U(w) = U_u + iU_v$,

$$\operatorname{grad}_w U = U_\xi\xi_u + U_\eta\eta_u + i(U_\xi\xi_v + U_\eta\eta_v).$$

The Cauchy–Riemann relations transform the imaginary part into

$$-U_\xi\eta_u + U_\eta\xi_u,$$

and with this expression $\operatorname{grad}_w U$ becomes

$$(\xi_u - i\eta_u)(U_\xi + iU_\eta) = \bar{f}'\operatorname{grad}_\zeta U.$$

Q.E.D.

Note that, with the selection of $h = f'$, the nonlinear part of Eq. (83) (regarding w') disappears as anticipated. The consequence of this selection is that $g = |f'|^2$, and the time transformation is related to the coordinate transformation by

$$dt/d\tau = |f'|^2. \tag{86}$$

The elegance and simplicity of Eq. (68) cannot but fill the reader with joy. The original equation of motion [Eq. (56)],

$$d^2\zeta/dt^2 + \zeta/|\zeta|^3 = 0,$$

is replaced by Eq. (68),

$$d^2w/d\tau^2 + 4Cw = 0,$$

the solution of which can be immediately written as

$$w = A\cos 2C^{1/2}\tau + B\sin 2C^{1/2}\tau \qquad \text{for} \quad C > 0 \tag{87}$$

or as

$$w = A\cosh 2(-C)^{1/2}\tau + B\sinh 2(-C)^{1/2}\tau \qquad \text{for} \quad C < 0, \tag{88}$$

and finally, for $C = 0$, we have

$$w = A + B\tau. \tag{89}$$

Such a uniform presentation of the problem of two bodies is, of course, not new and the uniformization of the introduction of universal variables was not the primary purpose but only a side product of this discussion. Nevertheless, the process of regularization also accomplished the introduction of such variables.

In summary we can make the following statements regarding the regularization of the problem of two bodies:

The transformations $\zeta = w^2$ and

$$t = 4\int_0^\tau |w|^2\, d\tau$$

give the equations of motion as

$$d^2w/d\tau^2 + 4Cw = 0,$$

where the constant

$$C = \frac{2}{|\zeta|} - |\dot{\zeta}|^2$$

is determined by the initial conditions. The initial conditions are transformed from the values ζ_0, $\dot{\zeta}_0$ to the values w_0, $w_0' = (dw/d\tau)_0$, according to $w_0 = \zeta^{1/2}$ and $w_0' = 2\dot{\zeta}_0\bar{w}_0$.

The following generalization of the preceding statements is also available and is based on the result given as Eq. (84).

The second-order differential equation

$$\ddot{\zeta} = \operatorname{grad}_\zeta U$$

can be transformed into

$$w'' = \operatorname{grad}_w |f'(w)|^2 U,$$

where the geometric transformation, $\zeta = f(w)$, and the time transformation, $dt = |f'(w)|^2\, d\tau$, are governed by the same function $f(w)$. If the time transformation is related to the function $f(w)$ in any other way, then the quadratic term in the first derivative will be present in the transformed equation.

In the above equation

$$\operatorname{grad}_w |f'(w)|^2 U = \frac{\partial}{\partial u}|f'|^2 U + i\frac{\partial}{\partial v}|f'|^2 U,$$

and the new (transformed) and original (physical) velocities are connected by

$$\dot{\zeta} = f'w'/|f'|^2 \qquad \text{or} \qquad w' = \bar{f}'\dot{\zeta}.$$

3.4 Local regularization of the restricted problem

The general formulation is not different from the problem of two bodies. We introduce the two transformations again as

$$z = f(w) \tag{90}$$

and

$$dt/d\tau = g(w) = |h(w)|^2,$$

similar to Eqs. (57) and (58). Note that Eq. (90) establishes a relation between z and w instead of between ζ and w since we deal with transformations of the nondimensional rotating coordinate system (z).

The equation of motion of the restricted problem in a complex form is

$$\ddot{z} + 2i\dot{z} = \operatorname{grad}_z U, \tag{91}$$

which, after performing the transformations, becomes

$$w'' + 2i\,g(w)\,w' = \operatorname{grad}_w \left|\frac{h^2}{f'}\right|^2 U - 2iw' \operatorname{Im}\left(w'\frac{d}{dw}\ln\frac{f'}{h}\right), \tag{92}$$

with

$$U \equiv \Omega - C/2.$$

The derivation of Eq. (92) is similar to that of Eq. (83) except that the term linear in w' enters into (92). It is once again observed that, if $f' = h$ or $g(w) = |f'|^2$, the equation of motion becomes

$$w'' + 2i|f'|^2 w' = \operatorname{grad}_w |f'|^2 U. \tag{92'}$$

Note that the square of the first derivative $(\mathbf{r}_i')$ appears as expected; nevertheless, Eq. (154) is regularized for binary collisions. The usefulness of such transformations must be decided by their applicability to computers.

An implication in the stellar dynamics of this regularized equation is that there appears to be a tendency of stars in a cluster to form binaries. This, of course, necessitates handling close approaches within the framework of the n-body problem.

3.12 Notes

The fundamental ideas regarding the regularization of the general problem of three bodies were contributed by Sundman [1] who clearly shows Poincaré's influence [2].

Section 3.2 treats the possible simplest case in order to free the reader from algebraic complications. Already Euler [2a] regularized the problem of collision between two bodies moving on the same straight line, as was called to my attention by Miss C. Williams. Note that Euler's method corresponds to the selection of $n = 1$ in Eq. (23) of Section 3.2.

The development of the basic ideas, with many references, can be found in the books by Whittaker [3] and by Leimanis and Minorsky [4]. Wintner's book [5] offers a somewhat decentralized treatment of the problem of regularization. Happel [6] dedicates a fourteen-page chapter to the subject.

Regularization may be achieved in the case of the general problem of two bodies (Section 3.3) without transformation of the coordinates. I am grateful to Dr. R. F. Arenstorf for his communication in which he offers a delightfully short proof that the time transformation

$$dt/d\tau = r$$

may be used to obtain from the conservation of angular momentum and from Eqs. (56) and (59) the following result:

$$\zeta'' + C\zeta = \text{const},$$

where prime stands for derivative with respect to τ. This is analogous to the regularization applied for the collision problem in Section 3.2 where the above time transformation and the eccentric anomaly as the regularizing variable are introduced in Eq. (40). See also [4] and [5, p. 193].

Levi-Civita's contributions to the subject are numerous [7]. The third and the last references quoted are preferred. Closely related is the transformation proposed by Bohlin [8]. With slight modification this

becomes what in the text (Section 3.4) is called Levi-Civita's transformation.

The history of the Thiele–Burrau transformation (Section 3.6) begins with Euler [9] who established the integrability of the problem of two centers of gravitation. This problem is discussed in the books by Whittaker [3, p. 97] and Charlier [10] as well as in Plummer's book [11], where the connection between the regularization of the restricted problem and Euler's problem is made clear. Euler's transformation replaces rectangular coordinates by elliptic coordinates and in this way the problem becomes of the Liouville type and can be integrated by elliptic functions. It is to be realized that, with either one of the force centers, conic-section orbits exist. When the two centers operate simultaneously, conic-section solutions still exist according to Bonnet's theorem [12] due to Legendre [13] and generalized by Egorov [14]. This theorem and its application to the problem of two fixed centers of gravitation and the application of this latter problem to the restricted problem are described by Szebehely [15]; for details see Section 10.5.1. The regularization of the restricted problem for $\mu = \frac{1}{2}$, using essentially Euler's transformation, was performed by Thiele [16] (without reference to Euler) and for arbitrary μ by Burrau [17]. The original problem was "oversolved" since local regularization would have allowed the continuation of Strömgren's class "a" orbits to their natural termination (see Chapter 9). In 1892, Thiele proposed the study of the motion around the third libration point (see Chapter 4), L_3, as an astronomy prize problem in the Danish Royal Academy. The small elliptic orbits around L_3 resulting from the solution of the linearized equations of motion will be treated in detail in Chapter 5. As the amplitude increases, a member of this family becomes a so-called ejection or collision orbit colliding with m_1. Thiele's problem was to establish this member of the family. The regularization of the restricted problem therefore was prompted by the necessity to complete a family of orbits.

The Thiele–Burrau regularization was adopted by the Strömgren school for their extensive numerical integrations, and the various transformation formulas were tabulated and presented in graphs and in nomograms. Inasmuch as this transformation was used mostly, and is recommended primarily, for hand calculations, only Lindow's [18] nomogram and Strömgren's (Burrau and Strömgren [19]) tables of the transformations are mentioned here.

Historically it is significant that Burrau had settled Thiele's prize problem before Thiele's application of Euler's transformation appeared. Reference should be made in this respect to Burrau's papers [20] in which he recognizes the above-mentioned fact that only local regularization is necessary for the solution of the problem and he discovers consequently the $\tau = at^{1/3}$ transformation discussed at the end of Section 3.2. The derivation given there is made more precise by Wintner [5], which is followed in Section 3.2, Part D.

The introduction of Birkhoff's transformation (Section 3.5) was much less eventful. An extensive discussion of his transformation is given in [21]. No numerical applications are offered and the transformation is used in connection with the topology of the dynamical problem. Note that on p. 13, Eq. 21 is in error and ρ_1 and ρ_2 should be interchanged. See also a later work by Birkhoff [22].

Prior to Birkhoff's result, Armellini's strange paper [23], evidently primed by Sundman, appeared and discussed regularization of the restricted problem by introducing $d\tau = (r_1 + r_2 - 1)^{-1}\,dt$.

It is interesting to review the series of publications associated with increasing the exponent of $k(w) = w$ from one to two: $k(w) = w^2$. This is the process from Birkhoff's to Lemaître's transformation (Section 3.7). The first to propose the transformation $z = (c/2)(w^2 + w^{-2})$ without specific applications to regularization was Plummer [24]. Several papers by Lemaître of the University of Louvain treat the regularization of the general problem of three bodies [25]. Application of this work to the global regularization of the restricted problem is contributed by Deprit in a series of papers [26, 26a]. Lemaître's 1955 paper and Deprit's excellent 1963 summary in Icarus are recommended primarily. The underlying principle of a symmetric treatment of the general plane problem of three bodies is given by Murnaghan [27], by Szebehely [28], by Deprit and Delie [29], by Deprit and Roels [30], and finally by Deprit and Delie [31]. Essentially the same method was proposed by Arenstorf in a remarkable and excellent paper with strong analytical orientation [32]. The mathematically inclined reader will find thorough enjoyment in this paper where the concept of dynamical equivalence (Section 3.4) is introduced.

The transformation in question was named after Lemaître by me in 1964 (see [35]) since at that time the earliest references (1954) I could locate were written by him (see [25]). I propose to preserve the terminology, in spite of Plummer's paper [24] of a much earlier date (1914), for two reasons: First, Lemaître's name is now associated with this transformation in the literature through several contributions by others in addition to myself. Second, Plummer's aim in introducing the transformation was not directed to regularization; in fact, he does not mention regularization in connection with this transformation, though he discusses other transformations in the same paper with regularization in mind, regarding applications. On the other hand, Lemaître's main purpose was to regularize the general problem of three bodies.

Broucke [34] pointed out that, with the function $h(w) = w^n$ mentioned

in Sections 3.8 and 3.9, not only is regularization affected but also certain multiple periodic orbits (with repeated loops around the origin) can be transformed into simple periodic orbits without loops. This also implies that series solutions obtained in the w plane for a given, say, satellite problem will be valid for more revolutions in the physical (q or z) plane than in the w plane.

The form in which the conformal mappings are presented in Figs. 3.7(b), 3.9(a), (b), and 3.13(a), (b) is conventional; see, for instance, Deprit's and Broucke's [26a] paper.

The importance of regularization in space mechanics is discussed by Szebehely [33], and a comparison of the various regularization methods mentioned in Section 3.9 is offered by the same author [35]. This latter paper discusses a general perturbation scheme for the solution of the equations of motion of the restricted problem after Birkhoff's regularizing transformation was performed. An alternate derivation of the regularized equations is given by Szebehely and Giacaglia [35a], in a paper in which the regularization of the elliptic restricted problem is also treated (see Section 10.3). Examples for general perturbation analyses are given by Pierce [36] using Levi-Civita's regularization. A derivation of Eq. (92) in Section 3.4 is offered by Szebehely [36a].

A discussion of universal variables mentioned in Section 3.3 in connection with Eqs. (68) and (87) to (89) is presented by Herrick [37] who — not from the point of view of regularization — introduced such variables first in 1945 [38].

A comprehensive treatise on regularization with a number of examples is available as an article by Szebehely (39).

The process of regularization within the framework of Hamiltonian dynamics will be presented in Chapter 7.

Section 3.10 discusses the question of the existence of solutions of the restricted problem. The outlines given for the theorems and for some of the proofs show that the analytical background is well established in the literature. The three major references regarding this section are by Leimanis (Leimanis and Minorsky [4, p. 59]), Wintner [5, pp. 269, 329, 356–359], and Arenstorf [32]. Wintner's fundamental theorem regarding the existence of solutions of the restricted problem is an application of one of Painlevé's results. The solution of the problem of establishing $\lim \tau \to t^*$ for the various regularization transformations is offered by Arenstorf [32]. The simple discussion given in Section 3.10 for the transformation introducing the eccentric anomaly is not entirely rigorous, in spite of the fact that every collision in the restricted problem is essentially a (straight-line) two-body collision. The last remark of Section 3.10 is at variance with Wintner's conclusion [5, p. 359] and has been first observed by Arenstorf [32].

Numerical results outlined in Section 3.11 and obtained for the restricted problem by use of regularized equations were first given by the Copenhagen school using the Thiele–Burrau transformation. This work, for which a general paper by Strömgren [40] is given as a representative reference, is discussed in considerable detail in Chapter 9. The Copenhagen school established several single-collision orbits and a few consecutive-collision orbits, but no families consisting solely of collision orbits were found.

Consecutive-collision orbits without periodicity but forming a well-defined family were established by V. Szebehely and co-workers. The first paper by Szebehely *et al.* [41] shows a family of trajectories connecting the earth's center with the moon's center. One member of this family is close to such an actual earth-to-moon trajectory which could be utilized for lunar landing missions. The two-body approximation of some of the computed consecutive collision orbits is established. The continuation of the family of orbits is described by Pierce and Standish [42]. Consecutive collision orbits of high complexity are systematically developed from the basic family in [42] and are discussed in Chapter 9.

Computation of collision trajectories in the restricted problem using Levi-Civita's regularization have been performed by Benedikt [43] and and Broucke [44].

Benedikt in the same paper discusses the power series solution for the problem of finding initial conditions which lead to collision. The same problem is treated by Levi-Civita [45], by Bisconcini [46], and by Whittaker [3, p. 424].

The general problem of three bodies mentioned in Section 3.11 is not the subject of this treatise; nevertheless, the fact that practitioners of celestial mechanics were reluctant to accept Sundman's developments cannot be ignored. Considering double collisions only, the general problem is solved in principle since Sundman [1] regularized this problem. (Simultaneous (i.e., triple) collisions are to be excluded by specifying nonzero angular momentum vector. This condition is only necessary but not sufficient, since triple collisions can be excluded by proper choice of the initial conditions even when the angular momentum is zero [5, p. 265]).

The same applies to the restricted problem, which, after a global regularization was affected, can be solved, at least in principle, by power series. Regarding the question of local and global regularization, see Wintner [47].

In an interesting article, Cesco [48] points out that astronomers take only limited interest in regularization because of its ill-chosen applications. Motion around the equilateral libration points with small amplitude, when not even close approaches occur, does not seem to be

a good test case for judging the merits of regularization. Cesco proposes an example with double collision which shows the power, in fact the necessity, of regularization. I believe the argument to be correct but the basic issue somewhat different. Close approaches do not usually occur in celestial mechanics, and this is the reason for not applying regularizing transformations. When collisions and close approaches enter the problem, such as in space mechanics and in stellar dynamics, regularization does become essential.

The regularization of the n-body problem discussed in Section 3.11 is treated in considerable detail by Leimanis (Leimanis and Minorsky [4, p. 94]). The qualitative aspects are discussed by Khilmi [49]. See also a note by Ebert [50] on the use of regularization to find a new invariant relation for the problem of n bodies.

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3.13 References

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