

THE THEORY  
OF  
ORBIT DETERMINATION

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## PREFACE

This book is a tool for our own teaching and an opportunity to rethink and reorganize the results of our own research. However, I think such a book can be useful to others, for two main reasons. First, spaceflight is not anymore the privilege of the few superpowers, but is becoming available to many nations and agencies. Orbit determination is an essential know how, both in the planning phase of mission analysis and in the operations of space missions. Thus its mathematical tools need to become widely available.

Second, the knowledge and skill used in orbit determination, for both natural and artificial celestial bodies, was available only among a restricted group of specialists. The prevailing attitude was a proprietary one: the knowledge and the software were protected by formal copyright and/or by secrecy, although protecting in this way the pure mathematical theory is, in the long run, impossible. This attitude might have been justified under the conditions of the world of 30–40 years ago, in the critical phases of the competition to achieve *space firsts*. Now it is time to teach and disseminate the knowledge, allowing the formation of a wider group of specialists.

I know that many of the *rules of thumb* and practical advice contained in this book will be rated as well known, even obvious, by the few experts, but this is not the point. Even well known results may need to be presented in a rational, rigorous and didactically effective new way, together with the outcome of recent innovative research. On the other hand, this book does not have the intent of providing a comprehensive review of all that has been done in this field, because the size would become impractical. This book is about making widely available the outcome of the research done by my group over many years, and includes methods for which there are rigorous mathematical arguments and which have been fully tested by us first hand, and found to be effective. In the last  $\simeq 15$  years there has been an enormous progress in this field, and several other research groups have given important contributions: we are in no way claiming that their methods would not work, we are just giving a list of methods which we know to work.

The above arguments may not be enough for the approval of all the people in this field, but I do think that to state the mathematical foundations and rules of orbit determination, thus removing a vague flavor of craftsmanship, can benefit also the already existing specialists. The orbit determination expert, in the very competitive environment in which space missions and large astronomical projects are selected today, is too often under pressure to endorse claims of wonderful results to be achieved with very limited means. By ignoring the rules of good practice it is possible to claim illusory precision and/or completeness for the solution, including the orbits and other parameters which can be operationally, technologically and scientifically relevant. Maybe being able to cite a textbook stating clearly what is appropriate and what is illusory can help in relieving this improper pressure.

This book is based on the experience accumulated in about 30 years of research by myself and my coworkers of the former *Space Mechanics Group* at the Department of Mathematics of the University of Pisa. Thus it contains, besides the formal mathematical theory and the examples for teaching purposes, a number of “case studies” based upon actual research projects. They are about space missions and about natural objects: one of the goals is to stress the common mathematics used in satellite geodesy and in dynamical astronomy, and at the same time to present clearly the main differences.

The preparation of this book has been made possible by the collaboration of my younger colleague, Dr. Giovanni F. Gronchi. Besides classical material and original results by myself and by Dr. Gronchi, this book contains the output of research done by the people who have been members of our group and by either regular or occasional external coworkers. Thus I would like to include a long, but still possibly incomplete, list of coworkers whose contribution has to be acknowledged: L. Anselmo, O. Arratia, S. Baccili, A. Boattini, C. Bonanno, M. Carpino, G. Catastini, L. Cattaneo, S.R. Chesley, S. Cicalò, L. Denneau, P. Farinella, D. Farnocchia, Z. Knežević, L. Iess, R. Jedicke, A. La Spina, M. de’ Michieli Vitturi, A.M. Nobili, A. Rossi, M.E. Sansaturio, G. Tommei, G.B. Valsecchi, D. Villani, D. Vokrouhlický.

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Andrea Milani Comparetti, Pisa, April 2008