MORE Orbit Determination Software status

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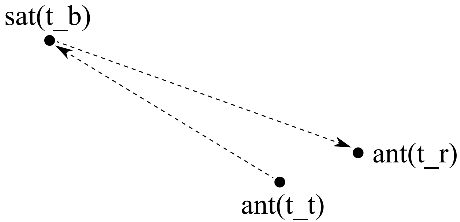
Outline

1. Current status of the Orbit14 software development: recent implementations and improvements
   - Relativistic framework for the fundamental dynamics and the computation of the MORE observables range and range-rate
   - Relativistic space-time coordinates transformations
   - Iterative method for the computation of the observables
2. Mercury rotation model (work in progress)

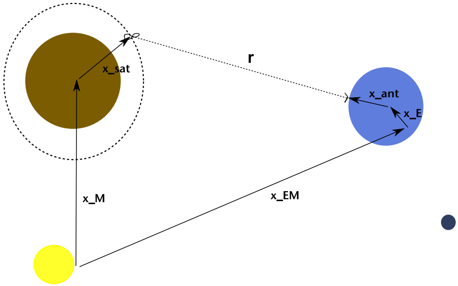
MORE observables and Dynamics



The observables of the MORE experiment are:

- The light time $\Delta t = t_r - t_t$ of the radar signal (t_t, t_r are the transmit and receive times), which is related to the range r, the distance between the ground antenna and the spacecraft
- The Doppler frequency shift f_D between the transmitted and the received signal, which is proportional to the range-rate $\dot{r}$

Notation



- $\mathbf{x}_{sat}$: Mercurycentric position of the satellite
- $\mathbf{x}_M$, $\mathbf{x}_{EM}$: Barycentric position of the Mercury and the Earth-Moon CoM
- $\mathbf{x}_E$: vector from the Earth-Moon CoM to the Earth CoM
- $\mathbf{x}_{ant}$: position of the ground antenna w.r.t. the Earth CoM

Definition of the observables range and range-rate

The Euclidean range can be geometrically computed by:

$$\rho = |(\mathbf{x}_{sat} + \mathbf{x}_M) - (\mathbf{x}_{EM} + \mathbf{x}_E + \mathbf{x}_{ant})|$$

while the total light time of the signal is the sum of the up-leg and down-leg light times $\Delta t = \Delta t_{up} + \Delta t_{do}$:

$$c\Delta t_{up} = |(\mathbf{x}_{sat}(t_b) + \mathbf{x}_M(t_b)) - (\mathbf{x}_{EM}(t_t) + \mathbf{x}_E(t_t) + \mathbf{x}_{ant}(t_t))| + S_{up}(\gamma)$$

$$c\Delta t_{do} = |(\mathbf{x}_{sat}(t_b) + \mathbf{x}_M(t_b)) - (\mathbf{x}_{EM}(t_r) + \mathbf{x}_E(t_r) + \mathbf{x}_{ant}(t_r))| + S_{do}(\gamma)$$

where t_b is the bounce time and $S(\gamma)$ is the parametrized Shapiro effect.

Conventionally defining the observable range as

$$r(t_r) \equiv \frac{c}{2}(\Delta t_{up} + \Delta t_{do}) = \frac{1}{2}(\rho_{up} + S_{up} + \rho_{do} + S_{do}),$$

in order to compute the Doppler effect we need the time derivative of r w.r.t the receive time t_r:

$$\dot{r} = \frac{1}{2}(\dot{\rho}_{up} + \dot{S}_{up} + \dot{\rho}_{do} + \dot{S}_{do})$$

Shapiro effect

The Shapiro correction for the computation of the range is (Moyer 2003):

$$S(\gamma) = \frac{(1+\gamma)\,\mu}{c^2}\,\ln\left(\frac{\rho_1+\rho_2+\rho+\frac{(1+\gamma)\,\mu}{c^2}}{\rho_1+\rho_2-\rho+\frac{(1+\gamma)\,\mu}{c^2}}\right)$$

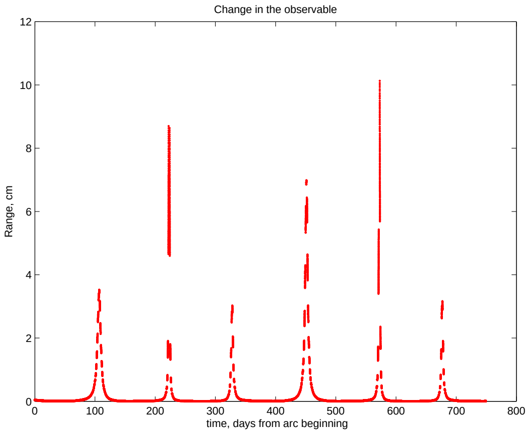
- μ is the gravitational parameter of the Sun
- ρ_1 is the distance between the barycenter of the Sun and the transmitter at the transmit time
- ρ_2 is the distances between the barycenter of the Sun and the receiver at the receive time
- ρ is the distance between the transmitter and the receiver

The Shapiro correction for the computation of the range-rate is:

$$\dot{S} = \frac{2(1+\gamma)\mu}{c^2}\left(\frac{-\rho(\dot{\rho}_1+\dot{\rho}_2)+\dot{\rho}(\rho_1+\rho_2+\frac{(1+\gamma)\mu}{c^2})}{(\rho_1+\rho_2+\frac{(1+\gamma)\mu}{c^2})^2-\rho^2}\right)$$

2-PN level correction is significant

The 2-PN corrective term $(1+\gamma)\mu/c^2$ in the Shapiro effect formula is significant for the MORE experiment:



1-PN vs 2-PN model for the Shapiro effect: differences significant (~ 10 cm) near conjuction.

Dynamics and Relativistic space-time coordinates

Dynamics with fixed parameters:

- The Geocentric dynamics of the antenna $\mathbf{x}_{ant}$

It is given by the IERS model. It is expressed in a Geocentric space-time reference system (GCRS), with time coordinate TDT (Terrestrial Time).

- The E-M-barycentric dynamics of the Geocenter $\mathbf{x}_E$

It is given by the JPL ephemerides. It is expressed in a Solar System Barycentric reference system (BCRS), with time coordinate TDB. The Barycentric Dynamical Time is the time in which the ephemerides are computed, it will be replaced by the new Barycentric Coordinate Time TCB, based on the SI second.

Dynamics and Relativistic space-time coordinates

Dynamics with parameters to solve for:

- The Mercurycentric dynamics of the Satellite $\mathbf{x}_{sat}$

It is computed by a suitable routine, the motion is expressed in a Mercurycentric space-time reference system with time coordinate TDM.

- The SS Barycentric dynamics of Mercury $\mathbf{x}_M$ and of the Earth-Moon barycenter $\mathbf{x}_{EM}$

It is computed by a suitable routine, the motion is expressed in a SS Barycentric space-time reference system with time coordinate TDB.

Computation of $\mathbf{x}_M$ and $\mathbf{x}_{EM}$

- Relativistic Lagrangian: corrective terms of Post-Newtonian (PN) order 1 in the small parameters $\frac{v_i^2}{c^2}$ and $\frac{G\, m_i}{c^2\, r_{ij}}$.
  - $v_i = |\overrightarrow{v_i}|$: barycentric velocity of body with mass m_i.
  - $r_{ij} = |\overrightarrow{r_j} - \overrightarrow{r_i}|$: mutual distance, appearing in the metric of the curved space-time.

$$L = L_{NEW} + L_{GR}$$

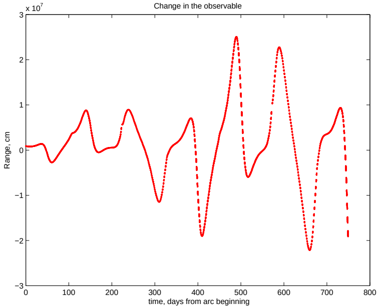
$$L_{GR} = \frac{1}{8\,c^2} \sum_i m_i v_i^4 - \frac{1}{2\,c^2} \sum_i \sum_{j \neq i} \sum_{k \neq i} \frac{G^2\, m_i\, m_j\, m_k}{r_{ij}\, r_{ik}} +$$

$$+ \frac{1}{2\,c^2} \sum_i \sum_{j \neq i} \frac{G\, m_i\, m_j}{r_{ij}} \left[\frac{3}{2}(v_i^2 + v_j^2) - \frac{7}{2}(\overrightarrow{v_i} \cdot \overrightarrow{v_j}) - \frac{1}{2\, r_{ij}^2} (\overrightarrow{r_{ij}} \cdot \overrightarrow{v_i})\, (\overrightarrow{r_{ij}} \cdot \overrightarrow{v_j}) \right]$$

Given the very large S/N ratio of the relativistic effects (see figure below), it is possible to test general relativity to great accuracy:

- Parameterized Post-Newtonian (PPN) formalism: relativistic equations of motion linearized with respect to the small parameters $\frac{v_i^2}{c^2}$ and $\frac{G\,m_i}{c^2\,r_{ij}}$, parameterized (with other relativistic effects) with constants having fixed values in Einstein theory, e.g. $\gamma, \beta, \zeta, \eta, \alpha_1, \alpha_2$;
- we solve for their value, together with the initial conditions and instrumental parameters, in the orbit determination procedure.

(Milani et al., 2009, IAUS 261)



Differences in range using a fully Relativistic and a Newtonian model.

$\Delta r = 4 \times 10^7$ cm

S/N $= \frac{\Delta r}{\sigma_r} = \frac{4\times10^7}{10} = 4 \times 10^6$

Relativistic space-time transformations

The five vectors involved in the dynamics are computed at different times (the epoch of the events) and in different space-time coordinates systems.

To perform the vector operations, all the vectors have to be converted to a common suitable space-time reference system, e.g. a Solar System Barycentric reference system (BCRS) with time TDB.

- x_M, x_{EM} and x_E are already in a BCRS as provided by numerical integration and external ephemerides.
- x_{ant} and x_{sat} have to be converted to the BCRS from the Earth-centered and Mercury-centered systems, respectively.

Note that the conversion of reference systems implies the conversion of both the space and the time coordinates.

Relativistic space-time transformations

Conversion of time coordinates

The differential equation giving the local time T (e.g. TDT or TDM) as a function of the TDB time t is:

$$\frac{dT}{dt} = 1 - \frac{1}{c^2}\left[U + \frac{v^2}{2} - L\right]$$

where U is the Newtonian gravitational potential generated by the other bodies than the Earth or Mercury at the planet center, v is the SSB velocity of the same planet and L is a suitable constant used to remove secular terms. In the case of the Earth it's $L = L_C = 1.48082686741 \times 10^{-8}$.

The time scale TDB will be replaced in the planetary ephemerides by the new TCB. When this happens, we will use a suitably defined Mercury Coordinate Time TCM such that the conversion equation will be exactly the same as the one for TDB to TDM.

Relativistic space-time transformations

Conversion of space coordinates from Geocentric to BCRS:
The geocentric coordinates of the antenna are transformed into TDB-compatible coordinates by:

$$\mathbf{x}_{ant}^{TDB} = \mathbf{x}_{ant}^{TDT} \left(1 - \frac{U}{c^2} - L_C\right) - \frac{1}{2} \left(\frac{\mathbf{v}_E^{TDB} \cdot \mathbf{x}_{ant}^{TDT}}{c^2}\right) \mathbf{v}_E^{TDB}$$

where U is the gravitational potential at the geocenter (excluding the Earth mass) and $\mathbf{v}_E^{TDB}$ is the barycentric velocity of the Earth.

$$\mathbf{v}_{ant}^{TDB} = \left[\mathbf{v}_{ant}^{TDT} \left(1 - \frac{U}{c^2} - L_C\right) - \frac{1}{2} \left(\frac{\mathbf{v}_E^{TDB} \cdot \mathbf{v}_{ant}^{TDT}}{c^2}\right) \mathbf{v}_E^{TDB}\right] \left[\frac{dT}{dt}\right] .$$

The above formula contains the effect on the velocities of the time coordinate change, which should be consistently used together with the coordinate change. T is the local time for Earth, that is TDT, and t is the corresponding TDB time.

Relativistic space-time transformations

Conversion from Mercurycentric to BCRS:
The mercurycentric coordinates of the orbiter are transformed into TDB-compatible coordinates by

$$\mathbf{x}_{sat}^{TDB} = \mathbf{x}_{sat}^{TDM} \left(1 - \frac{U}{c^2} - L_{Cmer}\right) - \frac{1}{2} \left(\frac{\mathbf{v}_M^{TDB} \cdot \mathbf{x}_{sat}^{TDM}}{c^2}\right) \mathbf{v}_M^{TDB}$$

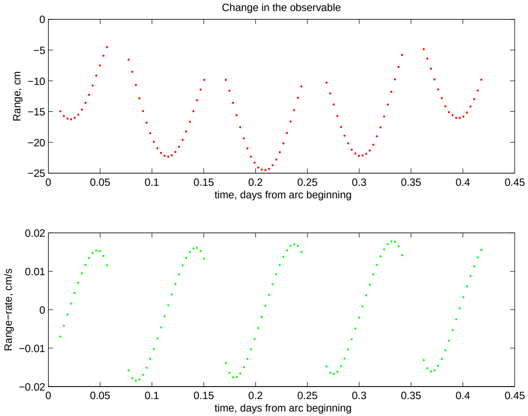
where U is the gravitational potential at the mercurycenter (excluding the Mercury mass) and L_{Cmer} is analogous to L_C, but for the moment we have set it equal to 0.

$$\mathbf{v}_{sat}^{TDB} = \left[\mathbf{v}_{sat}^{TDM} \left(1 - \frac{U}{c^2} - L_{Cmer}\right) - \frac{1}{2} \left(\frac{\mathbf{v}_M^{TDB} \cdot \mathbf{v}_{sat}^{TDM}}{c^2}\right) \mathbf{v}_M^{TDB}\right] \left[\frac{dT}{dt}\right] ,$$

where in this case T is TDM, and t is TDB.

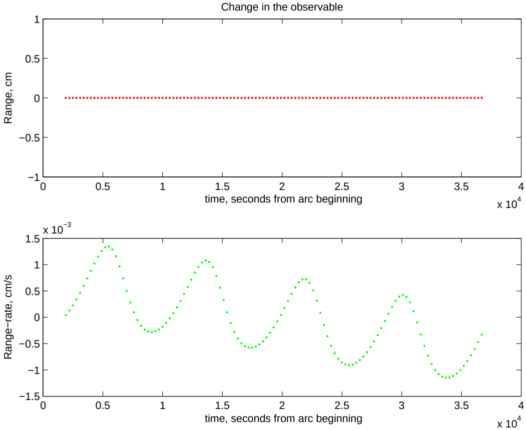
We have checked that the equations of motion for the S/C around Mercury do not need relativistic terms, the important relativistic effects are hidden in the coordinate transformations.

Effect of the relativistic correction to the transformations



The effect on the observables of the relativistic correction to the space-time coordinates transformation for Mercury and the Earth positions and velocities.

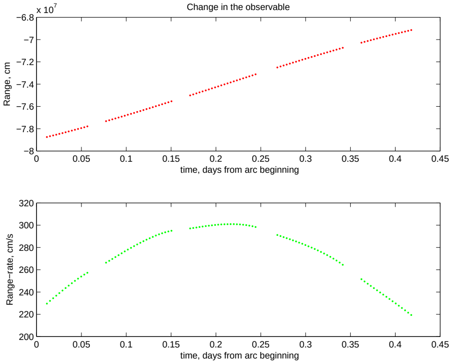
Effect of the relativistic correction to the transformations



The effect on the observables of the term dT/dt in the transformation of the velocities of the Earth and of Mercury.

Effect of the relativistic correction to the transformations

To allow for exchange of orbit data between MORE and ESOC/other experiments we must agree on the space-time reference system. E.g., a mistake in the time scale of the initial conditions can lead to a catastrophic error (large enough to stop convergence of orbit determination).



The effect on the observables of making a mistake in the type of time coordinate TDM or TDB for the time t_0 of the satellite's initial conditions.

Computation of the observable range (I)

Once the five vectors are available at the appropriate times and in a consistent BCRS system, the two different light-times, the down-leg $\Delta t_{do} = t_r - t_b$ and the up-leg $\Delta t_{up} = t_b - t_t$ are computed iteratively, using the following fixed point equations (t_r is known):

1. Computation of t_b:

$$t_b = t_r - \frac{1}{c}|(\mathbf{x}_{sat}(t_b) + \mathbf{x}_M(t_b)) - (\mathbf{x}_{EM}(t_r) + \mathbf{x}_E(t_r) + \mathbf{x}_{ant}(t_r))| - \frac{S_{do}(\gamma)}{c}$$

2. Once $t_b = t_b(t_r)$ is computed, we can compute t_t by the equation:

$$t_t = t_b - \frac{1}{c}|(\mathbf{x}_{sat}(t_b) + \mathbf{x}_M(t_b)) - (\mathbf{x}_{EM}(t_t) + \mathbf{x}_E(t_t) + \mathbf{x}_{ant}(t_t))| - \frac{S_{up}(\gamma)}{c}$$

Computation of the observable range (II)

The computed light-times are expressed in the time attached to the BCRS, which is TDB. Thus these times have to be converted back in the time system applicable at the receiveing station where the time measurement is performed, which is TDT.

t_r is already available in TDT, while t_t needs to be converted back from TDB to TDT. The difference between these two TDT times is Δt_{TDT}, from which we can conventionally define the observable range:

$$r(t_r) \equiv \frac{c}{2} \Delta t_{TDT}.$$

Δt_{TDT} is significantly different from $t_r - t_t$ in TDB, by an amount of the order of 10^{-7} s, while the sensitivity of the BC MORE experiment is of the order of 10^{-9} s.

Computation of the observable range rate (I)

After the two iterations providing at convergence t_b and t_t are complete, we can proceed to compute the range-rate. Derivating $r = r(t_r)$ w.r.t. the receive time t_r, we obtain:

$$\dot{r} = \frac{c}{2}(1 - \dot{t}_t) = \frac{1}{2}(\dot{\rho}_{do} + \dot{S}_{do} + \dot{\rho}_{up} + \dot{S}_{up})$$

The values $\dot{\rho}_{up}$ and $\dot{\rho}_{do}$ are given by the following implicit equations, using a fixed point iteration method:

$$\dot{\rho}_{do}(t_r) = \frac{1}{\rho_{do}(t_r)}\ [\mathbf{x}_{Ms}(t_b) - \mathbf{x}_{Ea}(t_r)] \cdot \left[\dot{\mathbf{x}}_{Ms}(t_b)\ \left(1 - \frac{\dot{\rho}_{do}(t_r) + \dot{S}_{do}}{c}\right) - \dot{\mathbf{x}}_{Ea}(t_r)\right]$$

$$\dot{\rho}_{up}(t_r) = \frac{1}{\rho_{up}(t_r)}\ [\mathbf{x}_{Ms}(t_b) - \mathbf{x}_{Ea}(t_t)] \cdot$$

$$\cdot \left[\dot{\mathbf{x}}_{Ms}(t_b)\ \left(1 - \frac{\dot{\rho}_{do}(t_r) + \dot{S}_{do}}{c}\right) - \dot{\mathbf{x}}_{Ea}(t_t)\left(1 - \frac{\dot{\rho}_{do}(t_r) + \dot{S}_{do}}{c} - \frac{\dot{\rho}_{up}(t_r) + \dot{S}_{up}}{c}\right)\right]$$

where $\mathbf{x}_{Ms} = \mathbf{x}_M + \mathbf{x}_{sat}$ and $\mathbf{x}_{Ea} = \mathbf{x}_E + \mathbf{x}_{ant}$.

Computation of the observable range-rate (II)

The computed range-rate, i.e. the quantity $d\,t_t/d\,t_r$, is expressed in the time attached to the BCRS, which is TDB. Thus it has to be converted in the time system applicable at the receiveing station on the Earth where the measurements are performed, which is TDT.

Indicating with T the time scale TDT and with t the time scale TDB, we need to compute:

$$\frac{d\,T_t}{d\,T_r} = \frac{d\,T_t}{d\,t_t}\,\frac{d\,t_t}{d\,t_r}\,\frac{d\,t_r}{d\,T_r}.$$

The first factor follows directly from the time coordinate transformation, while the third is obtained by the corresponding inverse transformation.

However, this correction is required only for consistency, since it has an order of magnitude of $10^{-7}\,cm/s^2$ and it is negligible for the sensitivity of MORE.

Derivatives for differential corrections

To build the differential corrector we need the partial derivatives of the observables w.r.t. the parameters we want to solve for.

Let's use as example the derivatives with respect to the S/C mercuricentric state vector initial conditions, the others are similar.

The preliminary version of our proposed implementation consists of the following formulas:

$$\frac{\partial\, r}{\partial\, \mathbf{x}^0_{sat}} = \frac{\partial\, r}{\partial\, \mathbf{x}_{sat}} \frac{\partial\, \mathbf{x}_{sat}}{\partial\, \mathbf{x}^0_{sat}} \qquad \frac{\partial\, r}{\partial\, \mathbf{v}^0_{sat}} = 0$$

$$\frac{\partial\, \dot{r}}{\partial\, \mathbf{x}^0_{sat}} = \frac{\partial\, \dot{r}}{\partial\, \mathbf{x}_{sat}} \frac{\partial\, \mathbf{x}_{sat}}{\partial\, \mathbf{x}^0_{sat}} + \frac{\partial\, \dot{r}}{\partial\, \mathbf{v}_{sat}} \frac{\partial\, \mathbf{v}_{sat}}{\partial\, \mathbf{x}^0_{sat}}$$

$$\frac{\partial\, \dot{r}}{\partial\, \mathbf{v}^0_{sat}} = \frac{\partial\, \dot{r}}{\partial\, \mathbf{x}_{sat}} \frac{\partial\, \mathbf{x}_{sat}}{\partial\, \mathbf{v}^0_{sat}} + \frac{\partial\, \dot{r}}{\partial\, \mathbf{v}_{sat}} \frac{\partial\, \mathbf{v}_{sat}}{\partial\, \mathbf{v}^0_{sat}}$$

The second partial derivatives in each term are computed via numerical integration of the Variational Equations (remember $\mathbf{x}_{sat} = \mathbf{x}_{sat}(t_b(t))$ and $\mathbf{v}_{sat} = \mathbf{v}_{sat}(t_b(t))$).

Derivatives for differential corrections

The first partial derivatives are computed by the following simple formulas:

$$\frac{\partial\, r}{\partial\, \mathbf{x}_{sat}} = \frac{1}{2c}\left(\frac{1}{\rho_{up}}\boldsymbol{\rho}_{up} + \frac{1}{\rho_{do}}\boldsymbol{\rho}_{do}\right)$$

$$\frac{\partial\, \dot{r}}{\partial\, \mathbf{x}_{sat}} = -\left[\frac{1}{\rho_{up}}\left(\frac{\dot{\rho}_{up}}{\rho_{up}}\boldsymbol{\rho}_{up} - \dot{\boldsymbol{\rho}}_{up}\right) + \frac{1}{\rho_{do}}\left(\frac{\dot{\rho}_{do}}{\rho_{do}}\boldsymbol{\rho}_{do} - \dot{\boldsymbol{\rho}}_{do}\right)\right]$$

$$\frac{\partial\, \dot{r}}{\partial\, \mathbf{v}_{sat}} = \left(\frac{1}{\rho_{up}}\boldsymbol{\rho}_{up} + \frac{1}{\rho_{do}}\boldsymbol{\rho}_{do}\right)$$

which do not include the relativitic corrections to the space-time transformations of coordinates. These corrections are not so important for the corrector as they are for the simulator. The effect of a lower accuracy in the partial derivatives is only a slowdown in the differential corrections iterative process, not affecting the accuracy of the solution.

Mercury rotation: status of work

We have received from our collaborators from the *University of Namur*, Belgium, a prototype software for the computation of the rotation state of Mercury in a self-consistent way, taking into account all the planetary perturbations.

This routine is expected to generate the Mercury rotation state ephemerides as a stack of rotation matrices at different times, to be eventually interpolated at the required times.

The parameters involved in the dynamics and to solve for, are the geopotential coefficients J_2 and C_{22}, the concentration coefficient C/MR^2 and the factor C_m/C (or, equivalently, the obliquity η and the libration in longitude amplitude ϵ).

We are in the process of including this model in our software, replacing the previous semiempirical model.