Equations of Motion and Variational Equations

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Contents

Notation

We will use the following notation:

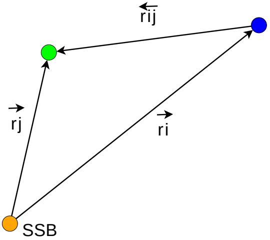
$\overrightarrow{r}_i = [x_{1i}, x_{2i}, x_{3i}]$

$(\overrightarrow{r}_i)_s = x_{s\,i}$

$\overrightarrow{r}_{i\,j} = \overrightarrow{r}_j - \overrightarrow{r}_i$

$(\overrightarrow{r}_{i\,j})_s = x_{s\,j} - x_{s\,i}$

$r_{i\,j} = \mid \overrightarrow{r}_{i\,j} \mid$



and an analogous notation hold for velocities $\overrightarrow{v}_i = [\dot{x}_{1i}, \dot{x}_{2i}, \dot{x}_{3i}]$, where all the positions and velocities are with respect to the Solar System Barycenter.

Equations of motion and Variational Equation

Let

$$\dot{X} = F(X) \qquad X(0) = X_0$$

be a generic Dynamical System with Flow $\Phi_{X_0}(t)$.

The Variational Equations for the System are:

$$\frac{d}{dt}\, A(t) = \frac{\partial\, F}{\partial\, X}\, (\Phi_{X_0}(t))\; A(t) \qquad A(0) = Id$$

where

$$A(t) = \frac{\partial \Phi_{X_0}(t)}{\partial\, X_0}$$

is needed for the Differential correction process.

Lagrangian formulation

The total Lagrangian function that we have used to describe the dynamics is:

$$L = L_{new} + L_{Pert}$$

$$L_{Pert} = L_{GR} + \bar{\gamma} L_{\gamma} + \bar{\beta} L_{\beta} + \zeta L_{\zeta} + J_{2\odot} L_{J2\odot} + \alpha_1 L_{\alpha_1} + \alpha_2 L_{\alpha_2} + \eta L_{\eta}$$

where $\bar{\gamma} = \gamma - 1$, $\bar{\beta} = \beta - 1$.

Then the Lagrangian equation for the k-th body is:

$$\frac{\partial L}{\partial \overrightarrow{r}_k} - \frac{d}{dt} \frac{\partial L}{\partial \overrightarrow{v}_k} = 0$$

Lagrangian function components

The total Lagrangian function L is constituted by the Newtonian n-body Lagrangian L_{new}

$$L_{new} = \frac{1}{2} \sum_i \mu_i \, v_i^2 + \frac{1}{2} \sum_i \sum_{j \neq i} \frac{\mu_i \mu_j}{r_{ij}}$$

where the bodies considered are the Sun, Mercury, Earth, Moon, Venus, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto.

The perturbative Lagrangian functions are the following:

- General Relativity: L_{GR}
- PPN γ, β: L_γ, L_β
- Variation of $\mu_\odot$: L_ζ
- Sun J_2 effect: $L_{J2\odot}$
- Preferred frame effect: L_{α_1}, L_{α_2}
- Possible violation of the equivalence principle: L_η

Equations of motion

Let L_P be any perturbative Lagrangian function, except the case of L_η which has to be discussed separately, then the perturbative acceleration of the $k-th$ body can be calculated as in the following.
If L_P depends only on positions $L_P = L_P(\overrightarrow{r})$:

$$\implies \quad \overrightarrow{a_k}^P = \frac{1}{\mu_k}\frac{\partial L_P}{\partial \overrightarrow{r_k}}$$

If L_P depends also on velocities and $L_P = L_P(\overrightarrow{r}, \overrightarrow{v}) = \frac{1}{c^2}L_P^0$:

$$\implies \quad \overrightarrow{a_k}^P \cong \frac{1}{\mu_k}\left[\frac{\partial L_P}{\partial \overrightarrow{r_k}} - \left(\frac{d}{dt}\frac{\partial L_P}{\partial \overrightarrow{v}_k}\right)\Bigg|_{\overrightarrow{a}=\overrightarrow{a}^{new}}\right]$$

where $\overrightarrow{a} = \overrightarrow{a}^{new}$ because of $O(c^{-2})$ approximation.

The Lagrangian formulation leads to the classic equations of motion in the form

$$\frac{d}{dt}\overrightarrow{v}_k = \overrightarrow{a}_k$$

where

$$\begin{aligned}\overrightarrow{a}_k &= \overrightarrow{a}_k^{new} + \overrightarrow{a}_k^{GR} + \bar{\gamma}\,\overrightarrow{a}_k^{\gamma} + \bar{\beta}\,\overrightarrow{a}_k^{\beta} + \zeta\,\overrightarrow{a}_k^{\zeta} + \\ &\quad + J_{2\odot}\,\overrightarrow{a}_k^{J_{2\odot}} + \alpha_1\,\overrightarrow{a}_k^{\alpha_1} + \alpha_2\,\overrightarrow{a}_k^{\alpha_2} + \eta\,\overrightarrow{a}_k^{\eta} + \frac{\dot{\overrightarrow{P}}_\alpha}{\mathcal{M}}\end{aligned}$$

Note that the term $\dot{\overrightarrow{P}}_\alpha/\mathcal{M}$ is the apparent acceleration due to the preferred frame effect, and it is the same for all bodies.

Variational equations

Once we have obtained the right hand side of the equations of motion, we can solve also for the Variational equation:

$$\frac{\partial \overrightarrow{a}_k}{\partial \overrightarrow{r}_l} \qquad \frac{\partial \overrightarrow{a}_k}{\partial \overrightarrow{v}_l}$$

which are the partial derivatives of the accelerations with respect to the positions and the velocities.

In particular the ones that we need are the partial derivatives of the accelerations of Mercury and of the Earth-Moon Barycenter, with respect to the positions and velocities of Mercury, Earth and Moon.

In reality we also have to compute the derivatives of the acceleration of the Sun because an indirect term is needed in the variational equations...

Indirect term due by the Sun in the Variational Equations

The Sun can be eliminated from the equations of motion by using the origin in the Solar System Barycenter:

$$\overrightarrow{r_0} = -\frac{\sum_{i\neq 0} \mu_i \overrightarrow{r}_i \left(1 + \frac{v_i^2}{2\,c^2} - \frac{U_i}{2\,c^2}\right)}{\mu_0 \left(1 + \frac{v_0^2}{2\,c^2} - \frac{U_0}{2\,c^2}\right)}$$

where $U_i = \sum_{k\neq i} \frac{\mu_k}{r_{ik}}$.

So we have to consider the following indirect term in the partial derivatives of the accelerations:

$$\frac{d\,\overrightarrow{a}_k}{d\,x_{p\,l}} = \frac{\partial\,\overrightarrow{a}_k}{\partial\,x_{p\,l}} + \frac{\partial\,\overrightarrow{a}_k}{\partial\,x_{p\,0}}\frac{\partial\,x_{p\,0}}{\partial\,x_{p\,l}}$$

$$\frac{d\,\overrightarrow{a}_k}{d\,\dot{x}_{p\,l}} = \frac{\partial\,\overrightarrow{a}_k}{\partial\,\dot{x}_{p\,l}} + \frac{\partial\,\overrightarrow{a}_k}{\partial\,\dot{x}_{p\,0}}\frac{\partial\,\dot{x}_{p\,0}}{\partial\,\dot{x}_{p\,l}}$$

Newtonian part

The Classic Newtonian acceleration for the k-th body is:

$$\overrightarrow{a}_k^{new} = \frac{1}{\mu_k}\frac{\partial L_{new}}{\partial \overrightarrow{r}_k} = \sum_{i\neq k} \frac{\mu_i}{r_{ki}^3}\overrightarrow{r}_{ki}$$

The Partial derivatives with respect to positions and velocities are:

$$\frac{\partial (\overrightarrow{a_k}^{new})_s}{\partial x_{pl}} = \sum_{i\neq k}\left[-\frac{3\,\mu_i}{r_{ki}^5}\,(\overrightarrow{r}_{ki})_p\,(\overrightarrow{r}_{ki})_s + \frac{\mu_i}{r_{ki}^3}\delta_{ps}\right](\delta_{l\,i} - \delta_{lk})$$

$$\frac{\partial (\overrightarrow{a_k}^{new})_s}{\partial \dot{x}_{pl}} = 0$$

General Relativity Lagrangian function

The Lagrangian function taking into account the perturbation terms due to General Relativity $O(c^{-2})$ is:

$$
\begin{aligned}
L_{GR} &= \frac{1}{8\,c^2}\sum_i \mu_i v_i^4 + \frac{1}{2}\sum_i\sum_{j\neq i}\frac{\mu_i\,\mu_j}{r_{ij}}\left[\frac{3}{2c^2}(v_i^2+v_j^2) + \right.\\
&- \left.\frac{7}{2\,c^2}(\overrightarrow{v_i}\cdot\overrightarrow{v_j}) - \frac{1}{2\,c^2}(\overrightarrow{n_{ij}}\cdot\overrightarrow{v_i})(\overrightarrow{n_{ij}}\cdot\overrightarrow{v_j})\right] + \\
&- \frac{1}{2\,c^2}\sum_i\sum_{j\neq i}\sum_{k\neq i}\frac{\mu_i\,\mu_j\,\mu_k}{r_{ij}\,r_{ik}}
\end{aligned}
$$

where $\overrightarrow{n}_{i\,j} = \overrightarrow{r}_{i\,j}/r_{i\,j}$.

General Relativity perturbative acceleration term

The perturbative acceleration due to General Relativity is:

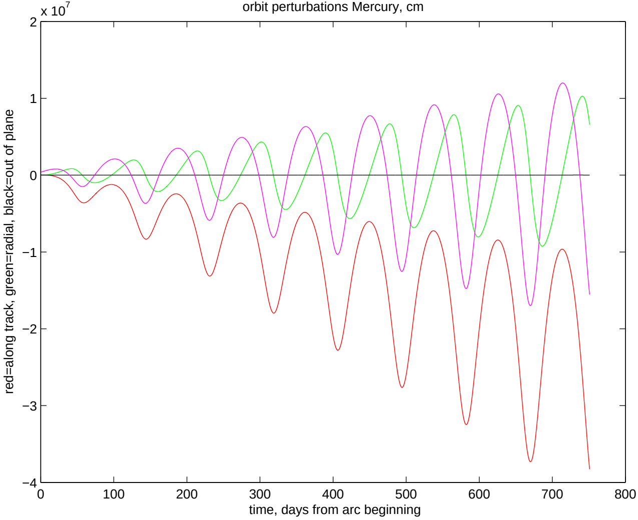
$$\overrightarrow{a}_k^{GR} = \frac{1}{c^2} \sum_{j \neq k} \frac{\mu_j}{r_{kj}^3} \overrightarrow{r_{kj}} \Bigg[-4 \sum_{l \neq k} \frac{\mu_l}{r_{kl}} - \sum_{l \neq j} \frac{\mu_l}{r_{jl}} + v_k^2 + 2 v_j^2 - 4 \overrightarrow{v_k} \cdot \overrightarrow{v_j} +$$

$$- \frac{3}{2} \left(\frac{\overrightarrow{r_{jk}} \cdot \overrightarrow{v_j}}{r_{kj}} \right)^2 + \frac{1}{2} \overrightarrow{r_{kj}} \cdot \overrightarrow{a}_j^{new} \Bigg] + \frac{1}{c^2} \sum_{j \neq k} \frac{\mu_j}{r_{kj}^3} \left[\overrightarrow{r_{jk}} \cdot (4 \overrightarrow{v_k} - 3 \overrightarrow{v_j}) \right] \overrightarrow{v_{jk}} +$$

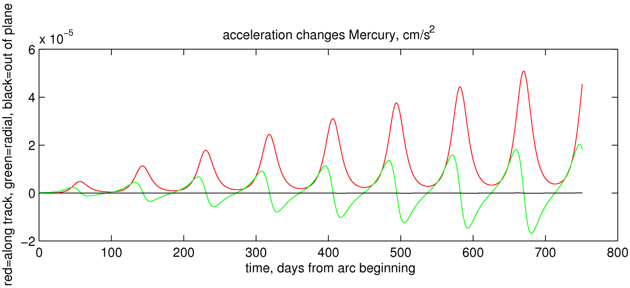
$$+ \frac{7}{2 c^2} \sum_{j \neq k} \frac{\mu_j}{r_{kj}} \overrightarrow{a}_j^{new}$$

We don't show the partial derivatives of $\overrightarrow{a}_k^{GR}$ with respect to positions and velocities because of the length of these formulas!

Effect of GR perturbation on the orbit of Mercury



Effect of GR perturbation on the acceleration of Mercury



PPN γ and β Lagrangian functions

The Lagrangian functions taking into account the effect of PPN γ and β are:

$$L_\gamma = \frac{1}{2\,c^2} \sum_i \sum_{j \neq i} \frac{\mu_i\,\mu_j}{r_{i\,j}} \left(\overrightarrow{v_j} - \overrightarrow{v_i}\right)^2$$

$$L_\beta = -\frac{1}{c^2} \sum_i \sum_{j \neq i} \sum_{k \neq i} \frac{\mu_i\,\mu_j\,\mu_k}{r_{ij}\,r_{ik}}$$

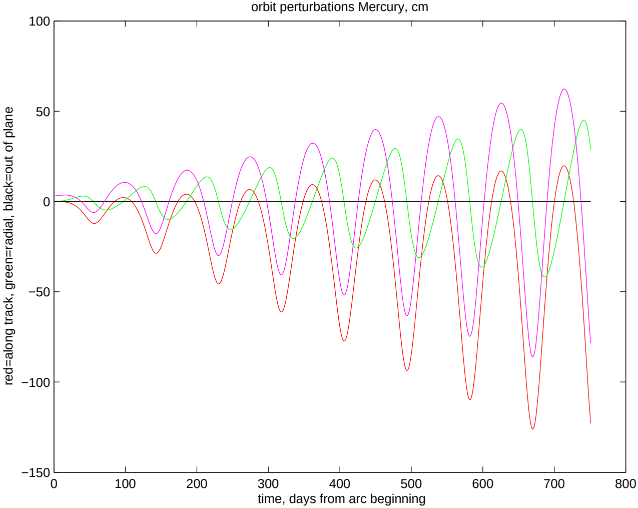
PPN parameters γ, β accelerations

The corresponding perturbative accelerations are:

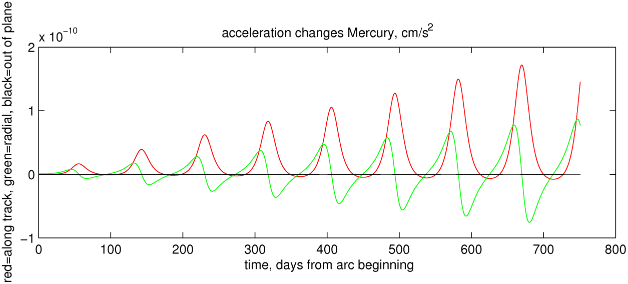
$$
\begin{aligned}
\overrightarrow{a}_k^{\gamma} \; &= \; \frac{1}{c^2} \sum_{j \neq k} \frac{\mu_j \, \overrightarrow{r_{kj}}}{r_{kj}^3} \left[-2 \sum_{l \neq k} \frac{\mu_l}{r_{kl}} + v_k^2 + v_j^2 - 2 \, \overrightarrow{v_k} \cdot \overrightarrow{v_j} \right] + \\
&+ \; \frac{2}{c^2} \sum_{j \neq k} \frac{\mu_j}{r_{kj}^3} \left(\overrightarrow{r_{jk}} \cdot \overrightarrow{v_{jk}} \right) \overrightarrow{v_{jk}} + \frac{2}{c^2} \sum_{j \neq k} \frac{\mu_j}{r_{kj}} \, \overrightarrow{a}_j^{new}
\end{aligned}
$$

$$
\overrightarrow{a}_k^{\beta} = -\frac{2}{c^2} \sum_{j \neq k} \frac{\mu_j}{r_{kj}^3} \left[\sum_{l \neq k} \frac{\mu_l}{r_{kl}} + \sum_{l \neq j} \frac{\mu_l}{r_{jl}} \right] \overrightarrow{r_{kj}}
$$

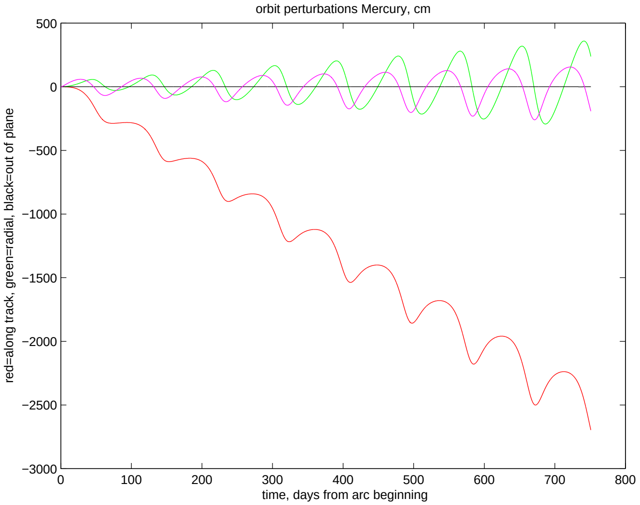
Effect of γ on the orbit of Mercury for $\bar{\gamma} = 10^{-5}$



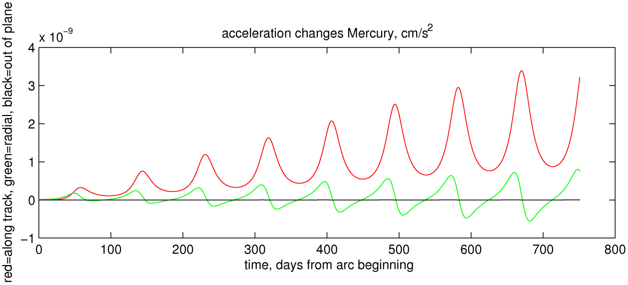
Effect of γ on the acceleration of Mercury for $\bar{\gamma} = 10^{-5}$



Effect of β on the orbit of Mercury for $\bar{\beta} = 10^{-4}$



Effect of β on the acceleration of Mercury for $\bar{\beta} = 10^{-4}$



Here we show the unfriendly formulas for the partial derivatives of the accelerations due to γ:

$$\frac{\partial\,(\overrightarrow{a}^{\gamma}_{k})_s}{\partial x_{p\,l}} = \frac{2}{c^2}\sum_{j\neq k}\Bigg[\Bigg(-\frac{3\mu_j}{r^5_{k\,j}}\,(\overrightarrow{r}_{k\,j})_p\,(\overrightarrow{r}_{k\,j})_s + \frac{\mu_j}{r^3_{k\,j}}\delta_{p\,s}\Bigg)\frac{v^2_{k\,j}}{2}\,(\delta_{l\,j}-\delta_{l\,k}) +$$

$$+\Bigg(-\frac{\mu_j}{r^3_{k\,j}}\,(\overrightarrow{r}_{k\,j})_p\left(\overrightarrow{a}^{new}_{k\,j}\right)_s + \frac{3\;\mu_j}{r^5_{k\,j}}\,(\overrightarrow{r}_{k\,j})_p\,(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{v}_{k\,j})\,(\overrightarrow{v}_{k\,j})_s +$$

$$-\frac{\mu_j}{r^3_{k\,j}}\,(\overrightarrow{v}_{k\,j})_p\,(\overrightarrow{v}_{k\,j})_s\Bigg)(\delta_{l\,j}-\delta_{l\,k}) + \frac{\mu_j}{r_{k\,j}}\left(\frac{\partial(\overrightarrow{a}^{new}_{j})_s}{\partial x_{p\,l}} - \frac{\partial(\overrightarrow{a}^{new}_{k})_s}{\partial x_{p\,l}}\right)\Bigg]$$

$$\frac{\partial\,(\overrightarrow{a}^{\gamma}_{k})_s}{\partial \dot{x}_{p\,l}} = \frac{2}{c^2}\sum_{j\neq k}\Bigg[\frac{\mu_j}{r^3_{k\,j}}\Big((\overrightarrow{r}_{k\,j})_s\,(\overrightarrow{v}_{k\,j})_p - (\overrightarrow{r}_{k\,j})_p\,(\overrightarrow{v}_{k\,j})_s\Big) +$$

$$-\frac{\mu_j}{r^3_{k\,j}}\,(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{v}_{k\,j})\;\delta_{p\,s}\Bigg](\delta_{l\,j}-\delta_{l\,k})$$

... and β:

$$\frac{\partial\left(\overrightarrow{a_k}^{\beta}\right)_s}{\partial x_{pl}} = -\frac{2}{c^2}\sum_{j\neq k}\left[\left(-\frac{\mu_j}{r_{kj}^5}\left(\overrightarrow{r}_{kj}\right)_p\left(\overrightarrow{r}_{kj}\right)_s + \frac{\mu_i}{r_{kj}^3}\delta_{ps}\right)\left(\delta_{lj}-\delta_{lk}\right)\cdot\right.$$

$$\left.\cdot\left(\sum_{h\neq k}\frac{\mu_h}{r_{kh}} + \sum_{h\neq j}\frac{\mu_h}{r_{jh}}\right)\right] + \frac{2}{c^2}\sum_{j\neq k}\frac{\mu_j}{r_{kj}^3}\left(\overrightarrow{r}_{kj}\right)_s\left[\sum_{h\neq k}\frac{\mu_h}{r_{kh}^3}\left(\overrightarrow{r}_{kh}\right)_p\left(\delta_{lh}-\delta_{lk}\right)+\right.$$

$$\left. + \sum_{h\neq j}\frac{\mu_h}{r_{jh}^3}\left(\overrightarrow{r}_{jh}\right)_p\left(\delta_{lh}-\delta_{lj}\right)\right]$$

$$\frac{\left(\overrightarrow{a_k}^{\beta}\right)_s}{\partial \dot{x}_{pl}} = 0$$

Parameters ζ, $J_{2\odot}$ Lagrangian functions

The Lagrangian L_ζ takes into account the variation of the Sun gravitational parameter $\mu_\odot$:

$$L_\zeta = (t - t_0) \sum_{i \neq 0} \frac{\mu_0 \mu_i}{r_{0i}}$$

The Lagrangian $L_{J_{2\odot}}$ is the usual perturbation term due to the oblateness of a body:

$$L_{J_{2\odot}} = -\frac{1}{2} \sum_{i \neq 0} \frac{\mu_0 \, \mu_i}{r_{0\,i}} \left(\frac{R_0}{r_{0\,i}} \right)^2 \left[3 \left(\overrightarrow{n}_{0\,i} \cdot \overrightarrow{e}_0 \right)^2 - 1 \right]$$

where $\overrightarrow{e}_0$ is the unit vector along the rotation axis of the Sun.

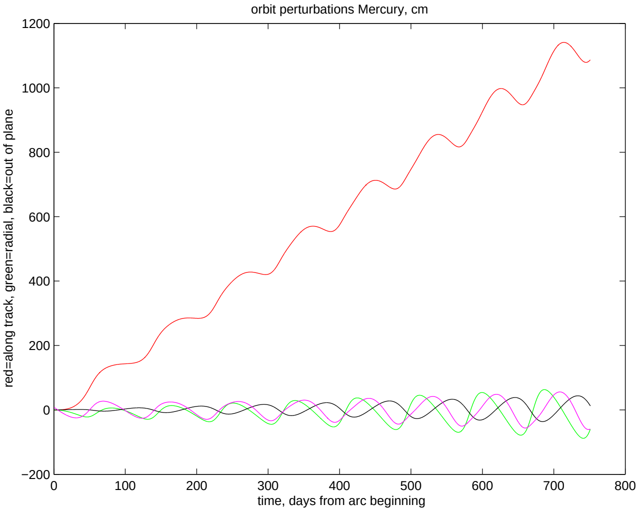
Accelerations due to parameters ζ, $J_{2\odot}$

For $k \neq 0$, i.e. a body different from the Sun, we have the following perturbative accelerations:

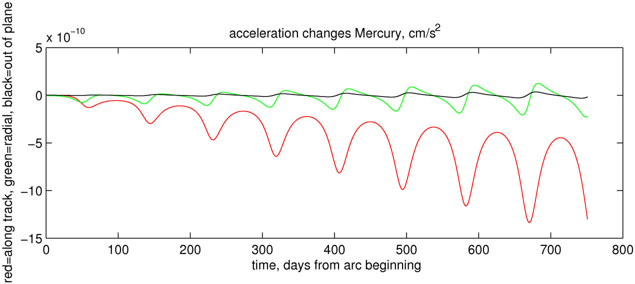
$$\overrightarrow{a}_k^{\zeta} = (t - t_0)\,\frac{\mu_0}{r_{k0}^3}\,\overrightarrow{r}_{k0}$$

$$\begin{aligned}
\overrightarrow{a}_k^{J_{2\odot}} &= -\frac{\mu_0\,R_0^2}{2}\left[-\frac{3}{r_{0k}^5}\left[3(\overrightarrow{n_{0k}}\cdot\overrightarrow{e_0})^2 - 1\right]\overrightarrow{r_{0k}} + \right.\\
&+ \left.\frac{6}{r_{0k}^4}(\overrightarrow{n_{0k}}\cdot\overrightarrow{e_0})(-(\overrightarrow{n_{0k}}\cdot\overrightarrow{e_0})\overrightarrow{n_{0k}} + \overrightarrow{e_0})\right]
\end{aligned}$$

Effect of $J_{2\odot}$ on the orbit of Mercury for $J_{2\odot} = 10^{-8}$



Effect of $J_{2\odot}$ on the acceleration of Mercury for $J_{2\odot} = 10^{-8}$



PPN parameters α_1, α_2 Preferred frame effect

$$L_{\alpha_1} = -\frac{1}{4\,c^2} \sum_j \sum_{i \neq j} \frac{\mu_i\,\mu_j}{r_{i\,j}} \left(\overrightarrow{z}_i \cdot \overrightarrow{z}_j\right)$$

$$L_{\alpha_2} = \frac{1}{4\,c^2} \sum_j \sum_{i \neq j} \frac{\mu_i\,\mu_j}{r_{i\,j}} \left[\left(\overrightarrow{z}_i \cdot \overrightarrow{z}_j\right) - \left(\overrightarrow{n_{i\,j}} \cdot \overrightarrow{z}_i\right)\left(\overrightarrow{n_{i\,j}} \cdot \overrightarrow{z}_j\right)\right]$$

where $\overrightarrow{z}_i = \overrightarrow{v}_i + \overrightarrow{w}$ and $\overrightarrow{w}$ is the velocity of the center of mass of the solar system with respect to the preferred frame.
Note that we also have to take into account the apparent acceleration $\dot{\overrightarrow{P}}_\alpha / \mathcal{M}$, where $\mathcal{M} = \sum_i \mu_i (1 + \frac{v_i^2}{2c^2} - \frac{U_i}{2c^2})$ and

$$\dot{\overrightarrow{P}}_\alpha = \alpha_1 \sum_k \frac{d}{dt} \frac{\partial L_{\alpha_1}}{\partial \overrightarrow{v}_k} + \alpha_2 \sum_k \frac{d}{dt} \frac{\partial L_{\alpha_2}}{\partial \overrightarrow{v}_k}$$

PPN α_1, α_2 perturbative accelerations

$$\overrightarrow{a}_k^{\alpha_1} = -\frac{1}{2\,c^2}\sum_{j\neq k}\frac{\mu_j}{r_{k\,j}^3}\overrightarrow{r_{k\,j}}\;(\overrightarrow{z}_k\cdot\overrightarrow{z}_j)+$$

$$+\frac{1}{2\,c^2}\sum_{j\neq k}\left(-\frac{\mu_j}{r_{k\,j}^2}\;(\overrightarrow{n}_{k\,j}\cdot\overrightarrow{z}_{k\,j})\;\overrightarrow{z}_j+\frac{\mu_j}{r_{k\,j}}\;\overrightarrow{a}_j^{new}\right)$$

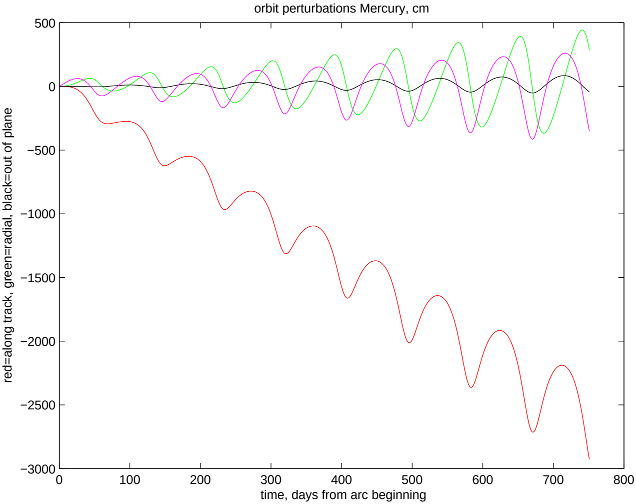
$$\overrightarrow{a}_k^{\alpha_2} = +\frac{1}{2\,c^2}\sum_{j\neq k}\frac{\mu_j}{r_{k\,j}^3}\overrightarrow{r_{k\,j}}\;(\overrightarrow{z}_k\cdot\overrightarrow{z}_j)+$$

$$-\frac{1}{2\,c^2}\sum_{j\neq k}\left(-\frac{\mu_j}{r_{k\,j}^2}\;(\overrightarrow{n}_{k\,j}\cdot\overrightarrow{z}_{k\,j})\;\overrightarrow{z}_j+\frac{\mu_j}{r_{k\,j}}\;\overrightarrow{a}_j^{new}\right)+$$

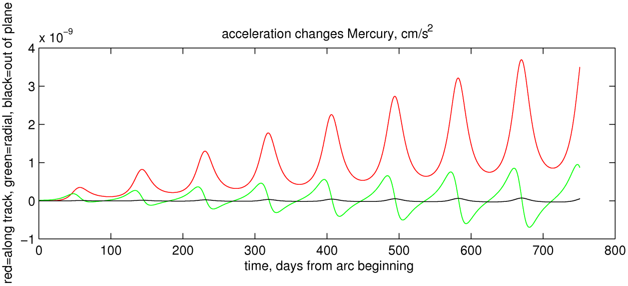
$$+\frac{1}{2\,c^2}\sum_{j\neq k}\frac{\mu_j}{r_{k\,j}^3}\Bigg[-3\,\overrightarrow{r}_{k\,j}(\overrightarrow{n}_{k\,j}\cdot\overrightarrow{z}_j)^2+\overrightarrow{z}_j(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{z}_j)+$$

$$+\overrightarrow{z}_j(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{z}_k)+\overrightarrow{r}_{k\,j}((\overrightarrow{z}_{k\,j}\cdot\overrightarrow{z}_j)+(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{a}_j^{new}))\Bigg]$$

Effect of α_1, α_2 on the orbit of Mercury for $\alpha_1 = 8 \times 10^{-6}\,\alpha_2 = 10^{-6}$



Effect of α_1, α_2 on the acceleration of Mercury for $\alpha_1 = 8 \times 10^{-6}\, \alpha_2 = 10^{-6}$



Partial derivatives of α_1 acceleration...

$$\frac{\partial\,(\overrightarrow{a}^{\,\alpha_1}_{k})_s}{\partial x_{p\,l}} = -\frac{1}{2\,c^2}\sum_{j\neq k}\left[-\frac{3\mu_j}{r^5_{k\,j}}\,(\overrightarrow{r}_{k\,j})_p\,(\overrightarrow{r}_{k\,j})_s+\frac{\mu_j}{r^3_{k\,j}}\delta_{p\,s}\right](\overrightarrow{z}_{k}\cdot\overrightarrow{z}_{j})(\delta_{l\,j}-\delta_{l\,k})+$$

$$+\frac{1}{2\,c^2}\sum_{j\neq k}\left[\frac{3\,\mu_j}{r^5_{k\,j}}\,(\overrightarrow{r}_{k\,j})_p\,(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{z}_{k\,j})-\frac{\mu_j}{r^3_{k\,j}}\,(\overrightarrow{z}_{k\,j})_p\right](\overrightarrow{z}_{j})_s\,(\delta_{l\,j}-\delta_{l\,k})+$$

$$+\frac{1}{2\,c^2}\sum_{j\neq k}\left[-\frac{\mu_j}{r^3_{k\,j}}\,(\overrightarrow{r}_{k\,j})_p\,(\delta_{l\,j}-\delta_{l\,k})\left(\overrightarrow{a}^{\,new}_{j}\right)_s+\frac{\mu_j}{r_{k\,j}}\frac{\partial\left(\overrightarrow{a}^{\,new}_{j}\right)_s}{\partial x_{p\,l}}\right]$$

$$\frac{\partial\,(\overrightarrow{a}^{\,\alpha_1}_{k})_s}{\partial \dot{x}_{p\,l}} = -\frac{1}{2\,c^2}\sum_{j\neq k}\frac{\mu_j}{r^3_{k\,j}}\,(\overrightarrow{r}_{k\,j})_s\,\left(\delta_{k\,l}\,(\overrightarrow{z}_{j})_p+(\overrightarrow{z}_{k})_p\,\delta_{j\,l}\right)+$$

$$+\frac{1}{2\,c^2}\sum_{j\neq k}\left(-\frac{\mu_j}{r^3_{k\,j}}\,(\overrightarrow{r}_{k\,j})_p\,(\overrightarrow{z}_{j})_s\,(\delta_{l\,j}-\delta_{l\,k})-\frac{\mu_j}{r^3_{k\,j}}\,(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{z}_{k\,j})\,\delta_{p\,s}\delta_{l\,j}\right)$$

...and very unfriendly partial derivatives of α_2 acceleration:

$$\frac{\partial\left(\overrightarrow{a}_{k}^{\alpha_2}\right)_s}{\partial x_{p\,l}} = \frac{1}{2\,c^2}\sum_{j\neq k}\left[-\frac{3\,\mu_j}{r_{k\,j}^5}\left(\overrightarrow{r}_{k\,j}\right)_p\left(\overrightarrow{r}_{k\,j}\right)_s + \frac{\mu_j}{r_{k\,j}^3}\delta_{p\,s}\right]\left(\overrightarrow{z}_k\cdot\overrightarrow{z}_j\right)\left(\delta_{l\,j}-\delta_{l\,k}\right)+$$

$$-\frac{1}{2\,c^2}\sum_{j\neq k}\left[\frac{3\,\mu_j}{r_{k\,j}^5}\left(\overrightarrow{r}_{k\,j}\right)_p\left(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{z}_{k\,j}\right)-\frac{\mu_j}{r_{k\,j}^3}\left(\overrightarrow{z}_{k\,j}\right)_p\right]\left(\overrightarrow{z}_j\right)_s\left(\delta_{l\,j}-\delta_{l\,k}\right)+$$

$$-\frac{1}{2\,c^2}\sum_{j\neq k}\left[-\frac{\mu_j}{r_{k\,j}^3}\left(\overrightarrow{r}_{k\,j}\right)_p\left(\delta_{l\,j}-\delta_{l\,k}\right)\left(\overrightarrow{a}_{j}^{new}\right)_s+\frac{\mu_j}{r_{k\,j}}\frac{\partial\left(\overrightarrow{a}_{j}^{new}\right)_s}{\partial x_{p\,l}}\right]$$

$$+\frac{1}{2\,c^2}\sum_{j\neq k}-\frac{3\,\mu_j}{r_{k\,j}^5}\left(\overrightarrow{r}_{k\,j}\right)_p\left(\delta_{l\,j}-\delta_{l\,k}\right)\Big[-3\,\left(\overrightarrow{r}_{k\,j}\right)_s\left(\overrightarrow{n}_{k\,j}\cdot\overrightarrow{z}_j\right)^2+$$

$$+\left(\overrightarrow{z}_j\right)_s\left(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{z}_j\right)+\left(\overrightarrow{z}_j\right)_s\left(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{z}_k\right)+\left(\overrightarrow{r}_{k\,j}\right)_s\left(\left(\overrightarrow{z}_{k\,j}\cdot\overrightarrow{z}_j\right)+\left(\overrightarrow{r}_{k\,j}\cdot\overrightarrow{a}_{j}^{new}\right)\right)\Big]+$$

$$+\frac{1}{2\,c^2}\sum_{j\neq k}\frac{\mu_j}{r_{k\,j}^3}\Big[\Big(-3\,\delta_{p\,s}\left(\overrightarrow{n}_{k\,j}\cdot\overrightarrow{z}_j\right)^2+\ldots\ldots\ldots$$

Tests for partial derivatives formulas

In order to minimize the risk of errors in the computation of the partial derivatives

$$L_P \longrightarrow \overrightarrow{a_k}^P \cong \frac{1}{\mu_k} \left[\frac{\partial L_P}{\partial \overrightarrow{r_k}} - \left(\frac{d}{dt} \frac{\partial L_P}{\partial \overrightarrow{v}_k} \right) \Bigg|_{\overrightarrow{a} = \overrightarrow{a}^{new}} \right]$$

$$\longrightarrow \frac{\partial \left(\overrightarrow{a}_k^P \right)_s}{\partial x_{p\,l}} \quad , \quad \frac{\partial \left(\overrightarrow{a}_k^P \right)_s}{\partial \dot{x}_{p\,l}}$$

we have tested the explicit, hand-calculated formulas, comparing them with the ones obtained using the Maple symbolic toolbox.

In practice we have evaluated both the hand-calculated and the computer-calculated formulas in some numerical values, and we have compared the results.