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RELATIVISTIC ORBIT DETERMINATION FOR MORE

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PLAN

1. Team
2. Orbit determination and relativity [Milani]
3. Lagrangian formulation for PN Relativity [Milani]

A. Equations of motion and variational equations [Cicalò, Latorre]

B. The MORE observables [Tommei]

4. Space-time reference systems [Milani]
5. Restricted ephemerides improvement [Milani]
6. Our documentation system and the certification process

1 Team

The MORE subteam working in Pisa to the Orbit Determination includes the following people, mostly from the Department of Mathematics, University of Pisa:

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2. Giovanni-Federico Gronchi (Researcher) gronchi@dm.unipi.it
3. Alessando Rossi (ISTI, CNR, Pisa; Researcher) alessandro.rossi@isti.cnr.it
4. Giacomo Tommei (PostDoc) tommei@dm.unipi.it
5. Emanuele Latorre (Contract Researcher) emanuele.latorre@gmail.com
6. Stefano Cicalò (PhD student) stefano.cicalo@gmail.com

Our research group includes two more people, Fabrizio Bernardi and Davide Farnocchia, who are not directly employed for BepiColombo but contribute to the overall know how and infrastructure.

2 Orbit determination and relativity

For the Relativity Experiment with the MORE data, we need to solve an orbit determination problem with a relativistic model to some PN order (TBD), including all the celestial bodies involved, namely the Earth, Mercury, and the MPO orbiter.

We do not handle a generic space-time, but the one where we are now. Thus we must include solutions for the initial conditions of all the bodies large enough to affect the measurements (at the expected level of accuracy of MORE).

We are here, eager to learn all the Relativity we need to use. However, the specialists of relativity need to understand the key problems in what we are doing, starting from the main problem of orbit determination: rank deficiency.

Rank deficiency d means that the normal matrix of the equations we are solving in a least squares fit has a kernel of dimension d, or an eigenvalue 0 of multiplicity d. Of course this implies that the normal system cannot be solved, and because of limited numerical accuracy even an approximate rank deficiency, that is d very small eigenvalues, is a problem. We shall first show this problem in the Newtonian case and its solutions.

PLAN:

2.1 Lagrangian formulation

2.2 Integrals of motion and rank deficiency

2.1 Lagrangian formulation

Hereafter we shall follow the notation of Moyer (2003):

$$
\begin{aligned}
\overrightarrow{r_{ij}} &= \overrightarrow{r_j} - \overrightarrow{r_i} \ , \ r_{ij} = \left|\overrightarrow{r_{ij}}\right| \ , \ \overrightarrow{a_{ij}} = \ddot{\overrightarrow{r_j}} - \ddot{\overrightarrow{r_i}} = \overrightarrow{a_j} - \overrightarrow{a_i} \\
\overrightarrow{v_{ij}} &= \dot{\overrightarrow{r_j}} - \dot{\overrightarrow{r_i}} = \overrightarrow{v_j} - \overrightarrow{v_i} \ , \ v_{ij} = \left|\overrightarrow{v_{ij}}\right|
\end{aligned}
$$

for $i, j = 0, N$, where 0 refers to the Sun, and use a Lagrangian formalism. The Newtonian $N+1$-body equations of motion are derived from a Lagrange function

$$
L_{NEW} = \frac{1}{2}\sum_i \mu_i v_i^2 + \frac{1}{2}\sum_i \sum_{j\neq i} \frac{\mu_i \mu_j}{r_{ij}} \ .
$$

The usual Lagrangian is multiplied by G, thus only the gravitational masses $\mu_i = G m_i$ appear in the overall Lagrangian; indeed the gravitational constant cannot be determined by any form of orbit determination (apart artificial systems). By Noether's theorem the symmetry of the Lagrangian with the isometry group of $\mathbf{R}^3$ implies 9 integrals of motion. The 3-parameter group of symmetries $\overrightarrow{r_j} \longrightarrow \overrightarrow{r_j} + \mathbf{h}$, $\mathbf{h} \in \mathbf{R}^3$ results in the vector integral of total linear momentum

$$
\overrightarrow{\mathbf{P}} = \sum_i \frac{\partial L_{NEW}}{\partial \overrightarrow{v_i}} = \sum_i \mu_i \overrightarrow{v_i} \ .
$$

2.2 Integrals of motion and rank deficiency

The linear motion of the center of mass (barycenter?) of the $N+1$-body system

$$\overrightarrow{b}(t) = \frac{1}{M}\sum_i \mu_i \overrightarrow{r_i}(t) = \frac{1}{M}\overrightarrow{\mathbf{P}}\,t + \overrightarrow{b}(0)$$

with $M = \sum_i \mu_i$. This implies that an orbit determination using as observations range/range-rate and angles between planets and a stellar reference frame, has a rank deficiency of 6: position and velocity of the barycenter are not observable.

There is only one solution to this problem, descoping, which can be obtained in two ways: either (1) the center of mass is assumed to be fixed, e.g., $\overrightarrow{b}(t) = \mathbf{0}$, or (2) it is constrained to remain fixed, by adding a priori observations of the form $\overrightarrow{b}(0) = \mathbf{0} \pm \sigma$ and $\dot{\overrightarrow{b}}(0) = \mathbf{0} \pm \sigma$, with a very small a priori uncertainty σ.

With solution (1) the equation of motion of the Sun is removed, and the position of the Sun is computed from the center of mass, that is $\overrightarrow{r_0}$ is replaced by $\overrightarrow{s}$ with

$$\overrightarrow{s} = -\frac{1}{M}\sum_{i=1}^{N} \mu_i \overrightarrow{r_i}\ .$$

By the same argument, the rotation group of symmetries implies the integral of total angular momentum, and if the observations where only range/range-rate between planets the rank deficiency would be of order 9.

3 Lagrangian formulation for PN Relativity

PLAN:

3.1 Center of mass in PN relativity

3.2 Three-body effects and oblateness of the Sun

3.3 Gravitational constant and mass of the Sun

3.4 Equations of motion for preferred frame effects

3.5 Violations of the strong equivalence principle

3 Lagrangian formulation for PN Relativity

The equations of motion of GR, to Post-Newtonian order 1, can be deduced from the relativistic Lagrangian

$$L = L_{NEW} + L_{GR0} + \beta L_\beta + \gamma L_\gamma$$

where γ, β are the Eddington parameters, both $= 1$ in GR, and L_{GR0} is the portion without free parameters (apart from G, discussed later) (why?).

$$\begin{aligned} L_{GR0} &= \frac{1}{8c^2}\sum_i \mu_i v_i^4 + \frac{1}{2\,c^2}\sum_i \sum_{j\neq i}\sum_{k\neq i}\frac{\mu_i\,\mu_j\,\mu_k}{r_{ij}\,r_{ik}} + \\ &+ \frac{1}{2\,c^2}\sum_i\sum_{j\neq i}\frac{\mu_i\,\mu_j}{r_{ij}}\left[\frac{1}{2}(v_i^2+v_j^2) - \frac{3}{2}(\vec{v_i}\cdot\vec{v_j}) - \frac{1}{2}(\vec{n_{ij}}\cdot\vec{v_i})\,(\vec{n_{ij}}\cdot\vec{v_j})\right] \end{aligned}$$

where $\vec{n_{ij}} = \vec{r_{ij}}/r_{ij}$.

$$L_\gamma = \frac{1}{2\,c^2}\sum_i\sum_{j\neq i}\frac{\mu_i\mu_j}{r_{ij}}\,(\vec{v_i}-\vec{v_j})^2 \quad , \quad L_\beta = -\frac{1}{c^2}\sum_i\sum_{j\neq i}\sum_{k\neq i}\frac{\mu_i\,\mu_j\,\mu_k}{r_{ij}\,r_{ik}}$$

3.1 Center of mass in PN Relativity

The relativistic Lagrangian L is also invariant by translation, thus by Noether's theorem there is a vector integral

$$\overrightarrow{\mathbf{P}} = \sum_i \frac{\partial L}{\partial \overrightarrow{v_i}} = \sum_i \mu_i \overrightarrow{v_i} + \sum_i \frac{\partial L_{GR0}}{\partial \overrightarrow{v_i}}$$

where the contributions from the derivatives of L_β vanish and the ones from L_γ cancel in the sum over i because they are antisymmetric. Thus

$$\begin{aligned}
\overrightarrow{\mathbf{P}} &= \sum_i \mu_i \overrightarrow{v_i} \left[1 + \frac{1}{2}\left(\frac{v_i}{c}\right)^2 - \frac{U_i}{2\,c^2}\right] - \frac{1}{2\,c^2} \sum_i \sum_{k \neq i} \frac{\mu_i \mu_k}{r_{ik}} (\overrightarrow{n_{ik}} \cdot \overrightarrow{v_k}) \overrightarrow{n_{ik}} \\
U_i &= \sum_{k \neq i} \frac{\mu_k}{r_{ik}}
\end{aligned}$$

and the vector

$$\overrightarrow{\mathbf{B}} = \sum_i \mu_i \overrightarrow{r_i} \left[1 + \frac{v_i^2}{2c^2} - \frac{U_i}{2c^2}\right]$$

is such that, neglecting PN order 2,

$$\frac{d\overrightarrow{\mathbf{B}}}{dt} = \overrightarrow{\mathbf{P}} \,.$$

3.1 Center of mass in PN Relativity (continued)

The relativistic analog of the total mass

$$\mathcal{M} = \sum_i \mu_i \left[1 + (v_i^2 - U_i)/(2c^2)\right]$$

is an integral to order 1PN (because the PN order 1 term is the Newtonian energy divided by c^2), thus we can define the relativistic center of mass

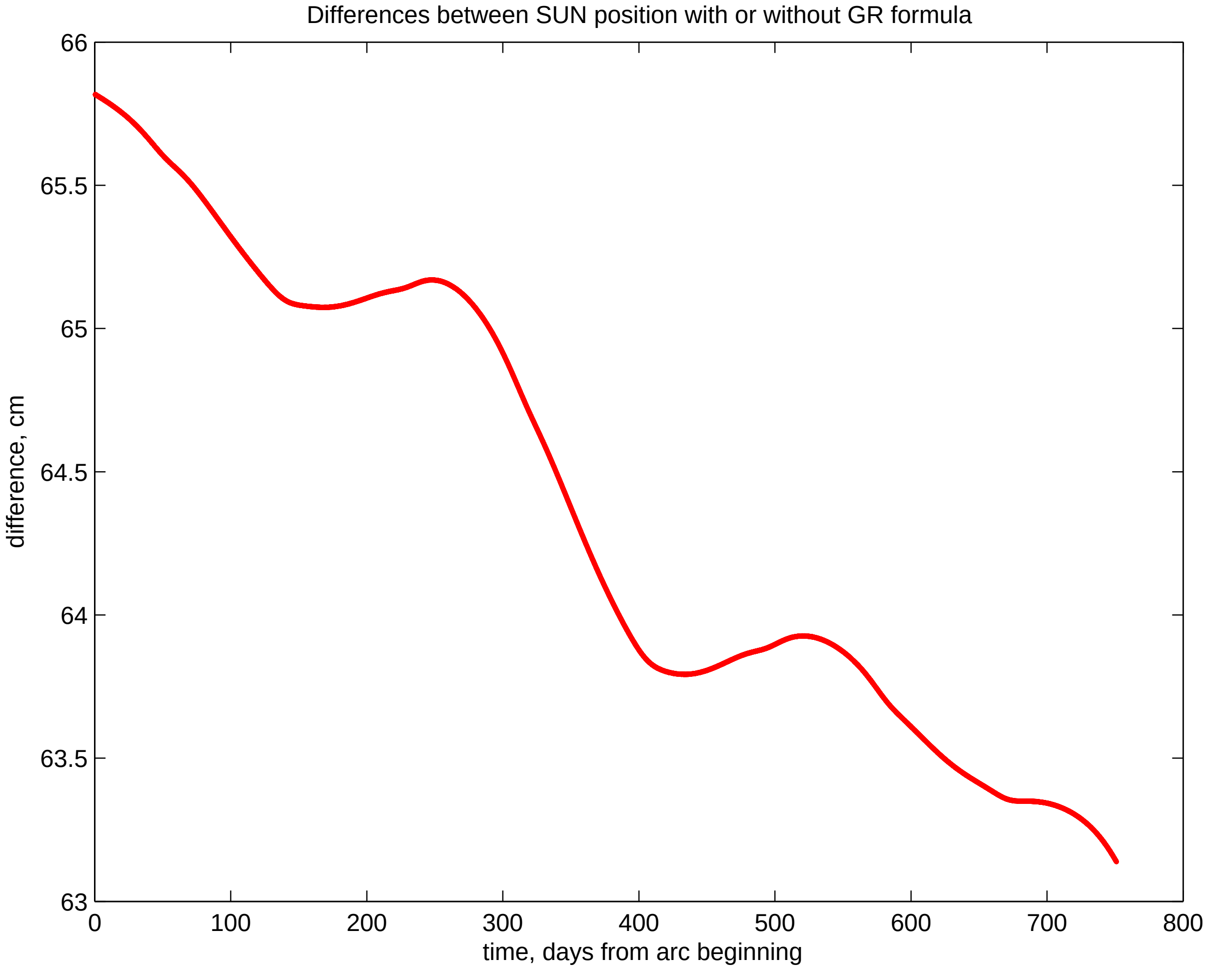
$$\overrightarrow{b} = \overrightarrow{\mathbf{B}}/\mathcal{M}$$

which is also a vector integral, and so is $\dot{\overrightarrow{b}}$. The rank deficiency problem is the same as in the Newtonian case. To solve it, we can either set $\overrightarrow{b}(t) = \mathbf{0}$ and solve for the position of the Sun from the ones of the planets

$$\overrightarrow{s} = -\frac{1}{\mu_0\left[1 + (v_0^2 - U_0)/2c^2)\right]} \sum_{i=1}^{N} \mu_i \left[1 + \frac{v_i^2 - U_i}{2c^2}\right] \overrightarrow{r_i} \ ,$$

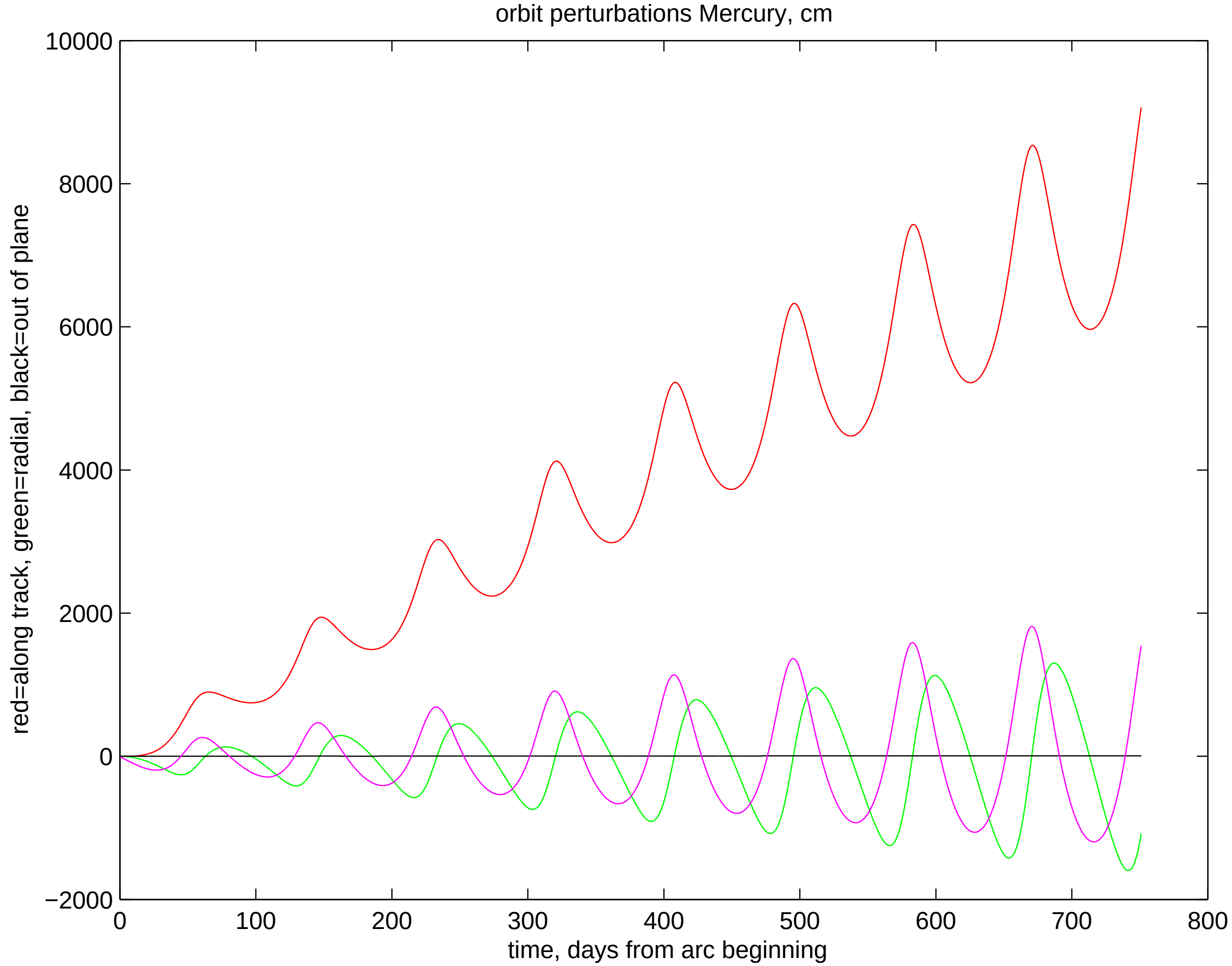
and similarly for the velocity. As an alternative, the equation of motion may include the one for the Sun, and a constraint on the center of mass can be added by means of a priori observations: this is somewhat more complicated because the constraints are nonlinear, but it is possible and the results must be the same.

3.1 Center of mass in PN Relativity (example)



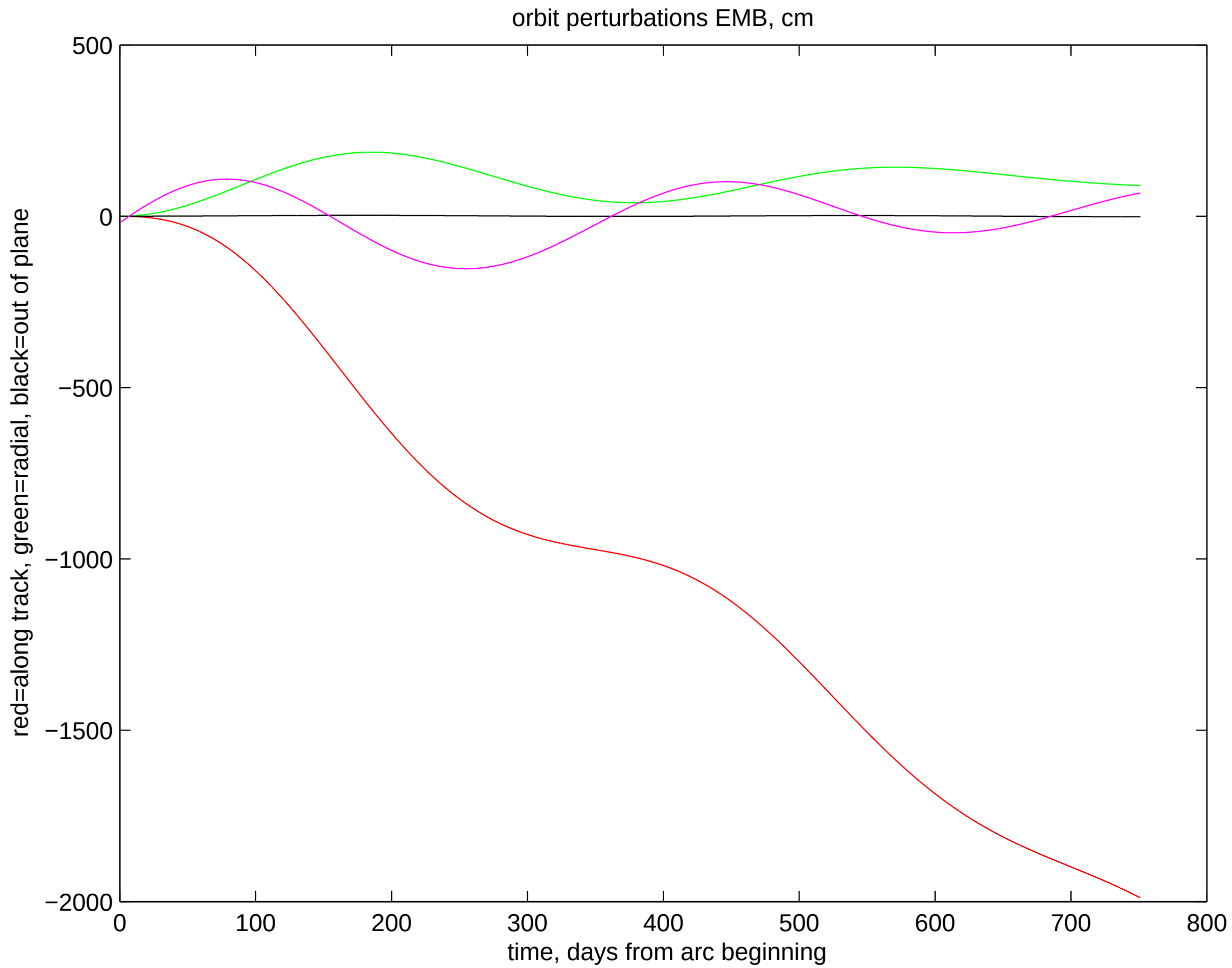
Distance (cm) between the barycentric position of the Sun, as computed by the Newtonian formula, and the relativistic 1PN formula, for the same position of the planets.

3.1 Center of mass in PN Relativity (example)



Differences in the orbit of Mercury due to the barycentric position of the Sun, as computed by the Newtonian formula, and the relativistic 1PN formula, for the same position of the planets.

3.1 Center of mass in PN Relativity (example)



Differences in the orbit of Earth-Moon barycenter due to the barycentric position of the Sun, as computed by the Newtonian formula, and the relativistic 1PN formula, for the same position of the planets.

3.2 Three body effects and oblateness of the Sun

The contribution of the oblateness of the Sun is $J_{2\odot} L_{J_{2\odot}}$ with

$$L_{J_{2\odot}} = -\frac{1}{2}\sum_{i\neq 0}\frac{\mu_0\mu_i}{r_{0i}}\left(\frac{R_0}{r_{0i}}\right)^2[3(\mathbf{n}_{0i}\cdot\mathbf{e}_0)^2-1]$$

where R_0 is the radius of the Sun, and $\mathbf{e}_0$ is the unit vector along the rotation axis of the Sun. The unit vector $\mathbf{e}_o$ is given in standard equatorial coordinates J2000 in Seidelmann et al. 2001: $\alpha_0 = 286^\circ.13$, $\delta_0 = 63^\circ.87$.

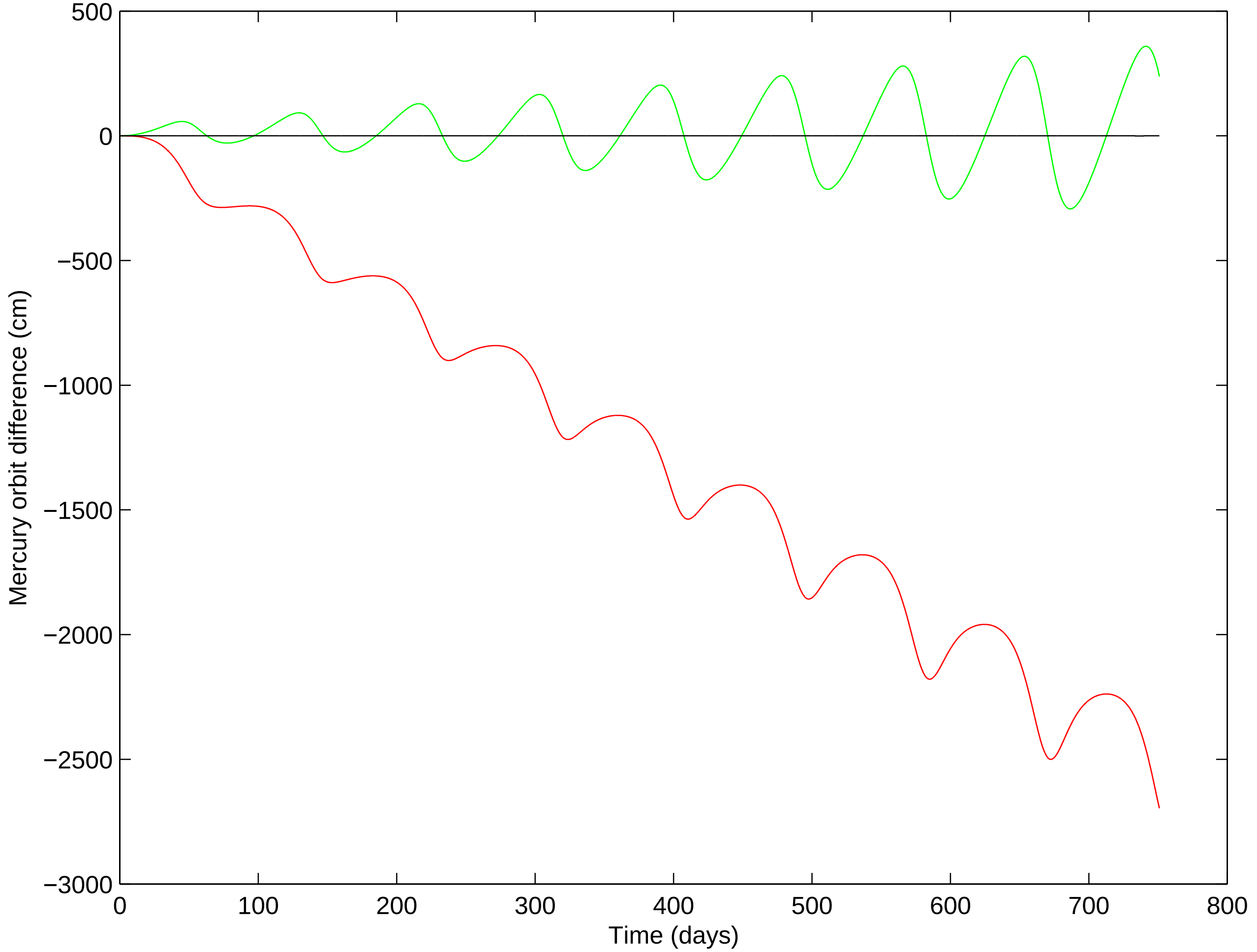
The main secular effect of the Sun's oblateness is the precession of the orbit of Mercury around the axis $\mathbf{e}_o$. However, β has a main effect which is a secular precession of the orbit of Mercury in its orbital plane (GR is isotropic). The angle between spin axis of the Sun and the orbital angular momentum of Mercury being only $\varepsilon = 3.^\circ 3$, thus $\cos\varepsilon = 0.998$ and $Corr(\beta, J_{2\odot}) = 0.997$ in our previous simulations of the Relativity experiment.

The results depend upon choices of the PPN parameters, especially from the possible use of the Nordtvedt equation, assuming a metric theory:

$$\eta = 4(\beta-1)-(\gamma-1)-\alpha_1-\frac{2}{3}\alpha_2$$

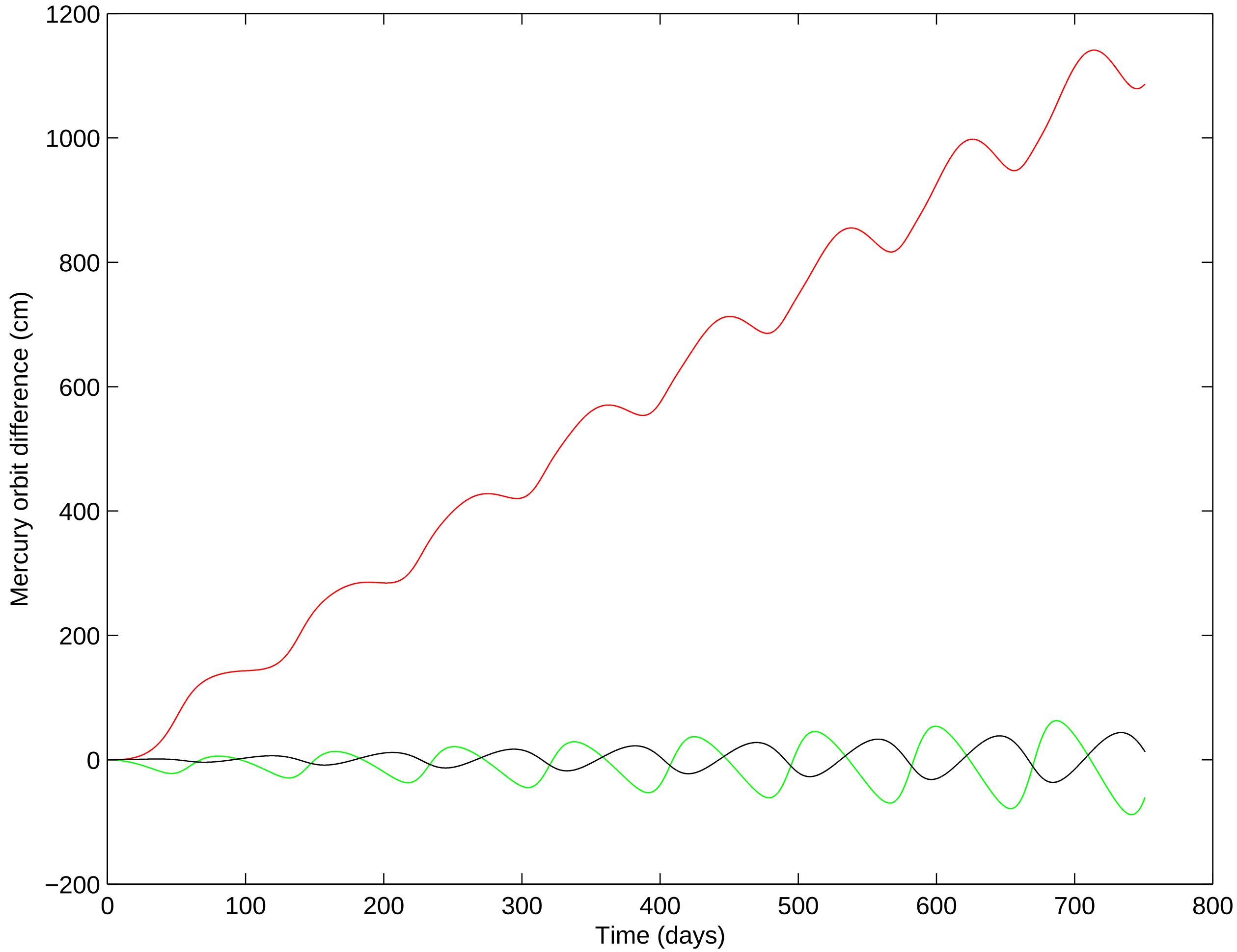
which removes the approximate symmetry between β and $J_{2\odot}$, provided both γ and η (SEP violation, see later) are known to $\simeq 10^{-5}$.

3.2 Three body effects and oblateness of the Sun (example)



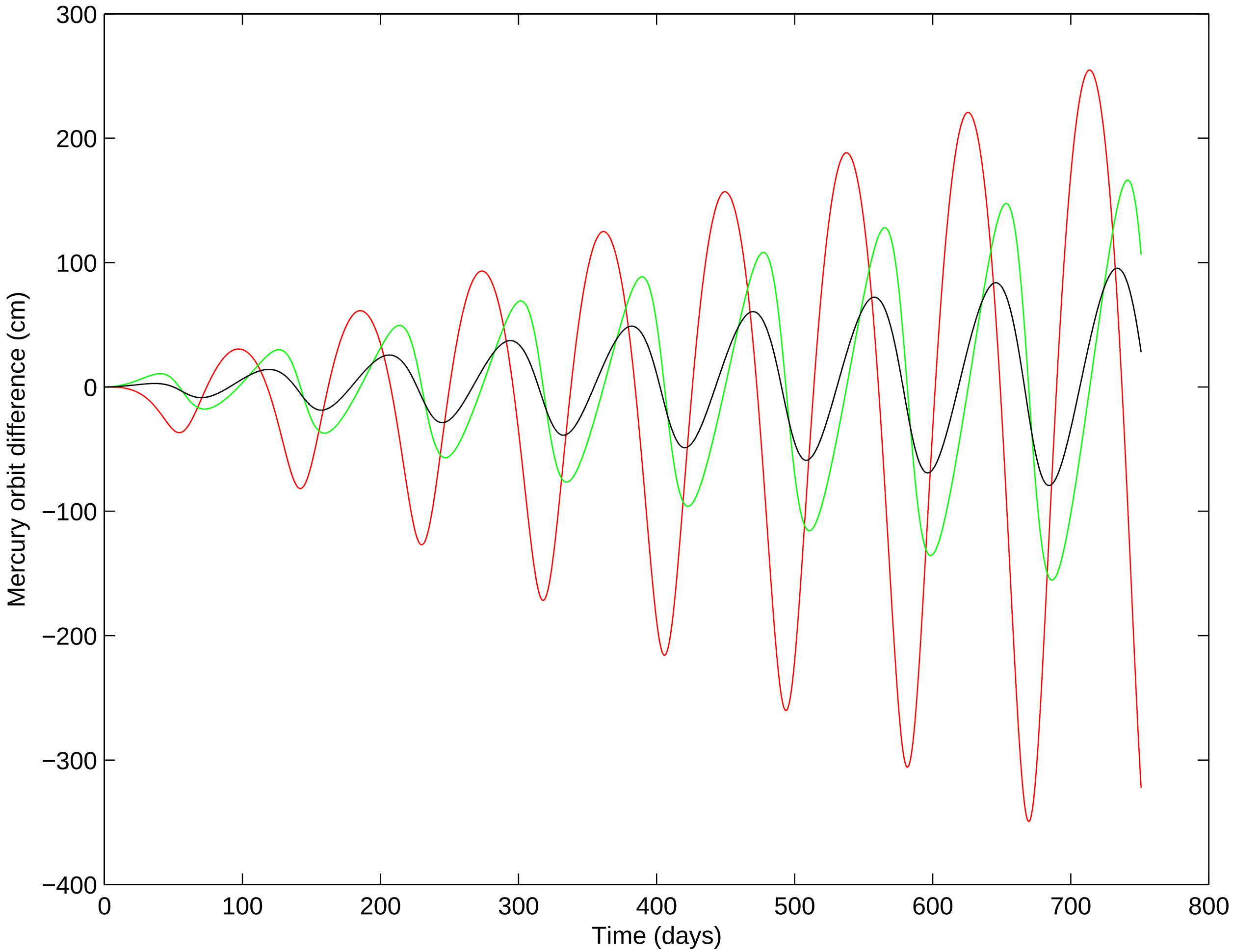
Perturbations on the orbit of Mercury due to $\beta = 10^{-4}$ with a 2 years extended mission: Red: transversal, Green: radial.

3.2 Three body effects and oblateness of the Sun (example)



Perturbations on the orbit of Mercury due to a change in $J_{2\odot}$ by 10^{-8} with a 2 years extended mission: Red: transversal, Green: radial.

3.2 Three body effects and oblateness of the Sun (example)



Perturbations on the orbit of Mercury due to a change in $J_{2\odot}$ by 2.19×10^{-8} and a $\beta = 10^{-4}$ with a 2 years extended mission. The correlation of the two transversal perturbations is 0.996.

3.2 Three body effects and oblateness of the Sun (the revenge of astrophysics)

I have been recently warned by a senior astrophysicist (Steve Shore, Univ. of Pisa) that the $J_{2\odot}$ results from circulation of matter inside the Sun, which may change during the 11.5 years cycle of solar activity.

If the value is $J_{2\odot} \simeq 2 \times 10^{-7}$ and the Nordtvedt equation was applicable, with good enough results on η, then according to the results of the previous simulations we might be able to measure the difference in $J_{2\odot}$ between one year and the next with an accuracy better than 1% of the $J_{2\odot}$ value. I neither know if such a change is likely, nor know what is the status of the solar cycle when BC is in orbit around Mercury.

3.3 Gravitational constant and mass of the Sun

A goal very interesting, especially for cosmologists, would be to measure the time variation of the gravitational constant G. From the orbit determination of the Moon with laser ranging we know already that $|\dot{G}/G| \leq 4 \times 10^{-12}$/yr. The Lagrange function terms are $(\dot{G}/G)\, L_{\dot{G}/G}$, where

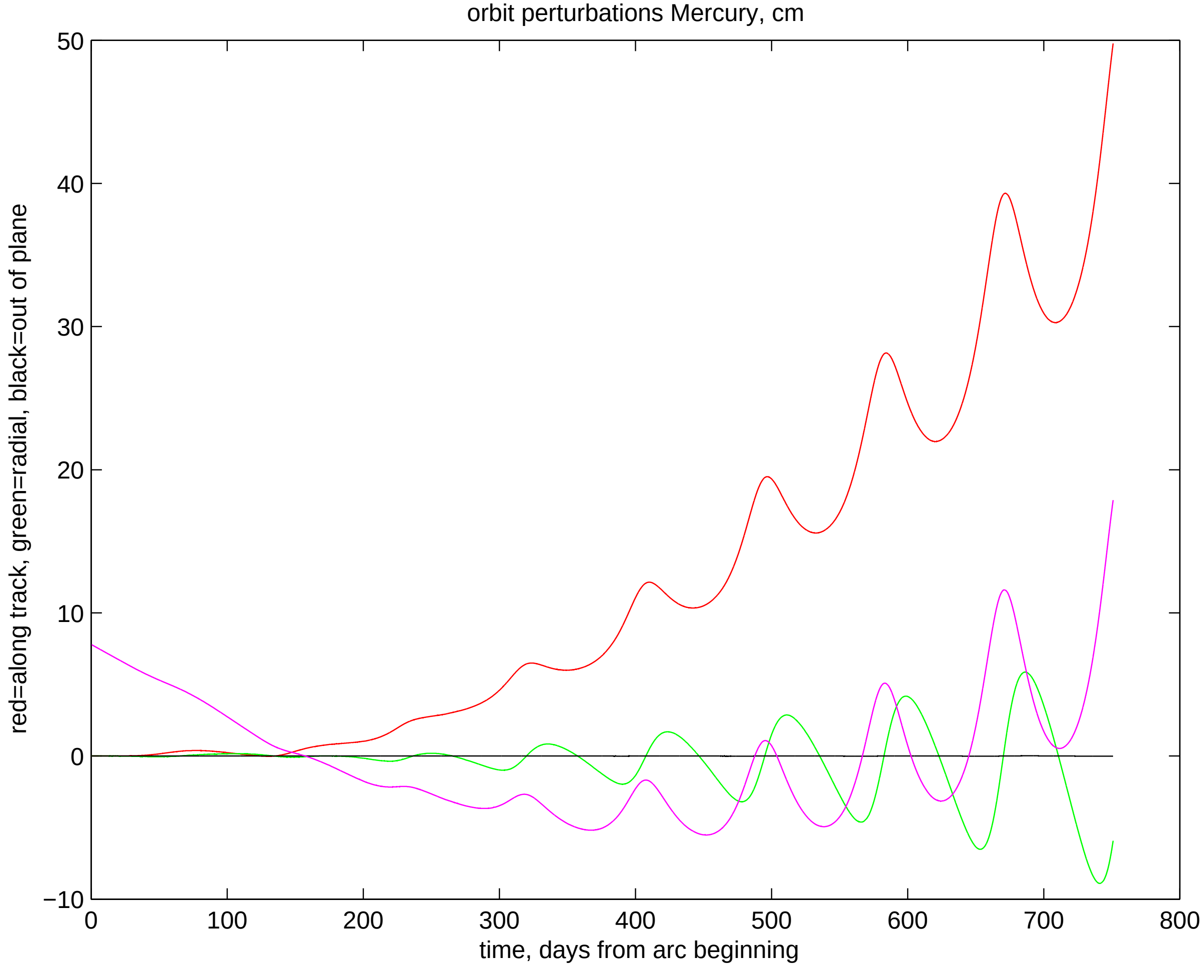
$$L_{\dot{G}/G} = \frac{t-t_0}{2} \sum_{i \neq j} \frac{\mu_i \mu_j}{r_{ij}} \ ,$$

but among these in practice the only terms with measurable effects are those with Sun's mass $\mu_0 = \mu_\odot$. Hence the parameter which can be determined and the corresponding Lagrange function term are

$$\zeta = \frac{d\mu_\odot}{dt} / \mu_\odot \ \ , \ \ L_\zeta = (t-t_0) \sum_{i \neq 0} \frac{\mu_0 \mu_i}{r_{0i}}$$

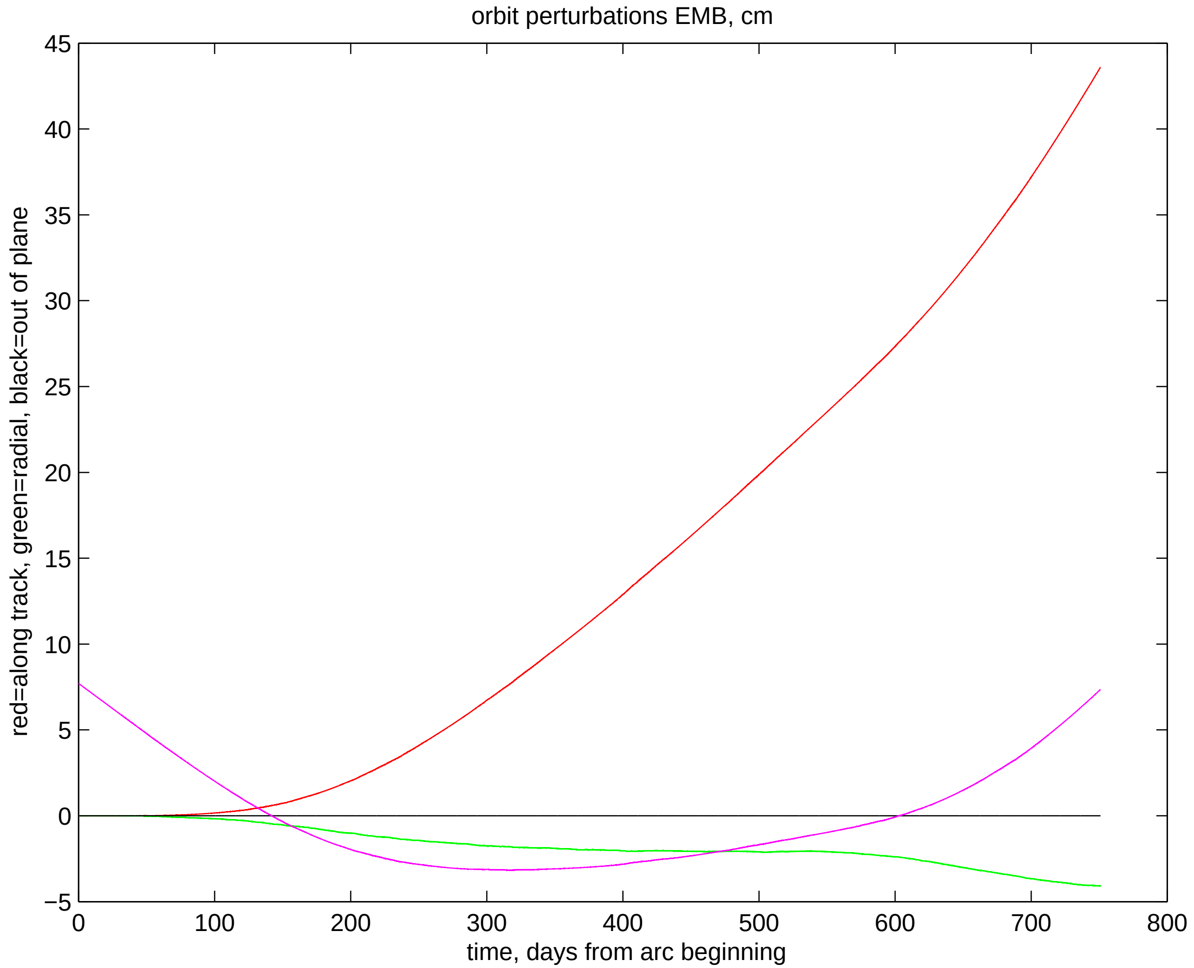
and it is not possible to discriminate the change with time of G from change with time of $m_\odot$. What should be measured is $\dot{m}_\odot / m_\odot \simeq 2 \times 10^{-13}$, due to mass shed as radiation. A contribution of the same order is the mass of charged particles emitted by the Sun, but the amount of the latter is poorly constrained, also because the flux of charged particles is not isotropic, thus the measurements from the neighborhood of the Earth are not representative.

Gravitational constant and mass of the Sun (example)



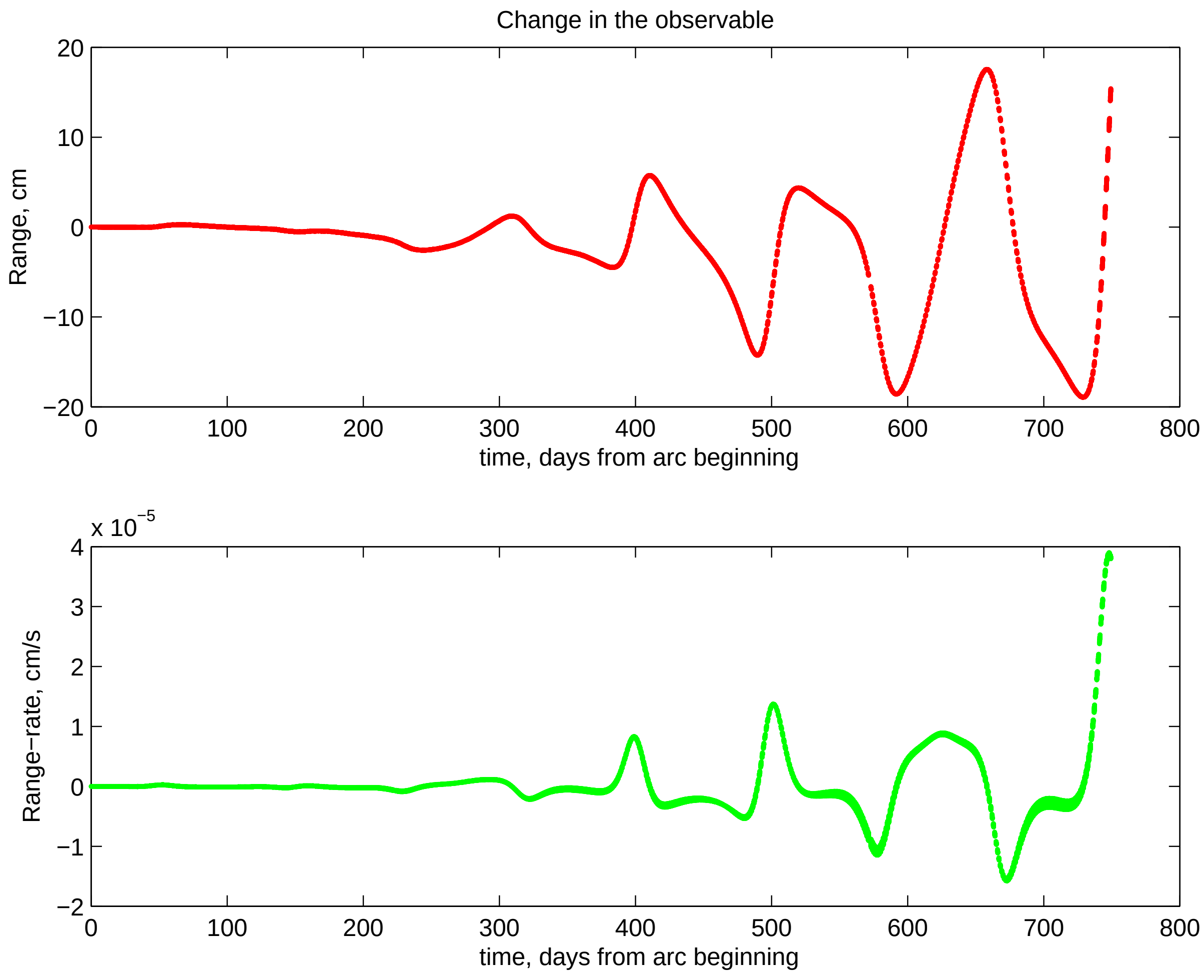
Perturbations on the orbit of Mercury due to $\zeta = 10^{-13}$ with a 2 years extended mission: Red: transversal, Green: radial.

Gravitational constant and mass of the Sun (example)



Perturbations on the orbit of the Earth-Moon barycenter due to $\zeta = 10^{-13}$ with a 2 years extended mission.

Gravitational constant and mass of the Sun (example)



Change in the signal due to $\zeta = 10^{-13}$ over 2 years. Above: range, the signal could be detected. Below: range-rate, well below the sensitivity of MORE. This signal is computed by assuming the initial conditions are not adjusted, the realistic difference is different in shape and size.

3.3 Gravitational constant and mass of the Sun (continued)

The main effect of a change of either the gravitational constant G or the mass of the Sun $m_\odot$ by a fraction 10^{-13} in one year is a quadratic perturbation along track, growing to $\simeq 50$ cm after two years for Mercury. If the range measurements contain a time-dependent bias with a quadratic signature, this results in a systematic error in the nominal solution for ζ. This argument was used to upgrade the requirements for the instrument to be used in the BepiColombo radioscience experiment, which now include an internal calibration loop to measure the transponder delay.

However, the determination of ζ from the orbit of Mercury is not a null experiment, but one in which there is a predicted value, although not a very accurate one. For changes in ζ of the order of few parts in 10^{-13} to discriminate between the new physics of a change in G and the known change in $m_\odot$ will be difficult.

The contribution of the change in $\mu_\odot$ to the motion of the center of mass:

$$\frac{d\overrightarrow{\mathbf{B}}}{dt} = \overrightarrow{\mathbf{P}} + \dot{\mu}_\odot \overrightarrow{s} \quad , \quad \frac{d^2\overrightarrow{\mathbf{B}}}{dt^2} = \dot{\mu}_\odot \dot{\overrightarrow{s}}$$

is negligible: $|\dot{\overrightarrow{s}}| \leq 20$ m/s, thus for $\dot{\mu}_\odot \leq 5 \times 10^{-13} \mu_\odot$ y^{-1} the acceleration is $\leq 2 \times 10^3\, 5 \times 10^{-13} / 3 \times 10^7 \simeq 3 \times 10^{-17}$ cm/s^2, a change in velocity by 10^{-5} μ/s in a year.

3.4 Preferred frame effects

The preferred frame effects are described by the contribution

$$L_\alpha = \alpha_1 L_{\alpha_1} + \alpha_2 L_{\alpha_2} \ ,$$

where

$$L_{\alpha_1} = -\frac{1}{4c^2} \sum_j \sum_{i \neq j} \frac{\mu_i \mu_j}{r_{ij}} \left(\vec{z_i} \cdot \vec{z_j} \right) \ .$$

The vector $\vec{z_i} = \vec{w} + \vec{v_i}$ is the velocity with respect to the preferred reference frame (assumed to be that of cosmic microwave background).

$$L_{\alpha_2} = \frac{1}{4c^2} \sum_j \sum_{i \neq j} \frac{\mu_i \mu_j}{r_{ij}} \left[\left(\vec{z_i} \cdot \vec{z_j} \right) - \left(\vec{n_{ij}} \cdot \vec{z_i} \right) \left(\vec{n_{ij}} \cdot \vec{z_j} \right) \right] \ .$$

To apply again Noether's theorem:

$$\frac{\partial L_\alpha}{\partial \vec{v_i}} = -\frac{\alpha_1 - \alpha_2}{2c^2} \mu_i \vec{z_i} \sum_{j \neq i} \frac{\mu_j}{r_{ij}} - \frac{\alpha_2}{2c^2} \sum_{j \neq i} \frac{\mu_i \mu_j}{r_{ij}^3} (\vec{r_{ji}} \cdot \vec{z_j}) \vec{r_{ji}} \ ,$$

$$\vec{\mathbf{P}}_\alpha = \sum_i \frac{\partial L_\alpha}{\partial \vec{v_i}}$$

3.4 Preferred frame effects (continued)

The Lagrangian L can be split into parts with and without the preferred frame effects $L = L_0 + L_\alpha$. The same is possible for the total linear momentum:

$$\overrightarrow{\mathbf{P}} = \overrightarrow{\mathbf{P}}_0 + \overrightarrow{\mathbf{P}}_\alpha \ , \quad \overrightarrow{\mathbf{P}}_0 = \sum_j \frac{\partial L_0}{\partial \overrightarrow{v_j}} \ ,$$

which is an integral, because also L_α is symmetric with respect to a constant translation. However, it is not invariant with respect to a time-dependent translation with constant velocity, and there is no center of mass integral. That is, from

$$\frac{d\overrightarrow{\mathbf{P}}}{dt} = \frac{d\overrightarrow{\mathbf{P}}_0}{dt} + \frac{d\overrightarrow{\mathbf{P}}_\alpha}{dt} = \mathbf{0}$$

we deduce the motion of the center of mass, defined ignoring L_α

$$\mathcal{M}\, \ddot{\overrightarrow{b}} = \dot{\overrightarrow{\mathbf{P}}}_0 = -\frac{d\overrightarrow{\mathbf{P}}_\alpha}{dt} \ .$$

This leads to the choice of a non-inertial reference system, obtained ignoring the preferred frame effects, which is accelerated.

3.4 Preferred frame effects (continued)

The accelerated reference frame results in an apparent force, giving the same acceleration on all the bodies:

$$\overrightarrow{a_{app}} = -\ddot{\vec{b}} = \frac{1}{\mathcal{M}}\frac{d\overrightarrow{\mathbf{P}}_\alpha}{dt}\;,\quad \frac{d\overrightarrow{\mathbf{P}}_\alpha}{dt} = \alpha_1 \sum_k \frac{d}{dt}\frac{\partial L_{\alpha_1}}{\partial \overrightarrow{v_k}} + \alpha_2 \sum_k \frac{d}{dt}\frac{\partial L_{\alpha_2}}{\partial \overrightarrow{v_k}}$$

$$\frac{d}{dt}\frac{\partial L_{\alpha_1}}{\partial \overrightarrow{v_k}} = -\frac{1}{2c^2}\sum_{j\neq k}\frac{\mu_j\mu_k}{r_{kj}}\left[-\overrightarrow{z_j}\,\frac{\overrightarrow{r_{kj}}\cdot\overrightarrow{z_{kj}}}{r_{kj}^2} + \overrightarrow{a_j}^{new}\right]$$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L_{\alpha_2}}{\partial \overrightarrow{v_k}} = {} & \frac{1}{2c^2}\sum_{j\neq k}\frac{\mu_j\mu_k}{r_{kj}}\left[-\overrightarrow{z_j}\,\frac{\overrightarrow{r_{kj}}\cdot\overrightarrow{z_{kj}}}{r_{kj}^2} + \overrightarrow{a_j}^{new}\right] + \\
& - \frac{1}{2c^2}\sum_{j\neq k}\frac{\mu_j\mu_k}{r_{kj}^3}\Big[-3\overrightarrow{r_{kj}}\left(\overrightarrow{n_{kj}}\cdot\overrightarrow{z_{kj}}\right)\left(\overrightarrow{n_{kj}}\cdot\overrightarrow{z_j}\right) + \\
& + \overrightarrow{z_{kj}}\left(\overrightarrow{r_{kj}}\cdot\overrightarrow{z_j}\right) + \overrightarrow{r_{kj}}\left(\overrightarrow{z_{kj}}\cdot\overrightarrow{z_j}\right) + \overrightarrow{r_{kj}}\left(\overrightarrow{r_{kj}}\cdot\overrightarrow{a_j}^{new}\right)\Big]
\end{aligned}$$

3.4 Preferred frame effects (example)

A description of this effect has not been found in the literature. The non-existence of the barycenter integrals is claimed (but we have found no formal proof).

The effect over 2 years of a the apparent force term associated with a preferred frame effects $\alpha_1 = 8 \times 10^{-6}$, $\alpha_2 = 10^{-6}$ has not been detected (within rounding off).

The acceleration values found for the apparent force term are very small: 10^{-17} cm/s^2. The formula should give exactly zero for $\overrightarrow{w} = \mathbf{0}$, but we have not been able to check this.

A common acceleration should not give equal effects on different planetary orbits. However, there might be a partial cancellation.

3.5 Violations of the strong equivalence principle

With the Lagrangian multiplied by G, the Newtonian kinetic energy is

$$T = \frac{1}{2}\sum_i \mu_i v_i^2 \ ,$$

that is we assume that the inertial mass and the gravitational mass are the same (at least exactly proportional). If some form of mass has a different gravitational coupling, there are for each body i two quantities μ_i and μ_i^I, one appearing in the gravitational potential (including the relativistic part) and the other appearing in the kinetic energy. If there is a violation of the strong equivalence principle involving body i, with a fraction Ω_i of its mass due to gravitational self-energy

$$\mu_i = [1+\eta\Omega_i]\,\mu_i^I \Longleftrightarrow \mu_i^I = [1-\eta\Omega_i]\,\mu_i + O(\eta^2)$$

with η a Post-Newtonian parameter for this violation. Neglecting $O(\eta^2)$ terms (also $O[\eta\,(\gamma-1)]$ etc.) this is expressed by a Lagrangian term $\eta\,L_\eta$ where

$$L_\eta = -\frac{1}{2}\sum_i \Omega_i\,\mu_i\,v_i^2 \ ,$$

with the effect of adding to the acceleration acting on body i:

$$\overrightarrow{a_i} = \overrightarrow{a_i}|_{\eta=0}\,[1+\eta\,\Omega_i] \ .$$

3.5 Violations of the strong equivalence principle (continued)

Since the fraction of gravitational self-energy is much larger, $\Omega_0 \simeq -3.5 \times 10^{-6}$, the largest effect of η is a change in the center of mass integral

$$\vec{\mathbf{P}} = \sum_j \frac{\partial L}{\partial \vec{v_j}} = \sum_j [1 - \eta\,\Omega_i]\,\mu_i\,\vec{v_j} \quad , \quad \frac{d\,\vec{\mathbf{P}}}{dt} = \mathbf{0}$$

and if we assume the center of mass is fixed at the origin

$$\vec{b} = \frac{1}{\mathcal{M}} \sum_j [1 - \eta\Omega_j]\,\mu_j = \mathbf{0}$$

the position of the Sun has to be corrected:

$$\vec{s} = \frac{-1}{\mu_0\,[1 - \eta\,\Omega_0]} \sum_{j \neq 0} [1 - \eta\Omega_j]\,\mu_j\,\vec{r_j} \ .$$

This results in indirect perturbations on some planet j, e.g., Mercury, because the Sun is displaced by $\eta\,\Omega_0\,\mu_i\,r_i/\mu_0$. If i refers to Jupiter, the Sun is moved by Jupiter by $\simeq 0.005$ AU $\simeq 7 \times 10^{10}$ cm, $\eta = 10^{-5}$ corresponds to a shift by $3.5 \times 10^{-6} \cdot 10^{-5} \cdot 7 \times 10^{10} \simeq 2$ cm.

3.5 Violations of the strong equivalence principle (continued)

The direct and indirect perturbations are

$$\overrightarrow{a_j} = (1+\eta\Omega_j)\left[\frac{\mu_0}{r_{j0}^3}\overrightarrow{r_{j0}} + \sum_{i\neq j,0}\frac{\mu_i}{r_{ji}^3}\overrightarrow{r_{ji}} + \ldots\right]\ ;$$

we compute the partial derivative of the acceleration for body j with respect to η:

$$\frac{\partial\overrightarrow{a_j}}{\partial\eta} = \Omega_j\left[\frac{\mu_0}{r_{j0}^3}\overrightarrow{r_{j0}} + \sum_{i\neq j,0}\frac{\mu_i}{r_{ji}^3}\overrightarrow{r_{ji}}\right] + \mu_0\frac{\partial}{\partial\overrightarrow{r_0}}\left[\frac{\mu_0}{r_{j0}^3}\right]\frac{\partial\overrightarrow{r_0}}{\partial\eta}$$

where the first term is the direct, the second the indirect (through the position of the Sun) perturbation due to $\eta \neq 0$.

$$\frac{\partial\overrightarrow{r_0}}{\partial\eta} = \sum_{i\neq 0}(\Omega_j - \Omega_0)\frac{\mu_i}{\mu_0}\overrightarrow{r_i}$$

By combining together and discarding smaller terms with $\Omega_i\,\mu_i$ (with $i \neq 0$) or η

$$\frac{\partial\overrightarrow{a_j}}{\partial\eta} = \Omega_j\,\mu_0\,\frac{\overrightarrow{r_{j0}}}{r_{j0}^3} - \Omega_0\frac{\partial}{\partial\overrightarrow{r_0}}\left[\frac{\mu_0}{r_{j0}^3}\right]\sum_{i\neq 0}\mu_i\,\overrightarrow{r_i}$$

with a direct (small parameter $\Omega_j\,\mu_0$) and an indirect part (small parameter $\Omega_0\,\mu_j$).

3.5 Violations of the strong equivalence principle (continued)

Orders of magnitude: on Mercury, the main term is from the Jupiter-Sun-Mercury 3-body problem, with negligible contribution from Ω_1

$$\frac{\partial \overrightarrow{a_1}}{\partial \eta} = -\Omega_0 \, \mu_5 \, \frac{\partial}{\partial \overrightarrow{r_0}} \left[\frac{\overrightarrow{r_0}}{r_{10}^3} \right] \overrightarrow{r_5}$$

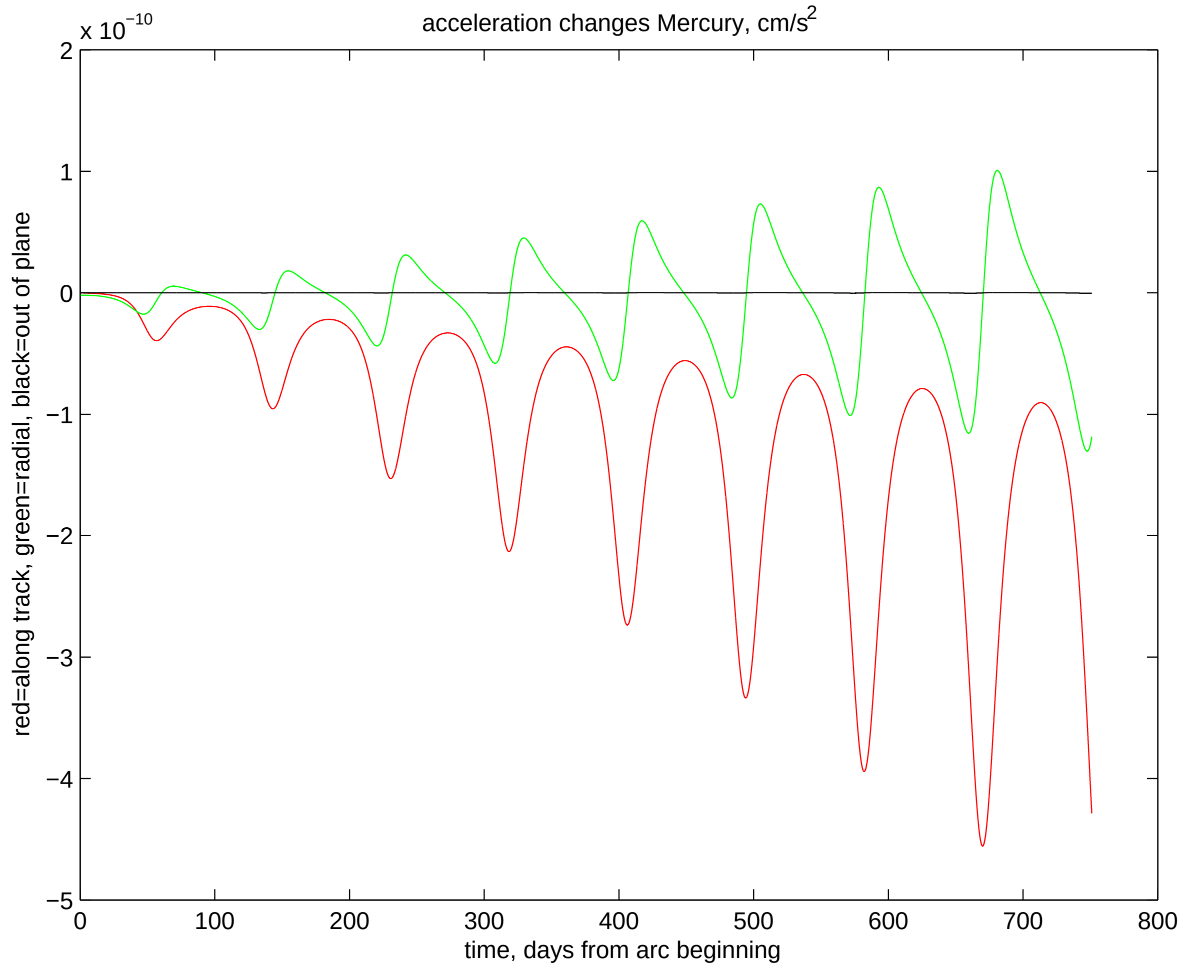
with small parameter $\Omega_0 \, \mu_5 / \mu_0 = -3.5 \times 10^{-9}$ and the vector size of the order of 13×4 cm/s^2.

On the Earth-Moon barycenter,

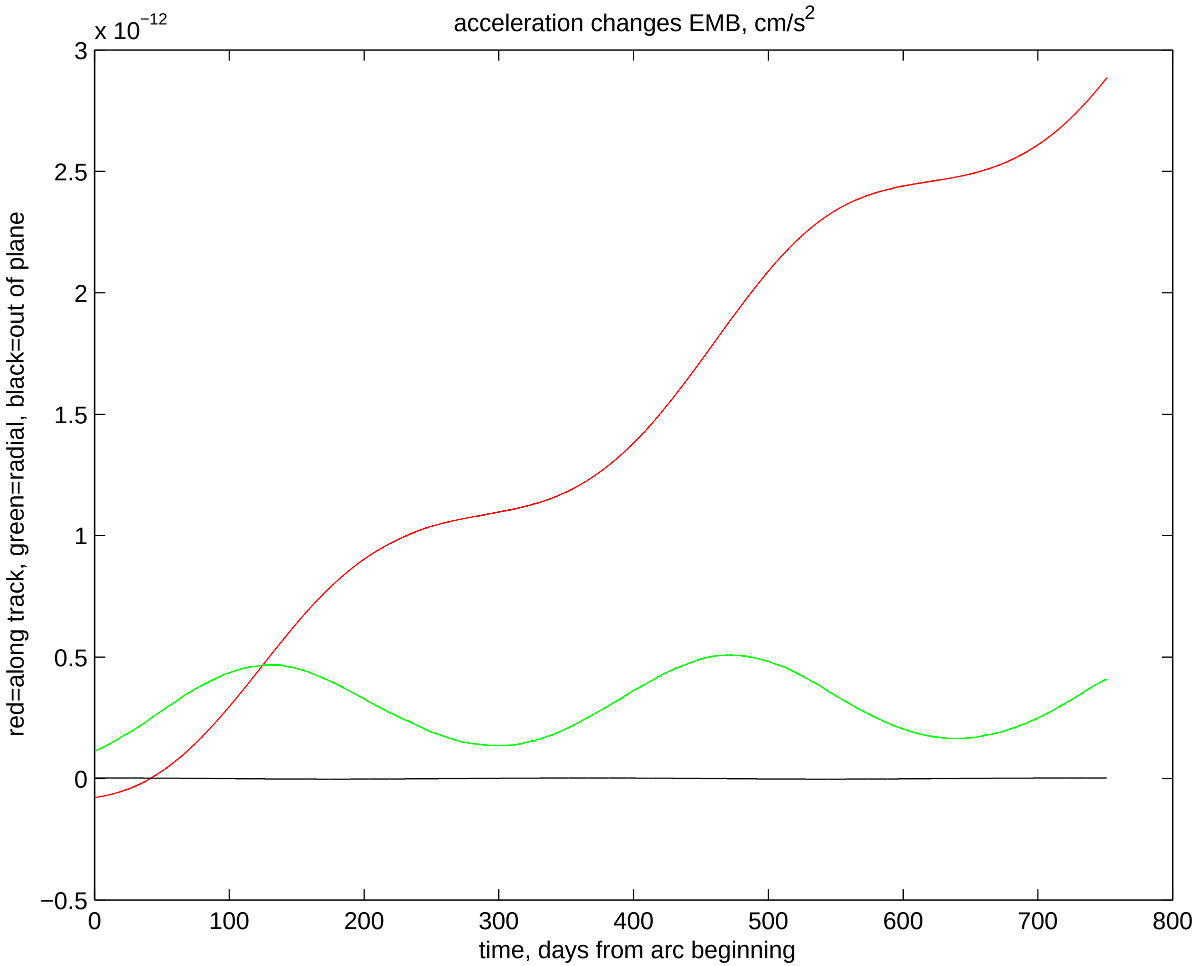
$$\frac{\partial \overrightarrow{a_3}}{\partial \eta} = \Omega_3 \, \mu_0 \, \frac{\overrightarrow{r_{30}}}{r_{30}^3} - \Omega_0 \, \mu_5 \, \frac{\partial}{\partial \overrightarrow{r_0}} \left[\frac{\overrightarrow{r_0}}{r_{30}^3} \right] \overrightarrow{r_5}$$

where the indirect term has small parameter $\Omega_0 \, \mu_5 / \mu_0$ and the vector size 5×0.6 cm/s. The direct term has small parameter $\Omega_3 \, \mu_0 / \mu_0 = \Omega_3 \simeq -5 \times 10^{-10}$ and vector size 0.6 cm/s^2.

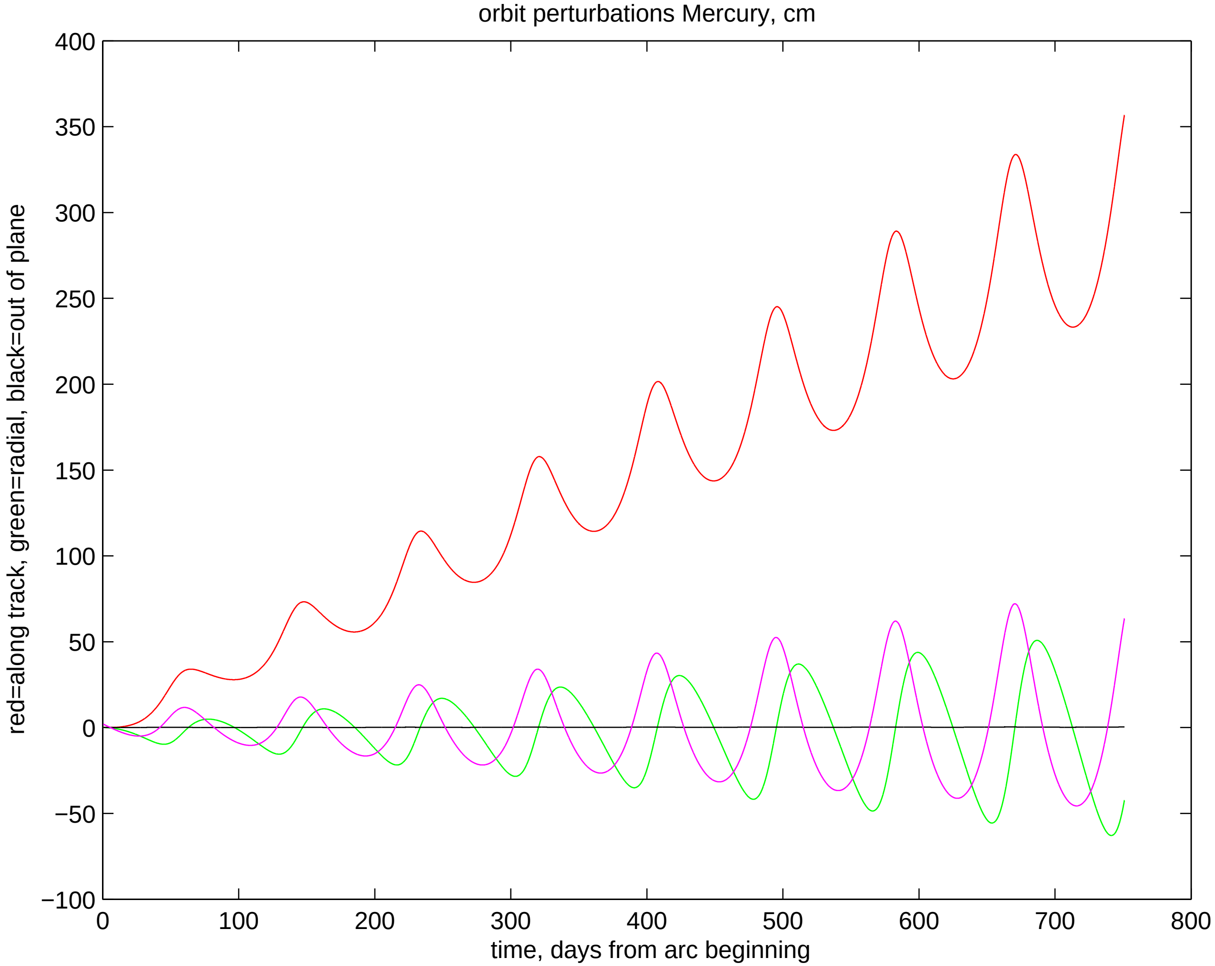
3.5 Violations of the strong equivalence principle
Indirect part, acceleration, Mercury



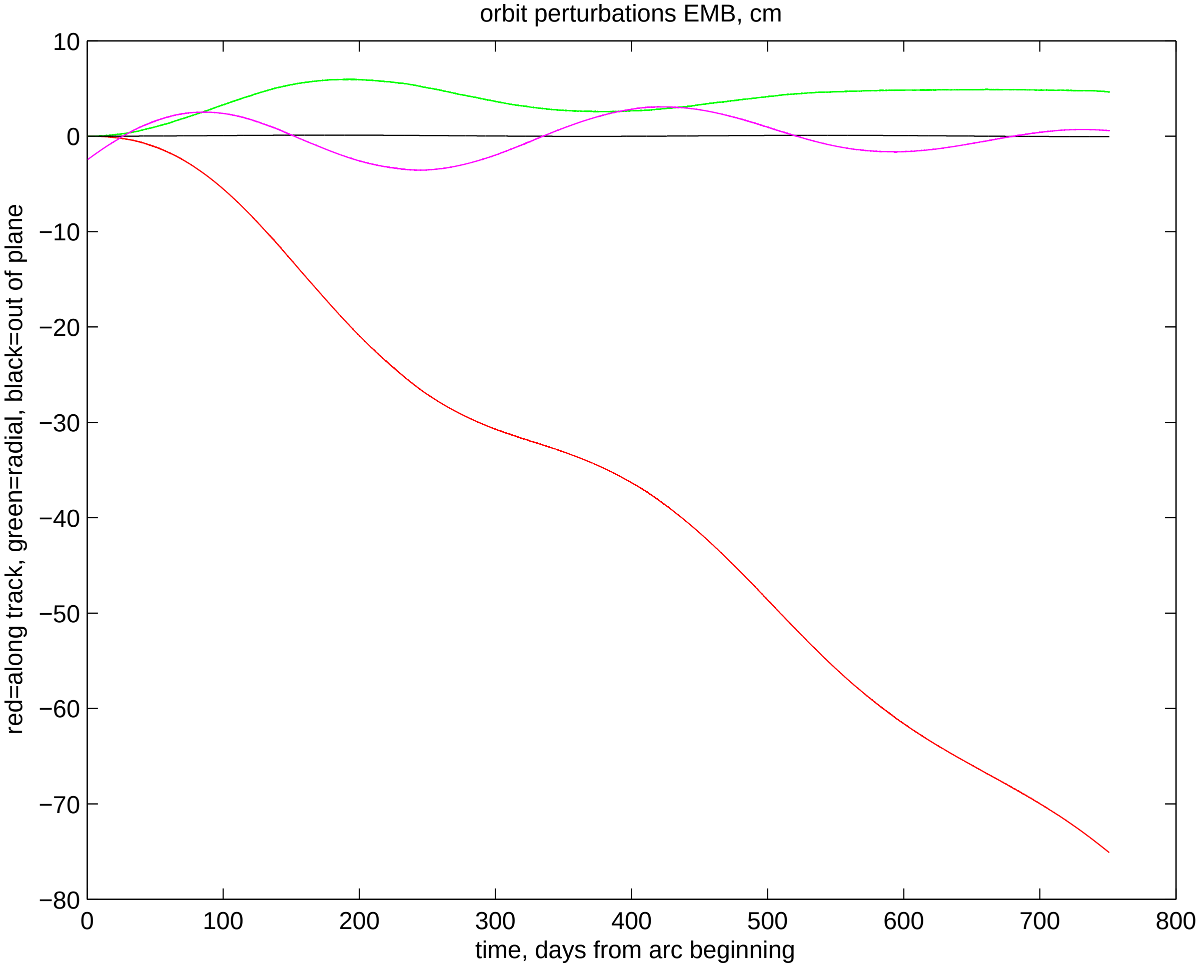
3.5 Violations of the strong equivalence principle
Indirect part, acceleration, Earth-Moon barycenter



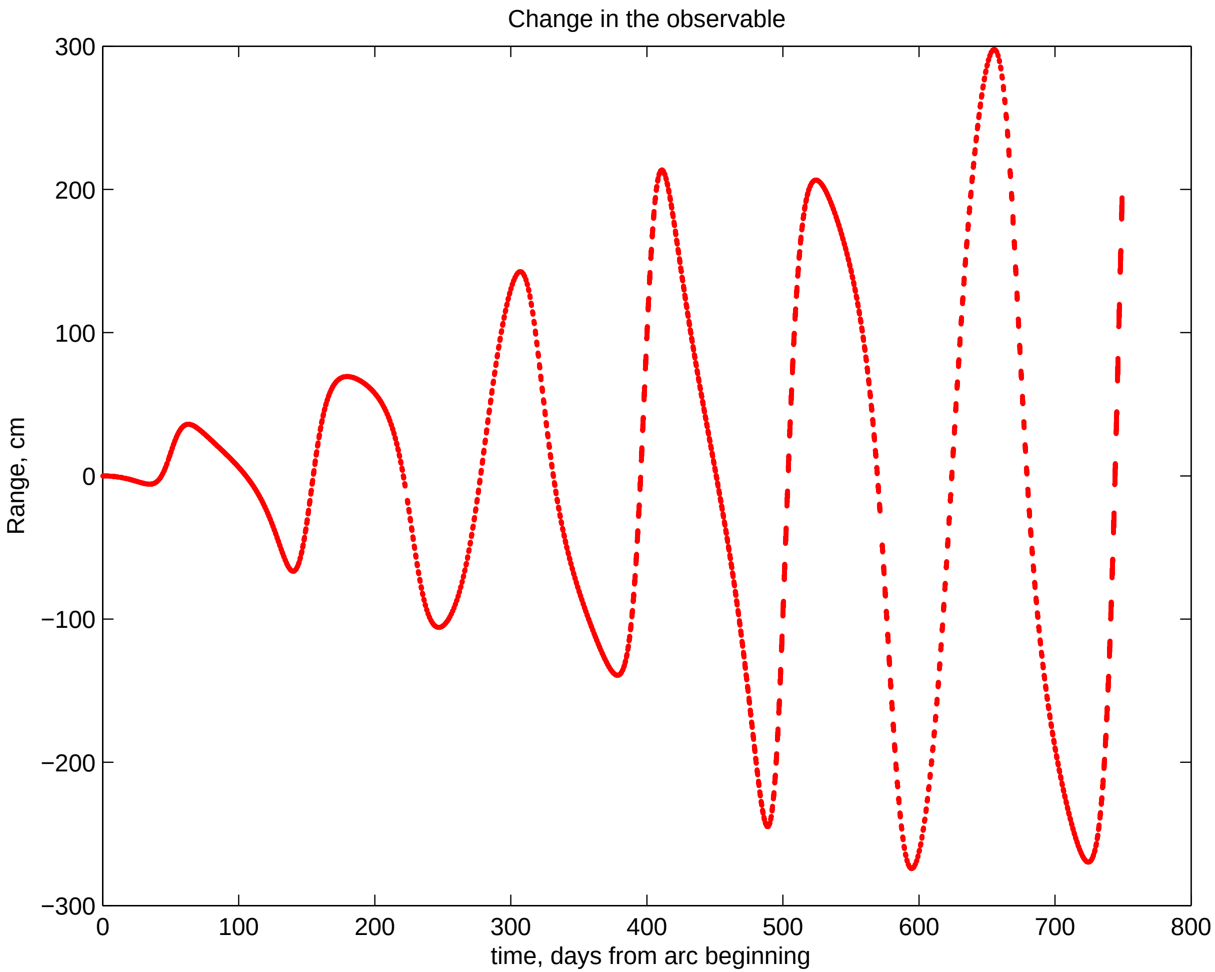
3.5 Violations of the strong equivalence principle
Indirect part, orbit, Mercury



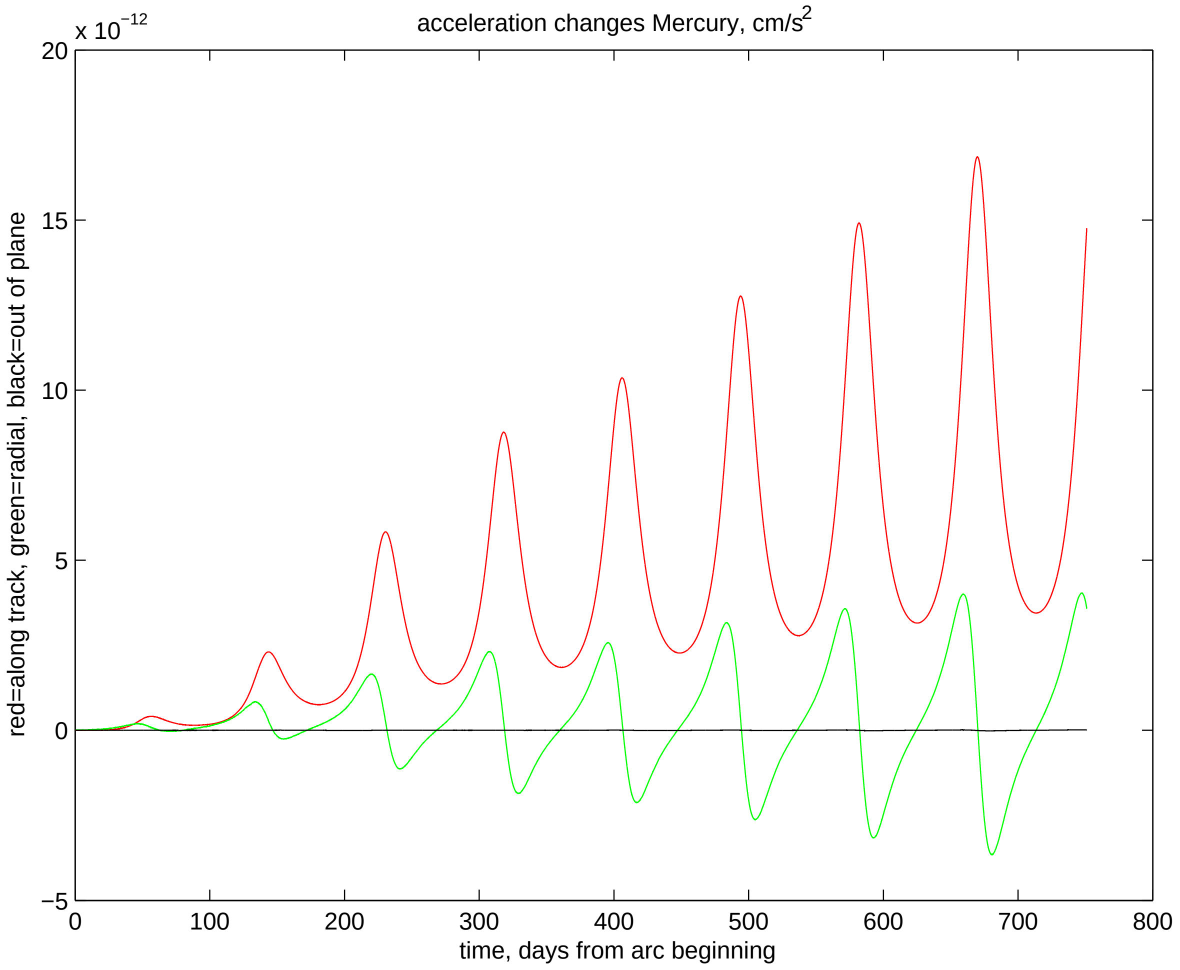
3.5 Violations of the strong equivalence principle
Indirect part, orbit, Earth-Moon barycenter



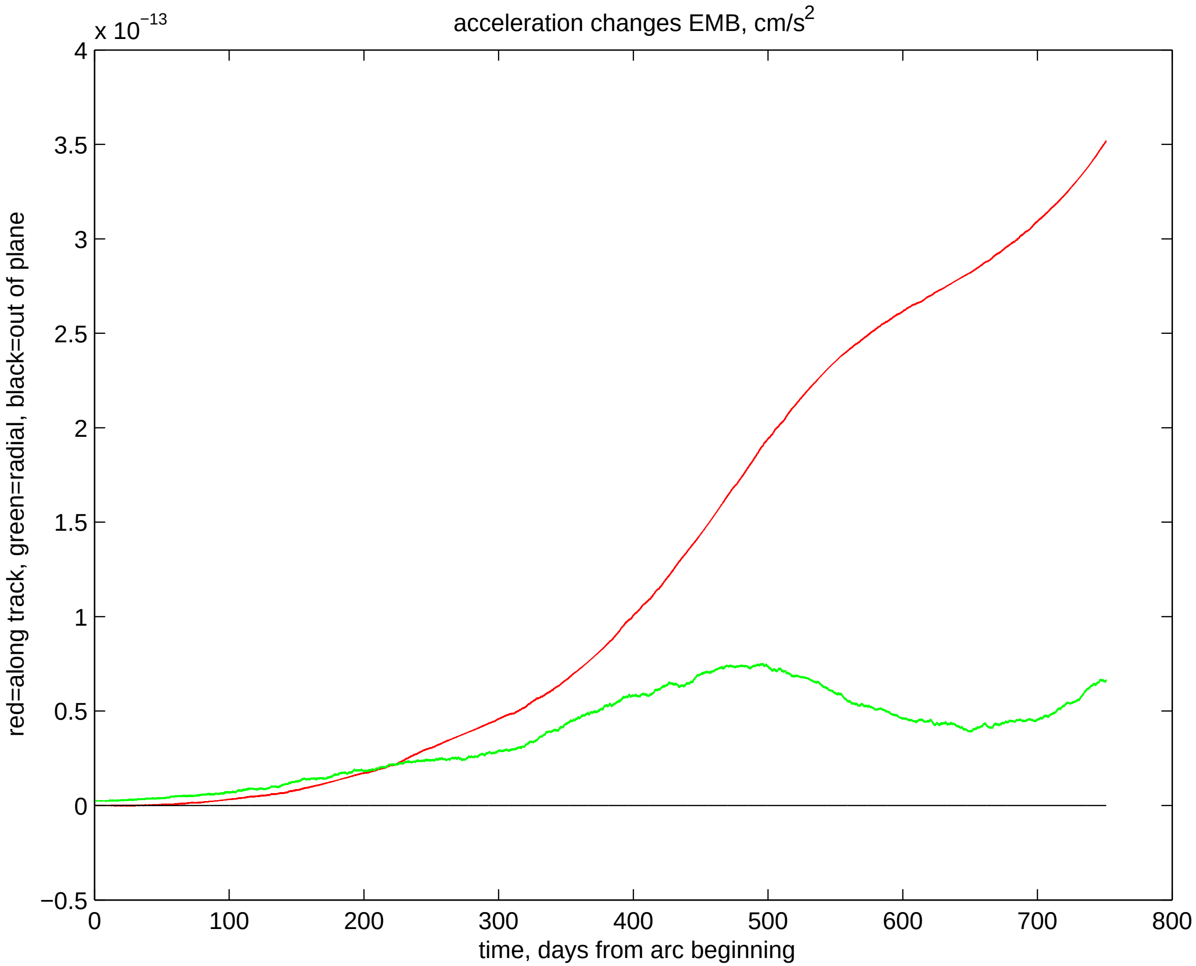
3.5 Violations of the strong equivalence principle
Indirect part, range signal



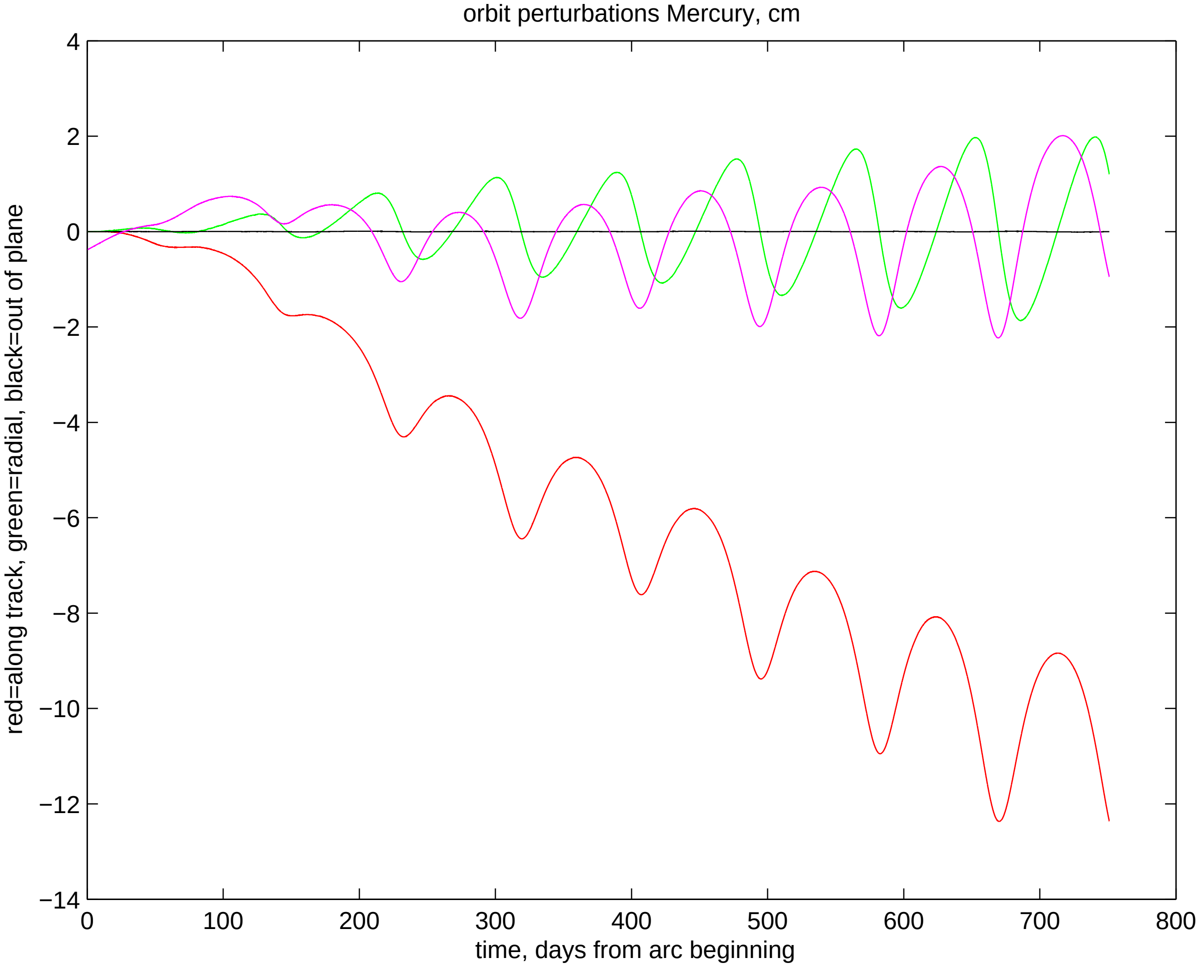
3.5 Violations of the strong equivalence principle
Direct part, acceleration, Mercury



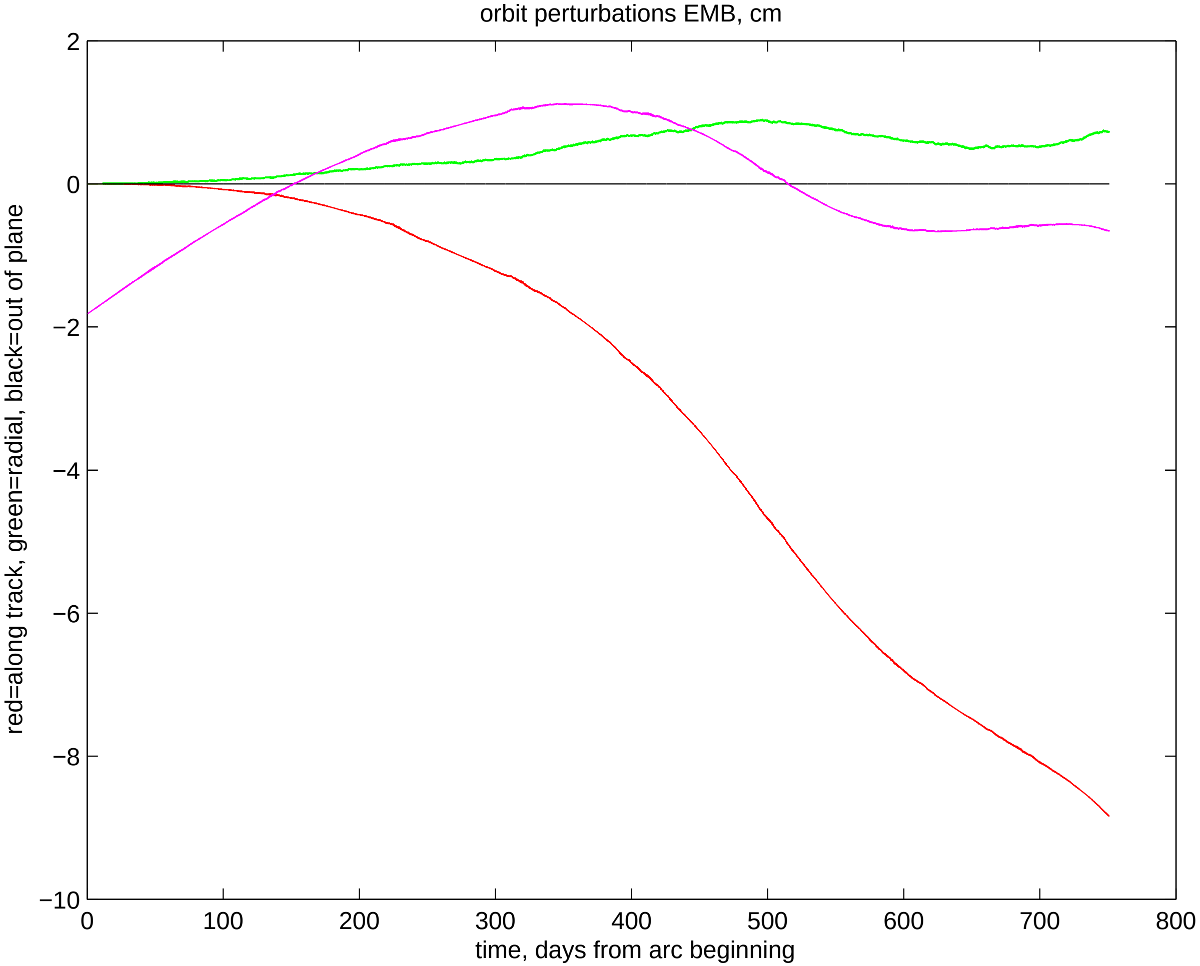
3.5 Violations of the strong equivalence principle
Direct part, acceleration, Earth-Moon barycenter



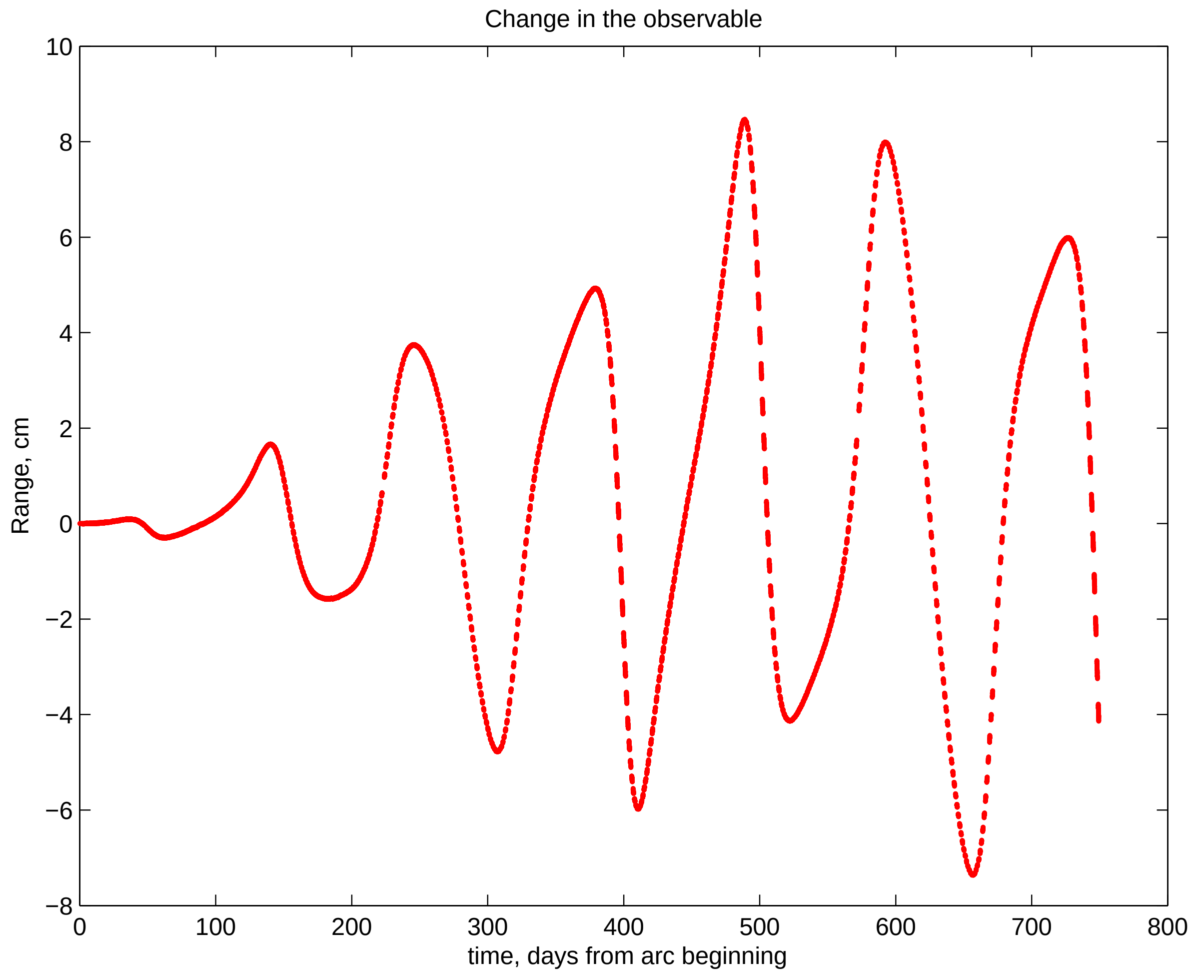
3.5 Violations of the strong equivalence principle
Direct part, orbit, Mercury



3.5 Violations of the strong equivalence principle
Direct part, orbit, Earth-Moon barycenter



3.5 Violations of the strong equivalence principle
Direct part, range signal



3.5 Violations of the strong equivalence principle- conclusions

That the order of magnitude of the signal is comparable to the accuracy of the measurements is a necessary, but not sufficient, condition for achieving a good result in the orbit determination. The figures we are showing are obtained by assuming the same initial conditions, while in the real orbit determination the initial conditions are adjusted, thus absorbing a good part of the signal. Still, if there is no signal even with the same initial conditions, the experiment must fail.

In conclusion, indirect perturbations give a much larger signal for a given η, thus before comparing results of the simulations we need to establish wether we agree on the physiscs, and on its implementation in software: that is, we need to compare these figures.

A new cycle of simulations, this time full scale and without shortcuts, will be possible in the next year or so.

4 Space-time reference systems

Not all implemented yet; some spatio-temporal coordinate changes written but not yet tested properly. We discuss what we have.

PLAN:

4.1 Dynamical Mercury Time

4.2 Masses, harmonic coefficients and tides

4.1 Dynamical Mercury Time

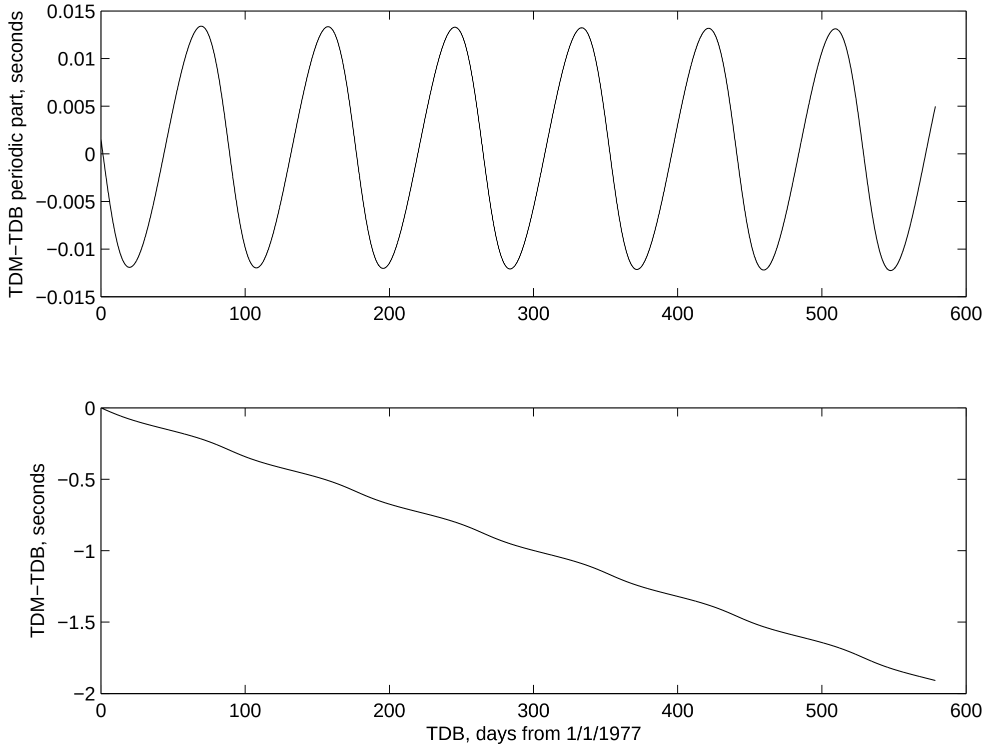
The relativistic equation of motion of a Mercury-centric satellite can be approximated to the required level of accuracy by a simpler equation of motion provided the independent variable of the equation is the proper time of Mercury. Thus, for the BepiColombo radioscience experiment, it is necessary to define a new time coordinate Dynamic Mercury Time (TDM) containing terms of post-Newtonian order 1 depending mostly upon the distance from the Sun r_{10} and velocity v_1 of Mercury. The relationship with the TDB scale, truncated to post-Newtonian order 1, is given by a differential equation

$$\frac{dt_{TDM}}{dt_{TDB}} = 1 - \frac{v_1^2}{2\,c^2} - \sum_{k\neq 1} \frac{\mu_k}{c^2\, r_{1k}}$$

which can be solved by a quadrature formula once the orbits of Mercury, the Sun and the other planets are known.

Caution must be used also in the space portion of the spacetime coordinate change, used to convert the spacecraft Mercury-centric orbit to solar system barycentric frame.

4.1 Dynamical Mercury Time (continued)



TDM as function of TDB shows a drift due to the non-zero average of the PN1 term; it could be removed by a change of scale of the dynamic time and of the mass of Mercury (defining a Mercury Time, TM). The periodic term, with the period of Mercury's orbit, is almost an order of magnitude larger than TT-TDB. The time derivative of the periodic correction is $\simeq 10^{-8}$, in computing the range-rate it is multiplied by $v_1 \simeq 50$ km/s, changing range-rate by up to 0.05 cm/s, $\simeq 30$ times larger than the accuracy of range-rate with an integration time of 30 s.

4.2 Mass, harmonic coefficients and tides

Cause	Formula	Parameters	Value cm/s^2
Mercury monopole	$GM_{☿}/r^2 = F_0$	$GM_{☿}$	$2.4 \cdot 10^2$
Mercury oblateness	$3\,F_0\,\overline{C}_{20}\,R_{☿}^2/r^2$	$\overline{C}_{20}$	$1.3 \cdot 10^{-2}$
Mercury triaxiality	$3\,F_0\,\overline{C}_{22}\,R_{☿}^2/r^2$	$\overline{C}_{22}$	$7.8 \cdot 10^{-3}$
Radiation pressure	$C_R\,F_{PR}$	C_R	$6.8 \cdot 10^{-5}$
Thermal emission	$4/9\,F_{PR}\,\alpha_{☿}\,\Delta T/T$	$\alpha_{☿}, \Delta T$	$3 \cdot 10^{-5}$
Sun tide	$2\,GM_{\odot}\,r/r_{\odot}^3$	$GM_{\odot}$	$2.3 \cdot 10^{-5}$
Effect of ε_1	$(9/2)\,\varepsilon_1\,F_0\,\overline{C}_{20}\,R_{☿}^2/r^2$	$\varepsilon_1\,\overline{C}_{20}$	$1.9 \cdot 10^{-5}$
Effect of ε_2	$(9/2)\,\varepsilon_2\,F_0\,\overline{C}_{22}\,R_{☿}^2/r^2$	$\varepsilon_2\,\overline{C}_{22}$	$3.3 \cdot 10^{-6}$
Solid tide	$3k_2\,GM_{\odot}\,R_{☿}^5/r_{\odot}^3\,r^4$	k_2	$2.8 \cdot 10^{-6}$
Mercury albedo	$C_R\,F_{PR}\,(1-\alpha_{☿})\,R_{☿}^2/(2\,r^2)$	$\alpha_{☿}, C_R$	$2.7 \cdot 10^{-6}$
Venus tide	$2\,GM_{♀}\,r/r_{☿♀}^3$	$GM_{♀}$	$4 \cdot 10^{-8}$
Relativistic Mercury	$F_0\,GM_{☿}/(c^2\,r)$	$GM_{☿}$	$1.9 \cdot 10^{-8}$

Accelerations acting on a spacecraft in orbit around Mercury, in a planetocentric reference frame, with $a = 3\,000$ km, $A/M = 0.05$ cm^2/g.

4.2 Mass, harmonic coefficients and tides (continued)

In writing the code, we have assumed so far that the Mercury-centric equations of motion are simply the Newtonian ones, provided the independent variable is TDM. This need to be certified by the experts on relativity, taking into account the requirements: that is, we are not interested in corrective terms which are far below the level of accuracy in the measurement of the non-gravitational accelerations acting on the orbiter.

As an example, the main relativistic term containing the mass of Mercury is negligible, as shown by the previous table. This does not prove that there are no relativistic corrections in the equation of motion, besides the large ones which are hidden in the change of time coordinate (which is equivalent to using a time-variable mass of Mercury).

Thus we need an estimate of the orders of magnitude of the accelerations introduced by a fully relativistic formulation of spherical harmonics, tides, etc. We also need confirmation that, once the independent variable is TDM, the mass of Mercury to be used in the equations of motion is constant to a good approximation (say 1 part in 10^9).

5 Restricted ephemerides improvement

PLAN:

5.1 The angular momentum integrals

5.2 Equations of motion for Earth-Moon Barycenter

5.3 Asteroids and the ephemerides comparison problem

5 Restricted ephemerides improvement

The level of accuracy of the measurements of MORE is incompatible with the use of the current planetary ephemerides, which have been solved by using lower accuracy measurmeents. It is less obvious, but true, that the data from MORE do not allow to to improve globally the planetary ephemerides.

Of the 5 vectors used in light-time equation, $\vec{x}_{ant}$ and $\vec{x}_E$ can be assumed known: their current knowledge cannot be improved by ranging to a Mercury orbiter. For the orbit of the Moon it is more effective to measure the range to the Moon, as it is done with **lunar laser ranging**. Both **navigation satellites** and **VLBI** give by far more information on the antenna position and on the rotation of the Earth.

The position of the Earth-Moon CoM and the position of Mercury can certainly be improved by the range measurements of MORE (the range-rate is less effective, because it is more accurate than range only over time scales $\leq 30,000$ s).

The question is about the other planets. The Mercury-centric orbit of the spacecraft is very weakly sensitive to the other planets. The orbit of the Earth-Moon CoM and of Mercury might be enough sensitive to contribute information on where Venus and Jupiter are; probably the main problem is confusion with the asteroid signal. For now we have chosen to keep all the planets and satellites fixed at the ephemerides position and solve only for Mercury and Earth-Moon CoM.

5.1 The angular momentum integrals

As already discussed, the MORE observable range and range-rate are invariant with respect to rotation of the orbits of both the Earth and Mercury (including the orbit of the spacecraft). If the positions of the other planets and of the Moon are read from the ephemerides, this symmetry is broken (also because of $J_{2\odot}$).

However, the orbits of Mercury and E-M barycenter are only weakly coupled to those of the other bodies, see the Roy-Walker small parameters: the short periodic perturbation of Jupiter on the orbit of the E-M barycenter is a fraction $\simeq 7 \times 10^{-6}$ of the monopole potential of the Sun.

Thus the normal system has an approximate rank deficiency which would not prevent the convergence of the solution but would decrease the accuracy, if not cured. It is possible to fix three coordinates, but also to add a 3 priori observations preventing all rotations of the initial conditions.

Another problem is the approximate symmetry from change of scale. It might be necessary to constrain also the equal scaling of the initial conditions, by adding an appropriate penalty and the corresponding a priori observation.

This part of the code has not been written yet, but we do not anticipate problems.

5.2 Equations of motion for Earth-Moon Barycenter

The equations of motion given so far are for all planets (and some satellites, e.g. the Moon). How can we compute the orbit of the Earth-Moon Barycenter $\overrightarrow{r_{\oplus☾}}$?

Newtonian formula: given the mass ratio of the E-M system $\mu = \mu_\oplus / \mu_☾ \simeq 81$

$$\overrightarrow{r_{\oplus☾}} = \frac{\mu}{1+\mu}\overrightarrow{r_\oplus} + \frac{1}{1+\mu}\overrightarrow{r_☾} \;\; , \;\; \overrightarrow{a_{\oplus☾}} = \frac{\mu}{1+\mu}\overrightarrow{a_\oplus} + \frac{1}{1+\mu}\overrightarrow{a_☾} \; .$$

In a relativistic framework, the accelerations $\overrightarrow{a_\oplus}, \overrightarrow{a_☾}$ contain the relativistic terms from $L_{GR0} + \beta L_\beta + \gamma L_\gamma + \ldots$, but the center of mass of the E-M system is given by a different mass ratio:

$$\mu_{GR} = (\mu_\oplus / \mu_☾) \left[1 + \frac{v_\oplus^2}{2c^2} - \frac{U_\oplus}{2c^2}\right] \cdot \left[1 + \frac{v_☾^2}{2c^2} - \frac{U_☾}{2c^2}\right]^{-1} \; .$$

The factors between square brackets in the relativistic mass ratio are close to 1 because the values of the velocities are $v_☾ / c \simeq v_\oplus / c \simeq 10^{-4}$ and of the potential are $U_☾ / c^2 \simeq U_\oplus / c^2 \simeq 10^{-8}$. Thus the two relativistic correction to the barycenter are both of the order of $10^{-8} \cdot 5,000$ km $= 5$ cm in position and $10^{-8} \cdot 0.5$ km/s $= 5 \times 10^{-3}$ cm/s.

5.2 Equations of motion for Earth-Moon Barycenter (continued)

Moreover, the relative velocity of the the Moon with respect to the E-M barycenter is only $\simeq 0.5$ km/s, the one of the Earth is just $\simeq 6$ m/s; the E-M distance is just $\simeq 1/400$ AU. Thus, the relativistic correction cancel each other! An order of magnitude estimate

$$|\mu_{GR}/\mu - 1| \leq 2\times 10^{-10} \Longrightarrow |\Delta\overrightarrow{r_{\oplus ☾}}| \leq 0.1\ \mathrm{cm}\ ,\ |\Delta\overrightarrow{v_{\oplus ☾}}| \leq 10^{-8}\ \mathrm{cm/s}$$

In conclusion, the acceleration $\overrightarrow{a_{\oplus ☾}}$ does not need to be corrected for the relativistic E-M barycenter.

Indirect oblation: the mutual attractions of Earth and Moon cancel out in $\overrightarrow{a_{\oplus ☾}}$, although they are not along the direction of $\overrightarrow{r_{\oplus ☾}}$.

5.3 Asteroids and the ephemerides comparison problem

One of the main problems in the validation of our software, and more generally in whatever comparison of ephemerides, is due to the open-ended complexity of the solar system dynamical structure. The solar system is an ∞-body problem: asteroids could be added in an arbitrary number (to a lesser extent, natural satellites).

The JPL ephemerides have been computed by including in the dynamical model two sets of asteroids. The first set of N_1 asteroids have each individual mass solved in the solar system orbit fit. The second set of N_2 asteroids have their mass collectively estimated by assuming the volume is known, from a radius (extracted from unspecified sources, probably IRAF) and three different densities estimated in the fit for the three spectral superclasses C, S and M.

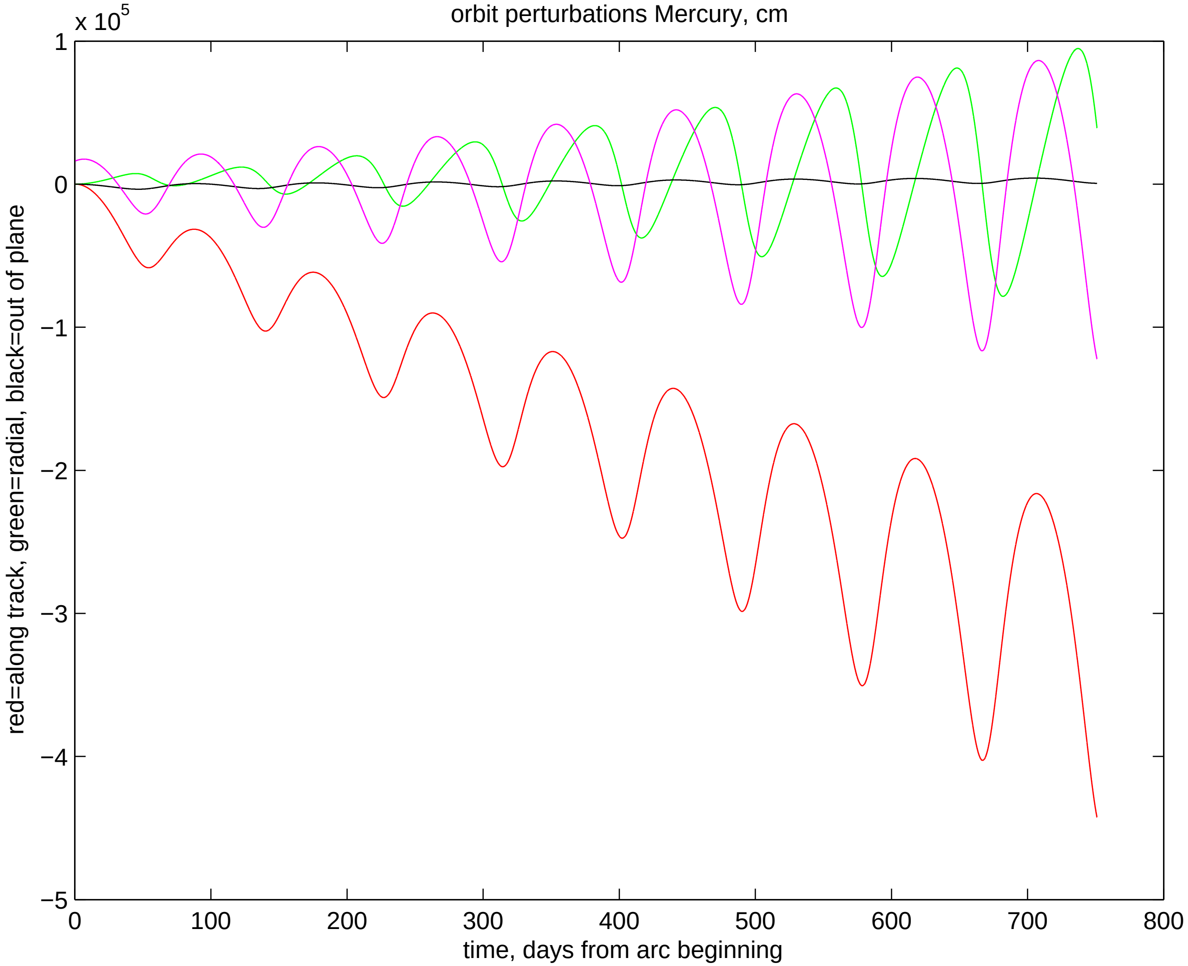
The JPL ephemerides do not contain Trans-Neptunian Objects, apart from Pluto, which is not the most massive of these.

As an example, the recent DE421 JPL ephemerides contain $N_1 = 67$ asteroids with individually estimated masses and $N_2 = 276$ asteroids with one of the three estimated densities, which are (in g/cm^3)

$$\rho_C = 1.093 \ , \ \ \rho_S = 3.452 \ , \ \ \rho_M = 4.221 \ .$$

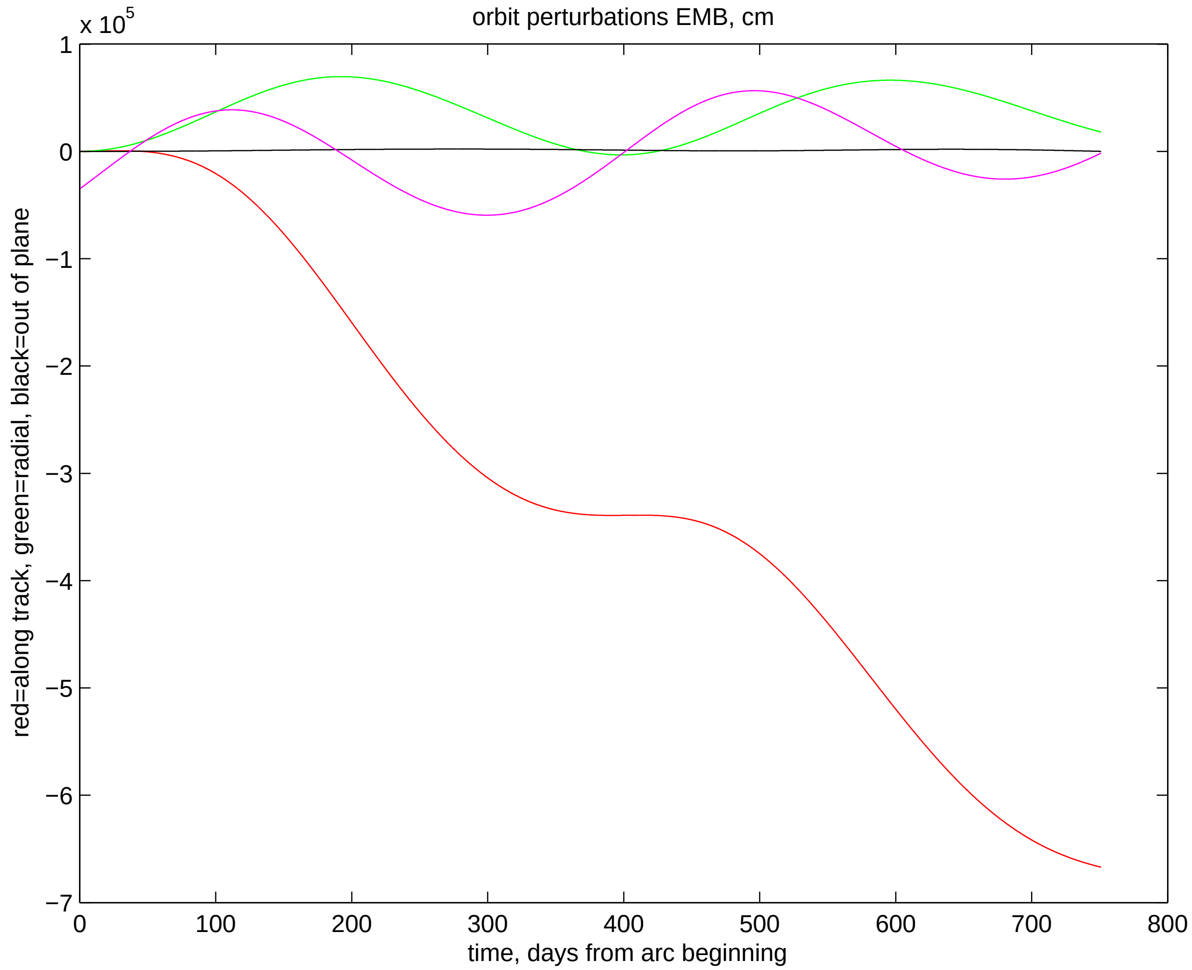
These values look a bit suspicious to experts of asteroid composition and porosity, but a fit is a fit. Also some individually estimated masses are weird.

5.3 Asteroids and the ephemerides comparison problem
67 asteroids perturbing Mercury



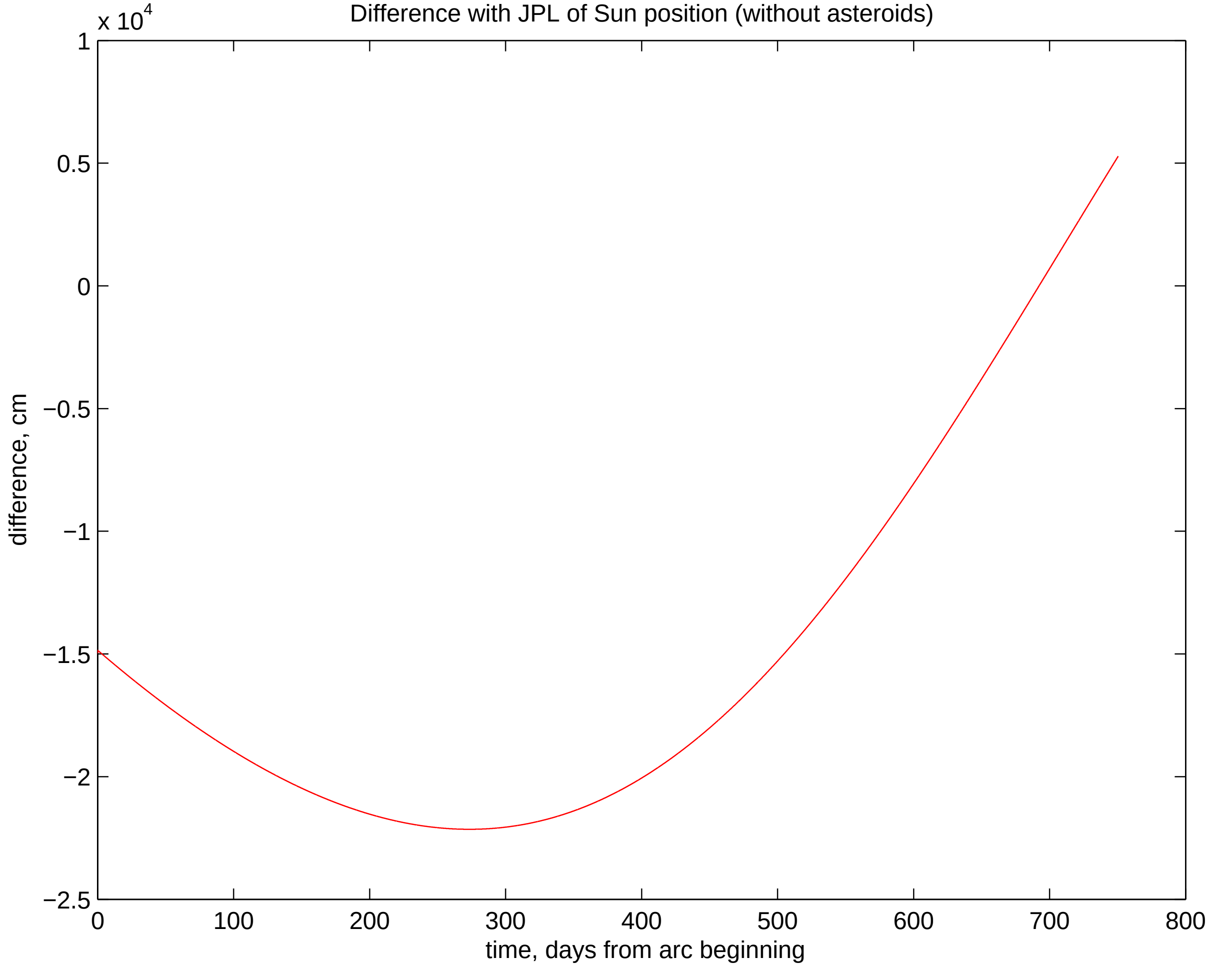
Perturbations on the orbit of Mercury due to the 67 asteroids of DE421 with a 2 years extended mission: Red: transversal, Green: radial, Black: out of plane, Purple: transveral with linear and quadratic trends removed.

5.3 Asteroids and the ephemerides comparison problem
67 asteroids perturbing EMB



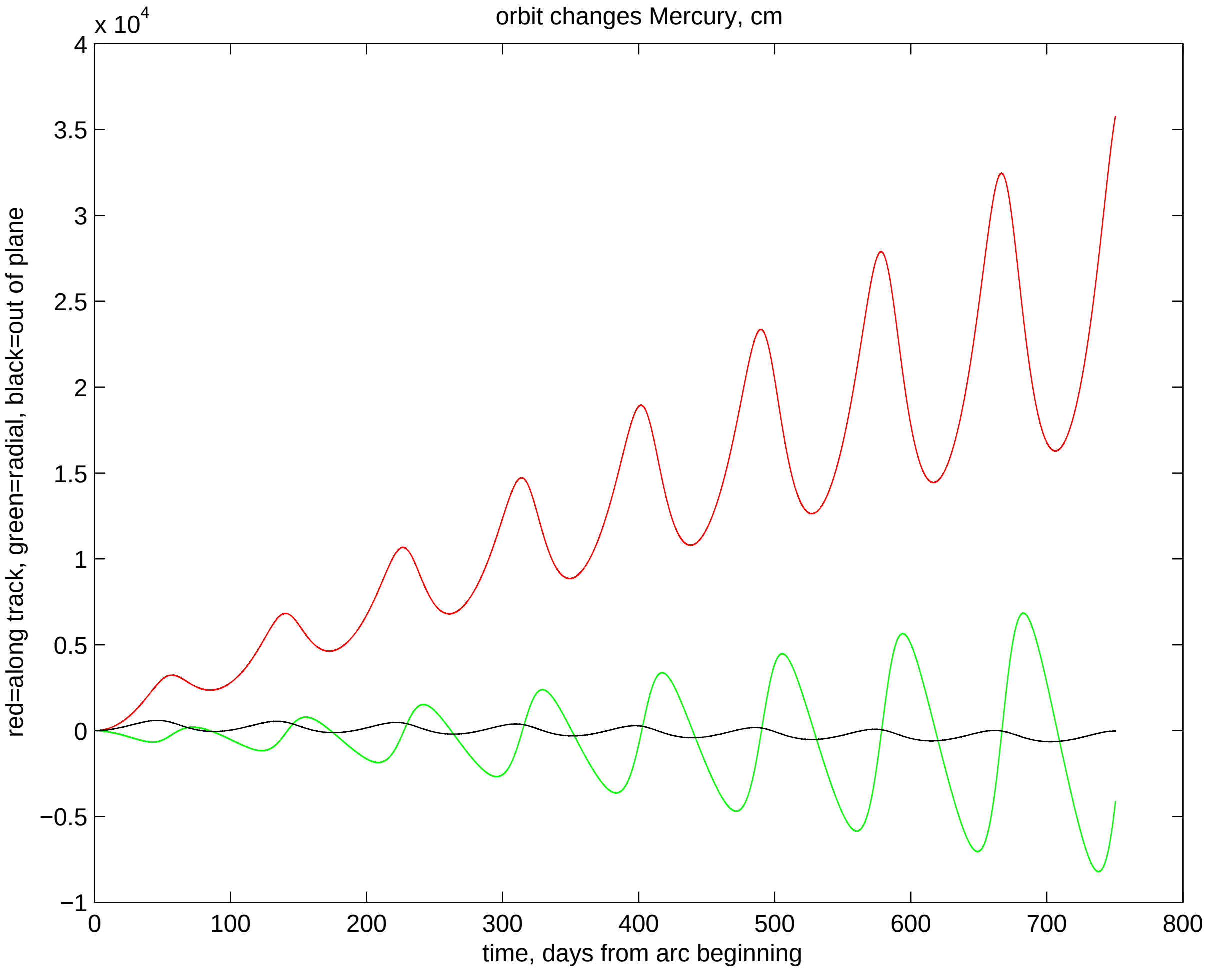
Perturbations on the orbit of Earth-Moon barycenter due to the 67 asteroids of DE421 with a 2 years extended mission: Red: transversal, Green: radial, Black: out of plane, Purple: transveral with linear and quadratic trends removed.

5.3 Asteroids and the ephemerides comparison problem difference with DE421 for the Sun



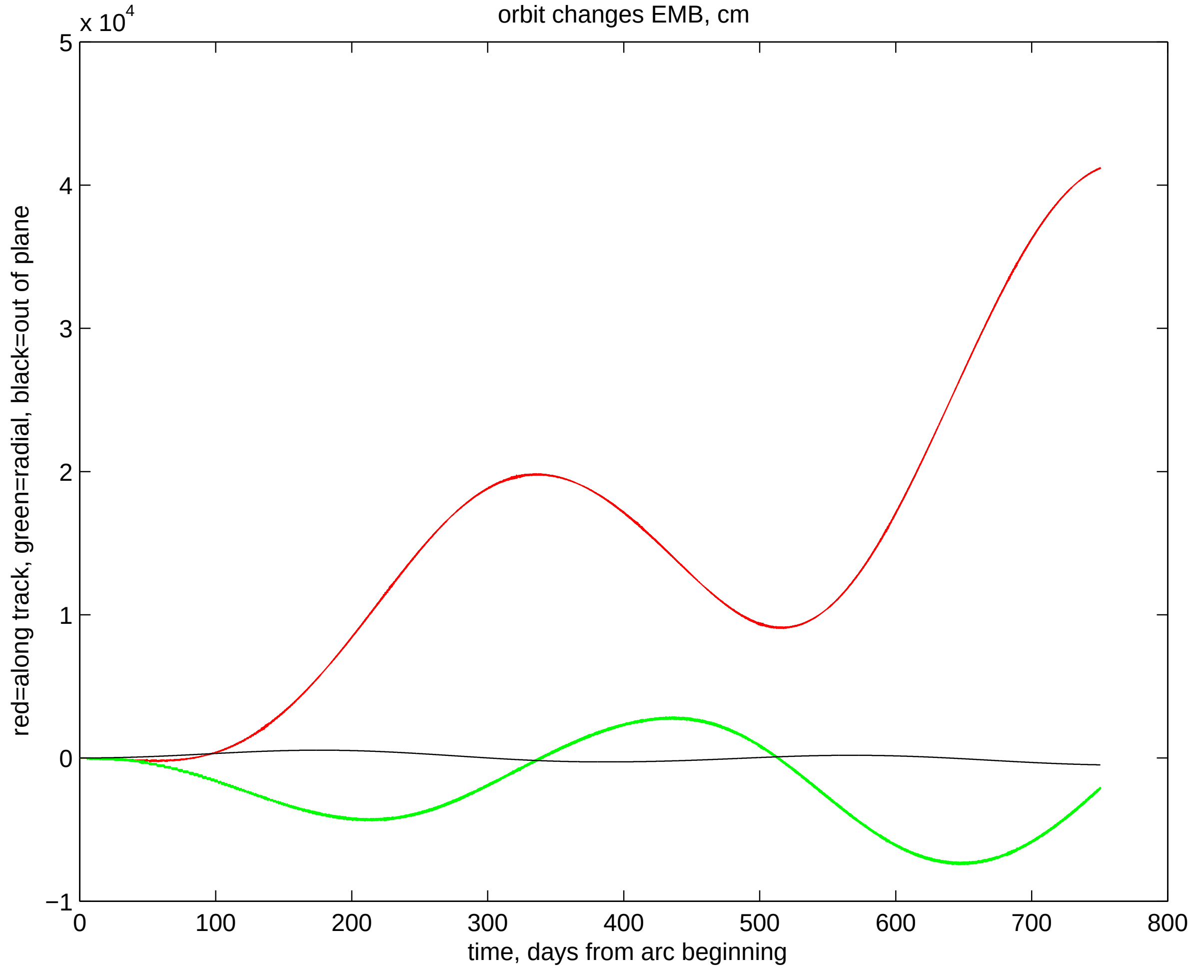
Difference in the motion of the Sun between our propagation without asteroids and DE421 with a 2 years extended mission: Red: distance. Note the size is comparable to the relativistic change in the barycenter.

5.3 Asteroids and the ephemerides comparison problem 67 asteroids, Mercury difference with DE421



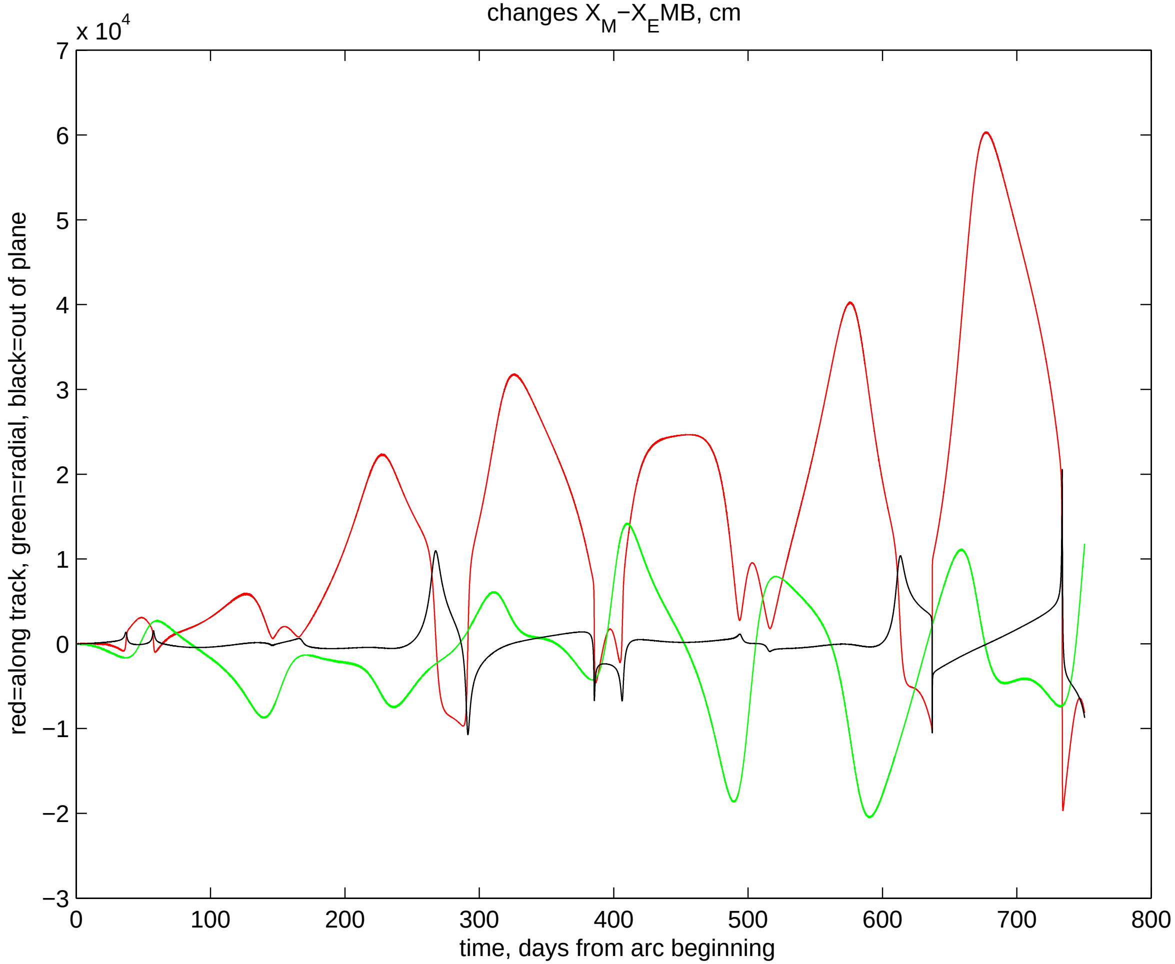
Difference between our model with the 67 asteroids of DE421 and the DE421 ephemerides, for Mercury, 2 years extended mission: Red: transversal, Green: radial, Black: out of plane, Purple: transveral with trends removed.

5.3 Asteroids and the ephemerides comparison problem
67 asteroids, EMB difference with DE421



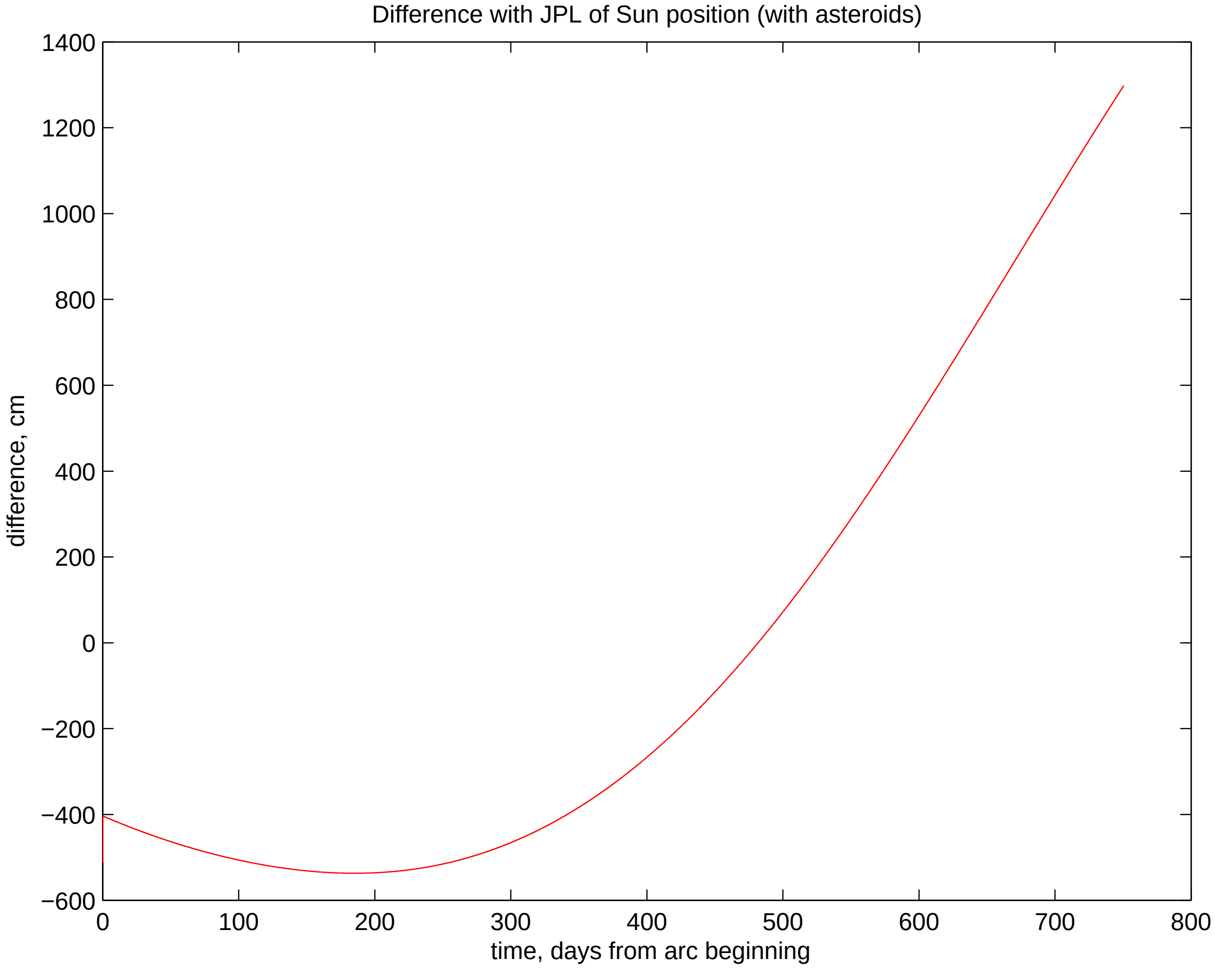
Difference between our model with the 67 asteroids of DE421 and the DE421 ephemerides, for relative position of EMB, 2 years extended mission: Red: transversal, Green: radial, Black: out of plane, Purple: transveral with trends removed.

5.3 Asteroids and the ephemerides comparison problem
67 asteroids, M-EMB difference with DE421



Difference between our model with the 67 asteroids of DE421 and the DE421 ephemerides, for relative position of Mercury and EMB: Red: transversal, Green: radial, Black: out of plane.

5.3 Asteroids and the ephemerides comparison problem
67 asteroids perturbing the Sun



Difference of the motion of the Sun with DE421, after implemeting the 67 asteroids of DE421: Red: distance.

5.3 Asteroids and the ephemerides comparison problem- Conclusions?

We have implemented the $N_1 = 67$ asteroids of DE421 in our dynamical model. The discrepancy with DE421 decreases by more than an order of magnitude with respect to the case without asteroids. However, it is still 3 to 4 orders of magnitude larger than the MORE accuracy requirements.

What should we do next?

1. Implement the other $N_2 = 276$ asteroids of DE421?
2. Implement the asteroid model of DE405, for which the mass and volume data are not completely available to us?
3. Implement TNO with known mass, changing the barycenter by hundreds of km?
4. Investigate other simplifications contained in the JPL ephemerides model?
5. Investigate different implementations of GR?
6. Ignore these differences because they will not matter in the final fit?
7. Write the new software interface needed to use IMCCE ephemerides? Which ones? With how many asteroids?

6 Documentation System

Our documentation system:

- One source file for code and documentation
- Mathematical specifications in LaTeX, in the source file; also test results, with graphics
- Hypertext structure for module dependencies and subroutine calls
- Printable documentation automatically updated with software changes and generated in real time
- Web-based documentation automatically updated available at `http://adams.dm.unipi.it/~mercury/private/` (protected with password)

Technology:

- f95totex, Pdflatex, Latex2html
- Makefile-controlled generation of printable documentation for each separate source file
- Perl scripts to add hyperlinks and to generate book structure
- A single command generating the full documentation (options for public/private documentation)

6.1 The certification procedure

One key difference between abstract, mathematical Celestial Mechanics and a version of this discipline suitable to obtain reliable results with real data is the following.

Formal perturbation theory always use a truncation to some order in the small parameters, such as in 1PM, 1.5PN, 2PN. The theories tend to be complete to some order, and totally missing higher order terms.

Prcatically applicable theories need to be truncated to some order of magnitude. Two terms of the same formal order, but with coefficients of a different order of magnitude, do not have the same importance.

Example: in computing some relativistic coordinate change, there are 1PN terms with U/c^2, but the potential U contains a term with the mass of the Sun (order of magnitude at the Earth: 10^{-8}) and one with the mass of Mercury, $\simeq 1.5 \times 10^{-7}$ times smaller. Thus there are 1PN terms which are negligible and must be removed from the software, to avoid a terrible slowdown of the computations, with no advantage in precision.

The opposite example is in the 2PN terms, which may be relevant under special circumstances, such as the Shapiro effect in a Superior Conjunction Experiment (see Tommei's presentation). Some 1.5PN terms may also be important.

6.1 The certification procedure (continued)

Thus, the certification procedure we are seeking requires the contribution from all the team to check the following:

1. Are we using the good physics, that is the appropriate assumptions, e.g., on what is previously known? E.g.: are the known masses gravitational masses?

2. Are we using the right mathematical formulae to express the physics? E.g., the Lagrange functions.

3. Are the equations we use properly derived? E.g., the equations of motion, the variational equations, the time and coordinates changes?

4. Are the formulae, as given in the mathematical specifications, correctly implemeted in software?

5. Are we using an adequate approximation, that is, are the effects of all the neglected terms below the measurement sensitivity? This requires to compute an order of magnitude estimate for whatever we neglect.

6. Are we neglecting what must be neglected, to avoid computational inefficiency and unnecessary rounding off?

6.1 The certification procedure (continued)

How we really would like the certification to be obtained and recoded?

A certification declaration is a statement which is written inside our single source file, containing the name(s) of the checkers and the date, possibly some limitation, e.g., *checked equations but not the software*.

We can make our omnicomprehensive documentation available online to the team (with/without the sorce code?), and receive the declaration we can include. But all this procedure needs to be formalized somehow.