The MORE observables

MORE Relativity Working Group

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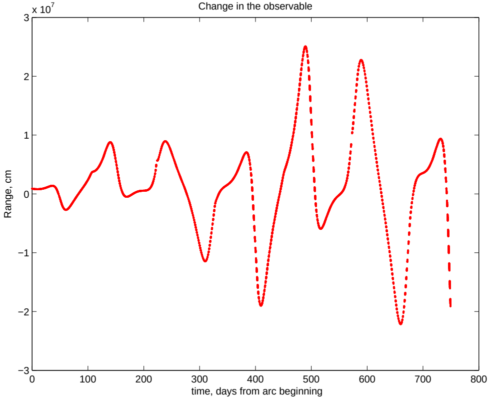
Definition

The observables of the MORE experiment are the distance r between the ground antenna and the spacecraft, and its time derivative $\dot{r}$. The range is computed using 5 state vectors:

$$r = |(\overrightarrow{x}_{sat} + \overrightarrow{x}_M) - (\overrightarrow{x}_{EM} + \overrightarrow{x}_E + \overrightarrow{x}_{ant})| + S(\gamma) \tag{1}$$

- $\overrightarrow{x}_{sat}$: mercurycentric position of the satellite
- $\overrightarrow{x}_M$: barycentric position of the CoM of Mercury
- $\overrightarrow{x}_{EM}$: barycentric position of the Earth-Moon CoM
- $\overrightarrow{x}_E$: vector from the Earth-Moon CoM to the CoM of the Earth
- $\overrightarrow{x}_{ant}$: position of the ground antenna center of phase with respect to the CoM of the Earth
- $S(\gamma)$: Shapiro effect.

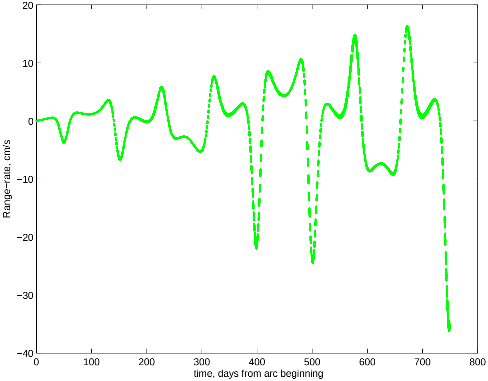
Range: fully Relativistic model vs Newtonian model



Differences in range using a fully Relativistic and a Newtonian model. The total Δr is 4×10^7 cm and

$\frac{\sigma r}{\Delta r} = \frac{10}{4 \times 10^7} = 2.5 \times 10^{-7}$

Range rate: fully Relativistic model vs Newtonian model



Differences in range rate using a fully Relativistic and a Newtonian model. The total $\Delta\dot{r}$ is 50 cm/s and

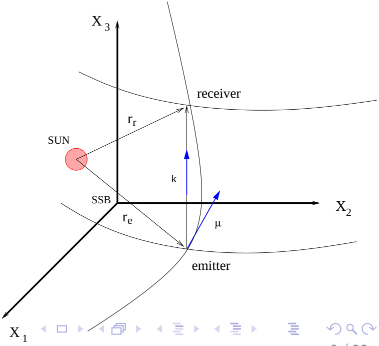
$\frac{\sigma_{\dot{r}}}{\Delta\dot{r}} = \frac{3\times10^{-4}}{50} = 6\times10^{-6}$

Shapiro effect: 1-PN level

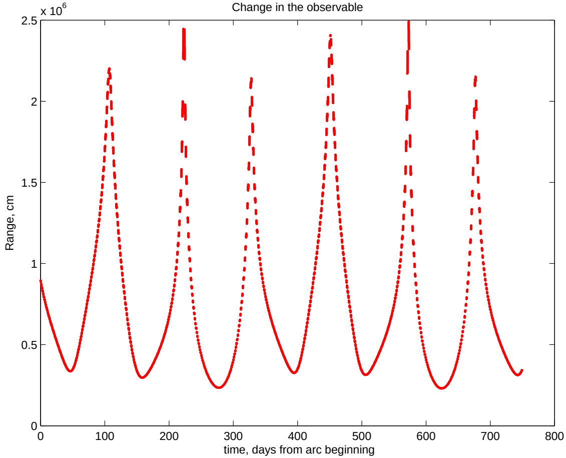
(References: Moyer 2003)

$$S(\gamma) = \frac{(1+\gamma)\,\mu_0}{c^2}\ \ln\left(\frac{r_r + \overrightarrow{k}\cdot\overrightarrow{r_r}}{r_t + \overrightarrow{k}\cdot\overrightarrow{r_t}}\right) = \frac{(1+\gamma)\,\mu_0}{c^2}\ \ln\left(\frac{r_e + r_r + r}{r_e + r_r - r}\right) \quad (2)$$

- $\frac{\mu_0}{c^2} \simeq 1.5$ Km is the Schwartzschild radius of the Sun
- $r_e = |\overrightarrow{r_e}|$, $r_r = |\overrightarrow{r_r}|$ are the heliocentric distances of the emitter and of the receiver
- $\overrightarrow{k}$ is the unit vector from the emitter to the receiver
- r is the range



Shapiro effect (1-PN level): range



Contribution of the Shapiro effect on the observable range: near superior conjuction the contribution is about 2.5×10^6 cm = 25 Km

Improving the model

(study done in collaboration with David Vokrouhlicky)

Ranging to Vikings on the surface of Mars provided $\sim 10^{-3}$ constraint on γ (Reasenberg et al. 1979). A more recent experiment using the Cassini spacecraft lead to $\sim 2.1 \times 10^{-5}$ result for γ (Bertotti et al. 2003).

A preliminary study of the ranging to BepiColombo has led to hopes reaching $\leq 10^{-6}$ level of accuracy in γ (Milani et al., 2002). With that, we need to revise necessary modelling tools, since the classical 1-PN monopole formula is insufficient.

We implemented two different levels of correction:

- 1.5-PN level (Will 2003, Klioner & Peip 2003), taking into account the motion of the Sun
- 2-PN level (Moyer 2003), taking into account the bending of the light path

Correction to 1.5-PN level

(References: Will, Klioner & Peip)
Taking into account a linear motion of the Sun

$$\overrightarrow{X_\odot}(t) = \overrightarrow{X_0^{ini}} + \overrightarrow{v_0}(t - t_{ref}) \tag{3}$$

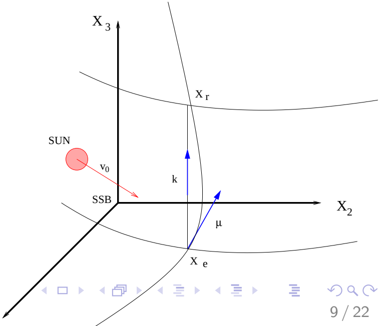
the Shapiro formula has to be corrected

$$S(\gamma) = \frac{(1+\gamma)\,\mu_0}{c^2}\left(1 - \overrightarrow{k}\cdot\frac{\overrightarrow{v_0}}{c}\right)\ln\left(\frac{|\overrightarrow{g_0}|\,r_r + \overrightarrow{g_0}\cdot\overrightarrow{r_r}}{|\overrightarrow{g_0}|\,r_e + \overrightarrow{g_0}\cdot\overrightarrow{r_e}}\right) \tag{4}$$

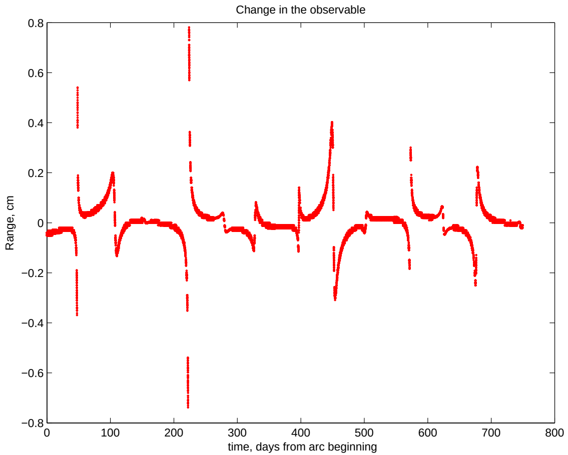
where

- $\overrightarrow{g_0} = \overrightarrow{\mu} - \frac{\overrightarrow{v_0}}{c}$, $\overrightarrow{\mu}$ is the unit vector tangent to the light ray at the emission
- $|\overrightarrow{g_0}| = 1 - \overrightarrow{k}\cdot\frac{\overrightarrow{v_0}}{c} + O(c^{-2})$

t_{ref} choice: Klioner & Peip argue that the best is to use $t_{ref} = t^{ca}$, time of the closest approach between the Sun and the unperturbed light ray



Shapiro effect: 1PN vs 1.5PN



1.5PN correction added with little effort, but does not seem to be important, since it affects γ at $< 10^{-7}$ level (Will 2003, Kopeikin 2008,...)

Discussion on 2-PN terms

A difficult issue is to determine 2PN-level terms in the *Shapiro* delay formula: here we present and we compare the corrections given by Moyer and Hellings

- Moyer (2003)
  Moyer proposes to add a term

$$\frac{(1+\gamma)\,\mu_0}{c^2} \tag{5}$$

both in the numerator and denominator of th argument of the natural logarithm.

$$\boxed{\ln\left(\frac{r_e + r_r + r}{r_r + r_r - r}\right) \to \ln\left(\frac{r_e + r_r + r + \frac{(1+\gamma)\,\mu_0}{c^2}}{r_e + r_r - r + \frac{(1+\gamma)\,\mu_0}{c^2}}\right)}$$

Evaluating the expression

$$\frac{r_e + r_r + r + \frac{(1+\gamma)\,\mu_0}{c^2}}{r_e + r_r - r + \frac{(1+\gamma)\,\mu_0}{c^2}}$$

near the conjuction configuration we got the approximation for the 2PN correction to the *Shapiro* formula

$$S_{2PN}^{Moy} \approx \frac{(1+\gamma)\,\mu_0}{c^2} \cdot \ln\left(1 - \frac{2\,r_r\,r_e}{b^2}\,\frac{(1+\gamma)\,\mu_0)}{c^2\,r}\right) \approx -(1+\gamma)^2\left(\frac{\mu_0}{c^2\,b}\right)^2 \frac{2\,r_r\,r_e}{r_r + r_e} \tag{6}$$

This result has been obtained also by Teyssandier et al and Klioner & Zschocke (2008).

- Hellings (1986)
  Hellings added a more complicated term of 2-PN level

$$S_{2PN}^{Hel} = \frac{1}{2}(1+\gamma)^2\left(\frac{\mu_0}{b\,c^2}\right)^2\left[r\left(1 + \frac{\left(\overrightarrow{k}\cdot\overrightarrow{r_e}\right)^2}{r_e^2}\right) + 2(\overrightarrow{k}\cdot\overrightarrow{r_e})\left(1 - \frac{r_r}{r_e}\right)\right] \tag{7}$$

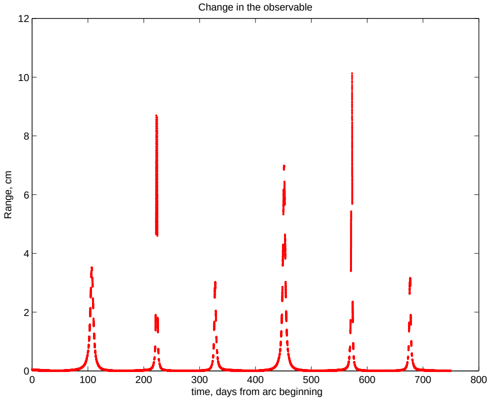
which in the conjuction approximation becomes

$$S_{2PN}^{Hel} \approx (1+\gamma)^2\left(\frac{\mu_0}{c^2\,b}\right)^2 2\,r_r \tag{8}$$

Considering the ratio between the 2-PN terms of Moyer and Hellings we have

$$\frac{S_{2PN}^{Moy}}{S_{2PN}^{Hel}} = -\frac{r_e}{r_e + r_r} \qquad \frac{S_{2PN}^{Moy} - S_{2PN}^{Hel}}{S_{2PN}^{Moy}} \approx \frac{12}{5}$$

Shapiro effect: 1PN vs 2PN



Differences in range using 1-PN model or 2-PN model for the Shapiro effect (we used the Moyer formulation and a minimum impact parameter $b = 3\,\mathbf{R}_\odot$): they are significant (~ 10 cm) near conjuction. The difference with Hellings formulation is larger by a factor $\simeq 2.4$. For larger values of b the effect decreases as $1/b^2$.

Range computation

$$r = |(\overrightarrow{x}_{sat} + \overrightarrow{x}_M) - (\overrightarrow{x}_{EM} + \overrightarrow{x}_E + \overrightarrow{x}_{ant})| + S(\gamma)$$

The 5 vectors have to be computed at the epoch of different events:

- $\overrightarrow{x}_{ant}$, $\overrightarrow{x}_{EM}$ and $\overrightarrow{x}_E$ at both the antenna transmit time t_t and the receive time t_r of the signal
- $\overrightarrow{x}_M$ and $\overrightarrow{x}_{sat}$ at the bounce time t_b, when the signal has arrived to the orbiter and is sent back, with corrections for the delay of the transponder.

Two different light-times:

- UP-LEG
  $$\Delta t_{up} = t_b - t_t + \Delta_{up}$$
  for the signal from the antenna to the orbiter
- DOWN-LEG
  $$\Delta t_{do} = t_r - t_b + \Delta_{do}$$
  for the return signal from the orbiter to the antenna

The two corrective terms Δ_{up}, Δ_{do} account for the Post-Newtonian corrections to the two different time scales, see later.

Given the vector differences down-leg and up-leg with their Shapiro effects

$$\overrightarrow{r_{do}}(t_r) = \overrightarrow{x}_{sat}(t_b) + \overrightarrow{x}_M(t_b) - \overrightarrow{x}_{EM}(t_r) - \overrightarrow{x}_E(t_r) - \overrightarrow{x}_{ant}(t_r)$$

$$\overrightarrow{r_{up}}(t_r) = \overrightarrow{x}_{sat}(t_b) + \overrightarrow{x}_M(t_b) - \overrightarrow{x}_{EM}(t_t) - \overrightarrow{x}_E(t_t) - \overrightarrow{x}_{ant}(t_t)$$

$$r_{do}(t_r) = |\overrightarrow{r_{do}}(t_r)| + S_{do}(\gamma) \quad , \quad r_{up}(t_r) = |\overrightarrow{r_{up}}(t_r)| + S_{up}(\gamma) \ ,$$

by definition of distance the light-times are

$$\Delta t_{do} = r_{do}/c \qquad \Delta t_{up} = r_{up}/c$$

If the measurement is at the receive time t_r, iterative procedure needs to start from the down-leg:

1. we compute $\overrightarrow{x}_{EM}$, $\overrightarrow{x}_E$ and $\overrightarrow{x}_{ant}$ at t_r;
2. we estimate t_b^0 for the bounce time;
3. we compute $\overrightarrow{x}_{sat}$ and $\overrightarrow{x}_M$ at t_b^0 and a first guess r_{do}^0;
4. we obtain a better estimate $t_b^1 = t_r - r_{do}^0/c$;
5. we repeat the prevoius steps computing r_{do}^1, and so on until convergence that is, until $r_{do}^k - r_{do}^{k-1}$ is smaller than the required accuracy.

After accepting the last value of t_b and r_{do} we start with another iterative procedure:

6. we compute the states $\overrightarrow{x}_{sat}$ and $\overrightarrow{x}_M$ at t_b;
7. we estimate t_t^0 for the transmit time;
8. we compute $\overrightarrow{x}_{EM}$, $\overrightarrow{x}_E$ and $\overrightarrow{x}_{ant}$ at epoch t_t^0 and r_{up}^0 is given by up-leg formula,
9. we obtain a better estimate $t_t^1 = t_b - r_{up}^0/c$;
10. we repeat the same procedure until convergence, that is to achieve a small enough $r_{up}^k - r_{up}^{k-1}$.

Then the 2-way range is just

$$r_{up} + r_{do}\,;$$

a 1-way range can be conventionally defined as

$$r(t_r) = (r_{up} + r_{do})/2\,.$$

Range rate computation

Let us examine how we compute the observable range rate for the down-leg (a similar procedure is used for the up-leg).
The range-rate is computed with the unit vector $\hat{r}_{do}$:

$$\dot{r}_{do}(t_r) = \hat{r}_{do} \cdot \dot{\overrightarrow{r_{do}}} + \dot{S}_{do}(\gamma) \ .$$

In order to compute $\dot{\overrightarrow{r_{do}}}$, a first approximation uses the velocities for each of the 5 vectors, at the times t_r and t_b, t_t obtained at convergence of the light-time iterations

$$\dot{\overrightarrow{r_{do}}} = (\dot{\overrightarrow{x}}_{sat} + \dot{\overrightarrow{x}}_{M}) - (\dot{\overrightarrow{x}}_{EM} + \dot{\overrightarrow{x}}_{E} + \dot{\overrightarrow{x}}_{ant}) \ .$$

However, this neglects that t_b, t_t depend on t_r also through r_{do}, r_{up}

$$\frac{dt_b}{dt_r} = 1 - \frac{\dot{r}_{do}}{c} + \frac{d\Delta_{do}}{dt_b}$$

$$\frac{dt_t}{dt_r} = 1 - \frac{\dot{r}_{do}}{c} - \frac{\dot{r}_{up}}{c} + \frac{d\Delta_{do}}{dt_b} + \frac{d\Delta_{up}}{dt_b}$$

The corresponding corrections to $\overrightarrow{\dot{r}_{do}}$

$$\overrightarrow{\dot{r}_{do}} = (\overrightarrow{\dot{x}}_{sat} + \overrightarrow{\dot{x}}_{M})\,(1 - \frac{\dot{r}_{do}}{c} + \frac{d\Delta_{do}}{dt_b}) - (\overrightarrow{\dot{x}}_{EM} + \overrightarrow{\dot{x}}_{E} + \overrightarrow{\dot{x}}_{ant})$$

are large, the first term being $\mathcal{O}(\dot{r}/c)$; the one due to Δ_{do} is smaller, but significant.

The improved value of $\overrightarrow{\dot{r}_{do}}$ has to be inserted in the range-rate equation, the correction recomputed and so on until convergence of the value $\dot{r}_{do}$.

Note that also the computation of $\dot{S}_{do}(\gamma), \dot{S}_{up}(\gamma)$ requires corrections $\mathcal{O}(\dot{r}/c)$, because the time derivative is with respect to t_r.

Conventionally

$$\dot{r}(t_r) = (\dot{r}_{up}(t_r) + \dot{r}_{do}(t_r))/2$$

is the instantaneous value.

However, an accurate measure of a Doppler effect requires to fit the difference in phase between carrier waves, the one generated at the station and the one returned from space, accumulated over some integration time Δ, typically between 10 and 1000 s. Thus the observable $\dot{r}$ is really obtained from a difference of ranges

$$\frac{r(t_b + \Delta/2) - r(t_b - \Delta/2)}{\Delta}$$

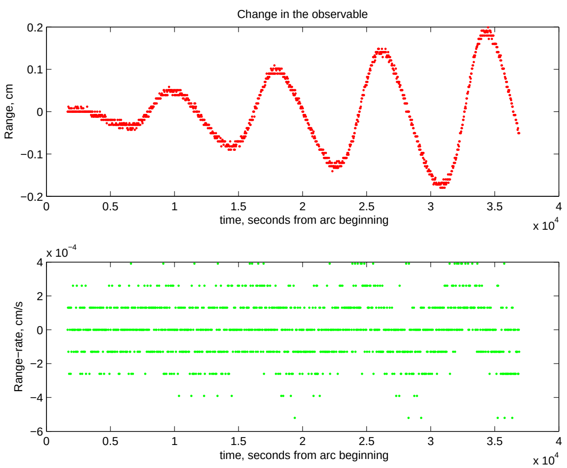
or, equivalently, an averaged value of range-rate over the integration interval, which can be computed with a quadrature formula:

$$\frac{1}{\Delta} \int_{t_b-\Delta/2}^{t_b+\Delta/2} \dot{r}(s)\, ds$$

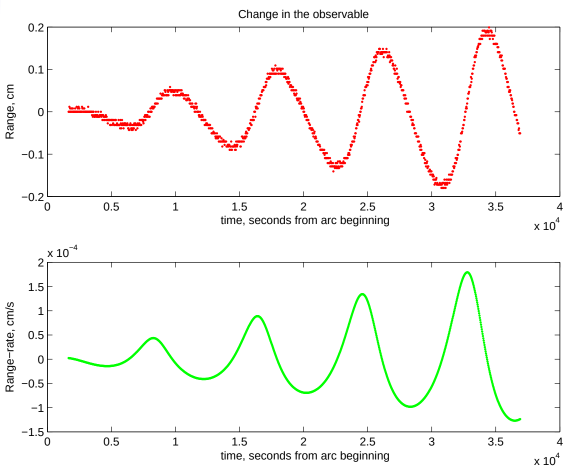
The two methods are not equivalent because of rounding off.

For MORE the accuracy of range-rate measurements can be 3×10^{-4} cm/s (over an integration time of $1\,000$ s). Let us take an integration time $\Delta = 30$ s, which is adequate for measuring the gravity field of Mercury. The accuracy over 30 s can be, by Gaussian statistics, $\simeq 3 \times 10^{-4}\ \sqrt{1\,000/30} \simeq 17 \times 10^{-4}$ cm/s.

The required accuracy in the difference $r(t_b + \Delta/2) - r(t_b - \Delta/2)$ is $\simeq 0.05$ cm. The distances can be as large as $\simeq 2 \times 10^{13}$ cm, thus the relative accuracy in the difference needs to be 2.5×10^{-15}. This is not possible with standard double precision, with rounding off relative accuracy 2.2×10^{-16} for a single operation.



The range-rate as average over the integration time of 30 s has been computed as range difference divided by the integration time. The difference due to a change by 10^{-11} of the C_{22} coefficient is obscured by the rounding off. This can be fixed only by performing the light-time computation in quadruple precision.



The range-rate computed as an integral is smooth. The difference is due to a change by 10^{-11} of the C_{22} harmonic coefficient and is marginally significant with respect to the accuracy (with integration time 30 s, about $17\mu/s$). The only problem is that the Shapiro effect in range-rate needs to be accurately computed.