On the singularities of generalized solutions to n–body type problems*

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Abstract

The validity of Sundman-type asymptotic estimates for collision solutions is established for a wide class of dynamical systems with singular forces, including the classical N–body problems with Newtonian, quasi–homogeneous and logarithmic potentials. The solutions are meant in the generalized sense of Morse (locally –in space and time– minimal trajectories with respect to compactly supported variations) and their uniform limits. The analysis includes the extension of the Von Zeipel's Theorem and the proof of isolatedness of collisions. Furthermore, such asymptotic analysis is applied to prove the absence of collisions for locally minimal trajectories.

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1 Introduction

Many systems of interacting bodies of interest in Celestial and other areas of classical Mechanics have the form

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}(t, x), \qquad i = 1, \dots, n \tag{1.1}$$

where the forces $\frac{\partial U}{\partial x_i}$ are undefined on a singular set Δ. This is for example the set of collisions between two or more particles in the n–body problem. Such singularities play a fundamental role in the phase portrait (see, e.g. [20]) and strongly influence the global orbit structure, as they can be held responsible, among others, of the presence of chaotic motions (see, e.g. [16]) and of motions becoming unbounded in a finite time [35, 54].

Two are the major steps in the analysis of the impact of the singularities in the n–body problem: the first consists in performing the asymptotic analysis along a single collision (total or partial) trajectory and goes back, in the classical case, to the works by Sundman ([50]), Wintner ([53]) and, in more recent years by Sperling, Pollard, Saari, Diacu and other authors (see for instance [43, 44, 47, 49, 24, 17]). The second step consists in blowing–up the singularity by a suitable change of coordinates introduced by McGehee in [36] and replacing it by an invariant boundary –the collision manifold– where the flow can be extended in a smooth manner. It turns out that, in many interesting applications, the flow on the collision manifold has a simple structure: it is a gradient–like, Morse–Smale flow featuring a few stationary points and heteroclinic connections (see, for instance, the

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surveys [16, 38]). The analysis of the extended flow allows us to obtain a full picture of the behavior of solutions near the singularity, despite the flow fails to be fully regularizable (except in a few cases).

The geometric approach, via the McGehee coordinates and the collision manifold, can be successfully applied also to obtain asymptotic estimates in some cases, such as the collinear three–body problem ([36]), the anisotropic Kepler problem ([13, 14, 23, 22]), the three–body problem both in the planar isosceles case ([15]) and the full perturbed three-body, as described in [16, 19]. Besides the quoted cases, however, one needs to establish the asymptotic estimates before blowing–up the singularity, in order to prove convergence of the blow–up family. The reason is quite technical and mainly rests in the fact the a singularity of the n–body problems needs not be isolated, for the possible occurrence of partial collisions in a neighborhood of the total collision. In the literature, this problem has been usually overcame by extending the flow on partial collisions via some regularization technique (such as Sundman's, in [15], or Levi–Civita's in [32]). Such a device works well only when partial collisions are binary, which are the only singularities to be globally removable. Thus, the extension of the geometrical analysis to the full n–body problems finds a strong theoretical obstruction: partial collisions must be regularizable, what is known to hold true only in few cases. Other interesting cases in which the geometric method is not effective are that of quasi–homogeneous potentials (where there is a lack of regularity for the extended flow) and that of logarithmic potentials (for the failure of the blow–up technique).

In this paper we extend the classical asymptotic estimates near collisions in three main directions.

(i) We take into account of a very general notion of solution for the dynamical system (1.1), which fits particularly well to solutions found by variational techniques. Our notion of solution includes, besides all classical noncollision trajectories, all the *locally minimal solutions* (with respect to compactly supported variations) which are often termed minimal the sense of Morse. Furthermore, we include in the set of *generalized solutions* all the limits of classical and locally minimal solutions.

(ii) We extend our analysis to a wide class of potentials including not only homogeneous and quasi-homogeneous potentials, but also those with weaker singularities of logarithmic type.

(iii) We allow potentials to strongly depend on time (we only require its time derivative to be controlled by the potential itself – see assumption (U1)). In this way, for instance, we can take into account models where masses vary in time.

Our main results on the asymptotics near total collisions (at the origin) are Theorems 2 and 3 (for quasi-homogeneous potentials) and Theorems 4 and 5 (when the potential is of logarithmic type) which extend the classical Sundman–Sperling asymptotic estimates ([50, 49]) in the directions above (see also [19, 21]).

As a consequence of the asymptotic estimates, the presence of a total collision prevents the occurrence of partial ones for neighboring times.

This observation plays a central role when extending the asymptotic estimates to the full n–body problem since it allows us to reduce from partial (even simultaneous) collisions to total ones by decomposing the system in colliding clusters. Our results also lead to the extension of the concept of singularity for the dynamical system (1.1) to the class of generalized solutions. We shall prove an extension of the Von Zeipel's Theorem: when the moment of inertia is bounded then every singularity of a generalized solution admits a limiting configuration, hence all singularities are collisions. The results on total collisions are then fully extended to partial ones in Theorem 6.

A further motivation for the study of generalized solutions comes from the variational approach to the study of selected trajectories to the n-body problem. Indeed the exclusion of collisions is a major problem in the application of variational techniques as it results in the recent literature, where many different arguments have been introduced to prove that the trajectories found in such a way are collisionless (see [1, 2, 4, 6, 7, 9, 11, 12, 26, 27, 39, 40, 41, 48, 51, 52]). As a first application we shall be able to extend some of these techniques in order to prove that action minimizing trajectories are free of collisions for a wider class of interaction potentials. For example in the case of quasi–homogeneous potentials, once collisions are isolated, the blow-up technique can be successfully applied to prove that locally minimal solutions are, in many circumstances, *free of collisions.* In order to do that we can use the method of *averaged variations* introduced by Marchal and developed in [34, 10, 27]. It has to be noticed that, when dealing with logarithmic–type potentials, the blow-up technique is not available since converging blow-up sequences do not exists; we can anyway prove that the average over all possible variations is negative by taking advantage of the harmonicity of the function $\log|x|$ in $\mathbb{R}^2$. With this result we can then extend to quasi–homogeneous and logarithmic potentials all the analysis of the (equivariant) minimal trajectories carried in [27].

Besides the direct method, other variational techniques –Morse and minimax theory– have been applied to the search of periodic solutions in singular problems ([1, 3, 33, 46]). In the quoted papers, however only the case of *strong force* interaction (see [28]) has been treated. Le us consider a sequence of solutions to penalized problems where an infinitesimal sequence of strong force terms is added to the potential: then its limit enjoys the same conservation laws as the generalized solutions. Hence our main results apply also to this class of trajectories. We believe that our study can be usefully applied to develop a Morse Theory that takes into account the topological contribution of collisions. Partial results in this direction are given in [5, 45], where the contribution of collisions to the Morse Index is computed.

The paper is organized as follows:

Contents

2 Singularities of locally minimal solutions

2.1 Locally minimal solutions

We fix a metric on the *configuration space* $\mathbb{R}^k$ and we denote by $I(x) = |x|^2$ the *moment of inertia* associated to the configuration $x \in \mathbb{R}^k$ and

$$\mathcal{E} := \{x \in \mathbb{R}^k : |x|^2 = 1\}$$

the inertia ellipsoid. We define the radial and "angular" variables associated to $x \in \mathbb{R}^k$ as

$$r := |x| = I^{\frac{1}{2}}(x) \in [0, +\infty), \quad s := \frac{x}{|x|} \in \mathcal{E}. \tag{2.1}$$

We consider the dynamical system

$$\ddot{x} = \nabla U(t, x), \tag{2.2}$$

on the time interval $(a, b) \subset \mathbb{R}$, $-\infty \leq a < b \leq +\infty$. Here U is a positive time-dependent potential function $U\colon\ (a, b) \times \left(\mathbb{R}^k \smallsetminus \Delta\right) \to \mathbb{R}^+$, and it is supposed to be of class $\mathcal{C}^1$ on its domain; by ∇U we denote its gradient with respect to the given metric.

Remark 2.1. In the case of n-body type systems as described in (1.1), given $m_1, \ldots, m_n$, $n \geq 2$ positive real numbers, we define the scalar product induced by the *mass metric* on the *configuration space* $\mathbb{R}^{nd}$ between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, as

$$x \cdot y = \sum_{i=1}^{n} m_i \langle x_i, y_i \rangle, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$. We denote by $|\cdot|$ the norm induced by the mass scalar product (2.3). Then $\nabla U(t, x)$ denotes the gradient of the potential, in the mass metric, with respect to the spatial variable x, that is:

$$\nabla U(t, x) = M^{-1} \frac{\partial U}{\partial x}(t, x),$$

where $\left(\frac{\partial U}{\partial x}\right)_i = \frac{\partial U}{\partial x_i}$, $i = 1, \ldots, n$, and $M = [M_{ij}]$, $M_{ij} = m_i \delta_{ij} \mathbf{1}_d$ ($\mathbf{1}_d$ is the d-dimensional identity matrix) for every $i, j = 1, \ldots, n$.

Furthermore we suppose that Δ is a singular set for U of an attractive type, in the sense that

(U0) $\lim_{x \to \Delta} U(t, x) = +\infty$, uniformly in t.

Borrowing the terminology from the study of the singularities of the n–body problem, the set Δ will be often referred as *collision set* and it is required to be a *cone*, that is

$$x \in \Delta \quad \Longrightarrow \quad \lambda x \in \Delta, \quad \forall \lambda \in \mathbb{R}.$$

We observe that being a cone implies that $0 \in \Delta$. When $x(t^*) \in \Delta$ for some $t^* \in (a, b)$ we will say that x has an *interior collision* at $t = t^*$ and that t^* is a *collision instant* for x. When $t^* = a$ or $t^* = b$ (when finite) we will talk about a *boundary collision.* In particular, if $x(t^*) = 0 \in \Delta$, we will say that x has a *total collision at the origin* at $t = t^*$. A collision instant t^* is termed *isolated* if there exists $\delta > 0$ such that, for every $t \in (t^* - \delta, t^* + \delta) \cap (a, b)$, $x(t) \notin \Delta$.

We consider the following assumptions on the potential U:

(U1) There exists a constant $C_1 \geq 0$ such that, for every $(t, x) \in (a, b) \times \left(\mathbb{R}^k \smallsetminus \Delta\right)$,

$$\left| \frac{\partial U}{\partial t}(t, x) \right| \leq C_1 \left(U(t, x) + 1 \right).$$

(U2) There exist constants $\tilde{\alpha} \in (0, 2)$ and $C_2 \geq 0$ such that

$$\nabla U(t, x) \cdot x + \tilde{\alpha} U(t, x) \geq -C_2.$$

We then define the *lagrangian action functional* on the interval (a, b) as

$$\mathcal{A}(x, [a, b]) := \int_a^b K(\dot{x}) + U(t, x)\, dt, \tag{2.4}$$

where

$$K(\dot{x}) := \frac{1}{2}|\dot{x}|^2, \tag{2.5}$$

is the *kinetic energy*. We observe that $\mathcal{A}(\cdot,[a,b])$ is bounded and $\mathcal{C}^2$ on the Hilbert space $H^1\left((a,b),\mathbb{R}^k\smallsetminus\Delta\right)$. In terms of the variables r and s introduced in (2.1), the action functional reads as

$$\mathcal{A}(rs,[a,b]) := \int_a^b \frac{1}{2}\left(\dot{r}^2 + r^2|\dot{s}|^2\right) + U(t,rs)\,dt$$

and the corresponding Euler–Lagrange equations, whenever $x \in H^1\left((a,b),\mathbb{R}^k\smallsetminus\Delta\right)$, are

$$\begin{aligned} -\ddot{r} + r|\dot{s}|^2 + \nabla U(t,rs)\cdot s &= 0 \\ -2r\dot{r}\dot{s} - r^2\ddot{s} + r\nabla_T U(t,rs) &= \mu s \end{aligned} \tag{2.6}$$

where $\mu = r^2|\dot{s}|^2$ is the Lagrange multiplier due to the presence of the constraint $|s|^2 = 1$ and the vector $\nabla_T U(t,rs)$ is the tangent components to the ellipsoid $\mathcal{E}$ of the gradient $\nabla U(t,rs)$, that is $\nabla_T U(t,rs) = \nabla U(t,rs) - \nabla U(t,rs)\cdot s$.

Definition 2.2. A path $x \in H^1_{loc}\left((a,b),\mathbb{R}^k\right)$ is a *locally minimal solution* for the dynamical system (2.2) if, for every $t_0 \in (a,b)$, there exists $\delta_0 > 0$ such that the restriction of x to the interval $I_0 = [t_0-\delta_0, t_0+\delta_0]$, is a local minimizer for $\mathcal{A}(\cdot, I_0)$ with respect to compactly supported variations (fixed-ends).

Remark 2.3. We observe that a priori a locally minimal solution x can have a large collision set, $x^{-1}(\Delta)$; this set, though of Lebesgue measure zero, can very well admit many accumulation points. For this reason the Euler–Lagrange equations (2.6) and the dynamical system (2.2) do not hold for a locally minimal trajectory.

Remark 2.4. When the potential is of class $\mathcal{C}^2$ outside Δ then every classical noncollision solution in the interval (a,b) is a a locally minimal solution.

Definition 2.5. A path x is a *generalized solution* for the dynamical system (2.2) if there exists a sequence x_n of locally minimal solutions such that

(i) $x_n \to x$ uniformly on compact subsets of (a,b);

(ii) for almost all $t \in (a,b)$ the associated total energy $h_n(t) := K(\dot{x}_n(t)) - U(t, x_n(t))$ converges.

We say that a (classical, locally minimal, generalized) solution x on the interval (t_1,t_2), has a *singularity* at t_2 (finite) if it is not possible to extend x as a (classical, locally minimal, generalized) solution to a larger interval (t_1,t_3) with $t_3 > t_2$.

In the framework of classical solutions to n-body systems, the classical Painlevé's Theorem ([42, 18]) asserts that the existence of a singularity at a finite time t^* is equivalent to the fact that the minimal of the mutual distances becomes infinitesimal as $t \to t^*$. This fact reads as:

Painlevé's Theorem. *Let $\bar{x}$ be a classical solution for the n-body dynamical system on the interval $[0,t^*)$. If $\bar{x}$ has a singularity at $t^* < +\infty$, then the potential associated to the problem diverges to $+\infty$ as t approaches t^*.*

Painlevé's Theorem does not necessarily imply that a collision (i.e. that is a singularity such that all mutual distances have a definite limit) occurs when there is a singularity at a finite time; indeed this two facts are equivalent only if each particle approaches a definite configuration (on this subject we refer to [43, 44, 47]). This result has been stated by Von Zeipel in 1908 (see [55] and also [37]) and definitely proved by Sperling in 1970 (see [49]): in the n-body problem the occurrence of singularities (in finite time) which are not collisions is then equivalent to the existence of an unbounded motion.

Von Zeipel's Theorem. *If $\bar{x}$ is a classical solution for the n-body dynamical system on the interval $[0, t^*)$ with a singularity at $t^* < +\infty$ and $\lim_{t\to t^*} I(\bar{x}(t)) < +\infty$, then $\bar{x}(t)$ has a definite limit configuration x^* as t tends to t^*.*

We will come back later on the proof of this result (in Corollary 2.17 and in Section 4). To our purposes, we give the following definition.

Definition 2.6. We say that the (generalized) solution $\bar{x}$ for the dynamical system (2.2) has a *singularity* at $t = t^*$ if

$$\lim_{t\to t^*} U(t, \bar{x}(t)) = +\infty.$$

Definition 2.7. The singularity t^* is a said to be a *collision* for the locally minimal solution $\bar{x}$ if it admits a limit configuration as t tends to t^*.

2.2 Approximation of locally minimal solutions

Let $\bar{x}$ be a locally minimal solution on the interval (a, b) and let $I_0 \subset (a, b)$ be an interval such that $\bar{x}$ is a (local) minimizer for $\mathcal{A}(\cdot, I_0)$ with respect to compactly supported variations. Generally local minimizers need not to be isolated; we illustrate below a penalization argument to select a particular solution from the possibly large set of local minimizers. To begin with, we define the auxiliary functional on the space $H^1\left(I_0, \mathbb{R}^k\right)$

$$\bar{\mathcal{A}}(x, I_0) := \int_{I_0} K(\dot{x}) + U(t, x) + \frac{|x - \bar{x}|^2}{2}\, dt. \tag{2.7}$$

When the interval I_0 is sufficiently small, $\bar{x}$ is actually the global minimizer for the penalized functional $\bar{\mathcal{A}}(\cdot, I_0)$ defined in (2.7). Of course we may assume that

$$\bar{\mathcal{A}}(\bar{x}, I_0) = \mathcal{A}(\bar{x}, I_0) < +\infty, \tag{2.8}$$

which is equivalent to require that $\bar{\mathcal{A}}(\cdot, I_0)$ takes a finite value at least at one point.

Proposition 2.8. *Let $\bar{x}$ be a locally minimal solution on the interval (a, b), let $\delta_0 > 0$ and $t_0 \in (a, b)$ be such that $\bar{x}$ is a local minimizer for $\mathcal{A}(\cdot, I_0)$ where $I_0 = [t_0-\delta_0, t_0+\delta_0] \subset (a, b)$. Then there exists $\bar{\delta} = \bar{\delta}(\bar{x}) > 0$ such that whenever $\delta_0 \le \bar{\delta}$, $\bar{x}$ is the unique global minimizer for $\bar{\mathcal{A}}(\cdot, I_0)$.*

Proof. For every $x \in H^1_{loc}\left(I_0, \mathbb{R}^k\right)$ the inequality $\mathcal{A}(x, I_0) \le \bar{\mathcal{A}}(x, I_0)$ holds true, and it is an equality only if $x = \bar{x}$. Since $\bar{x}$ is a local minimizer for $\mathcal{A}(x, I_0)$, one easily infers, by a simple convexity argument, the existence of $\varepsilon > 0$ such that

$$\|x - \bar{x}\|_\infty < \varepsilon \quad \Longrightarrow \quad \mathcal{A}(\bar{x}, I_0) \le \mathcal{A}(x, I_0).$$

We conclude that, for every $x \in H^1_{loc}\left(I_0, \mathbb{R}^k\right)$, such that $0 < \|x - \bar{x}\|_\infty < \varepsilon$, the following chain of inequalities holds

$$\bar{\mathcal{A}}(\bar{x}, I_0) = \mathcal{A}(\bar{x}, I_0) \le \mathcal{A}(x, I_0) < \bar{\mathcal{A}}(x, I_0);$$

hence $\bar{x}$ is a strict local minimizer for $\bar{\mathcal{A}}(\cdot, I_0)$, independently on δ_0.

In order to complete the proof we show that $\bar{\mathcal{A}}(\bar{x}, I_0) < \bar{\mathcal{A}}(x, I_0)$ also for those functions $x \in H^1_{loc}\left(I_0, \mathbb{R}^k\right)$ such that $\|x - \bar{x}\|_\infty \ge \varepsilon$, provided δ_0 is sufficiently small. Indeed, since the Sobolev space $H^1_{loc}\left(I_0, \mathbb{R}^k\right)$ is embedded in the space of absolutely continuous functions, we can compute, by Hölder inequality,

$$\begin{aligned} |(x - \bar{x})(t)| &\le \int_{I_0} |\dot{x}(s)| ds + \int_{I_0} |\dot{\bar{x}}(s)| ds \\ &\le \sqrt{2\delta_0} \left(\sqrt{\int_{I_0} |\dot{x}(s)|^2 ds} + \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2 ds} \right). \end{aligned} \tag{2.9}$$

By taking the supremum at both sides of (2.9) it follows that

$$\frac{\|x-\bar{x}\|_\infty}{\sqrt{2\delta_0}} - \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2\, ds} \leq \sqrt{\int_{I_0} |\dot{x}(s)|^2\, ds},$$

and therefore, for every $x \in H^1\left(I_0, \mathbb{R}^k\right)$,

$$\begin{aligned} \bar{\mathcal{A}}(x, I_0) \geq \int_{I_0} |\dot{x}(s)|^2\, ds &\geq \left(\frac{\|x-\bar{x}\|_\infty}{\sqrt{2\delta_0}} - \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2\, ds}\right)^2 \\ &\geq \left(\frac{\varepsilon}{\sqrt{2\delta_0}} - \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2\, ds}\right)^2 \end{aligned} \tag{2.10}$$

Hence, by choosing δ_0 such that $2\delta_0 < \varepsilon\left(\sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2 ds} + \sqrt{\bar{\mathcal{A}}(\bar{x}, I_0)}\right)^{-2}$, it follows that

$$\bar{\mathcal{A}}(x, I_0) \geq \left(\frac{\varepsilon}{\sqrt{2\delta_0}} - \sqrt{\int_{I_0} |\dot{\bar{x}}(s)|^2 ds}\right)^2 > \bar{\mathcal{A}}(\bar{x}, I_0)$$

also for those paths $x \in H^1_{loc}\left(I_0, \mathbb{R}^k\right)$ such that $\|x - \bar{x}\|_\infty \geq \varepsilon$. This concludes the proof. □

We now wish to approximate the singular potential U with a family of smooth potentials $U_\varepsilon \colon (a,b) \times \mathbb{R}^k \to \mathbb{R}^+$, depending on a parameter $\varepsilon > 0$. To this aim consider the function

$$\eta(s) = \begin{cases} s & \text{if } s \in [0,1] \\ \dfrac{-s^2+6s-1}{4} & \text{if } s \in [1,3] \\ 2 & \text{if } s \geq 3; \end{cases}$$

notice that $\eta \in \mathcal{C}^1\left(\mathbb{R}^+, \mathbb{R}^+\right)$ and, for every $s \in [0, +\infty)$,

$$\dot{\eta}(s)\, s \leq \eta(s) \qquad \text{and} \qquad \dot{\eta}(s) \leq 1.$$

Now let us define, for $\varepsilon > 0$,

$$\eta_\varepsilon(s) := \frac{1}{\varepsilon}\eta(\varepsilon s);$$

then the following inequalities hold for every $s \in [0, +\infty)$

$$\dot{\eta}_\varepsilon(s)\, s \leq \eta_\varepsilon(s), \qquad \text{and} \qquad \dot{\eta}_\varepsilon(s) \leq 1. \tag{2.11}$$

By means of the family η_ε we can regularize the potential U in the following way:

$$U_\varepsilon(t,x) = \begin{cases} \eta_\varepsilon\left(U(t,x)\right), & \text{if } x \in \mathbb{R}^k \setminus \Delta, \\ 2/\varepsilon, & \text{if } x \in \Delta. \end{cases} \tag{2.12}$$

It is worthwhile to understand that each $U_\varepsilon(t,x)$ coincides with $U(t,x)$ whenever $U(t,x) \leq 1/\varepsilon$; in fact

$$\eta_\varepsilon(s) = \frac{1}{\varepsilon}\eta(\varepsilon s) = s$$

whenever $\varepsilon s \in [0,1]$, that is $s \in [0, 1/\varepsilon]$. Next, we consider the associated family of boundary value problems on the interval $I_0 \subset (a,b)$

$$\begin{cases} \ddot{x} = \nabla U_\varepsilon(t,x) + (x - \bar{x}), \\ x|_{\partial I_0} = \bar{x}|_{\partial I_0} \end{cases} \tag{2.13}$$

where, as usual, $\nabla U_\varepsilon(t,x)$ is the gradient, in the mass metric, with respect to the spatial variable x. Solutions of (2.13) are critical points of the action functional

$$\bar{\mathcal{A}}_\varepsilon(x, I_0) := \int_{I_0} K(\dot{x}) + U_\varepsilon(t,x) + \frac{|x-\bar{x}|^2}{2}\, dt. \tag{2.14}$$

We observe that $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ is bounded and $\mathcal{C}^2$ on $H^1_{loc}\left(I_0, \mathbb{R}^k\right)$, since U_ε is smooth on the whole $\mathbb{R}^k$. We also remark that the infimum of $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ is achieved, for $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ is a positive and coercive functional on $H^1_{loc}\left(I_0, \mathbb{R}^k\right)$.

In the next proposition we prove that a locally minimal solution has the fundamental property to be the limit of a sequence of global minimizers for the approximating functionals $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$, provided the interval $I_0 \subset (a,b)$ is chosen so small that the restriction of the minimal solution to I_0 is the unique global minimizer for $\bar{\mathcal{A}}(\cdot, I_0)$. This result is crucial, indeed, as observed in Remark 2.3, the Euler–Lagrange equations and the dynamical system hold for a locally minimal solution; we will anyway be able to use the ones corresponding to the approximating global minimizers (for the regularized problems) to prove the fundamental properties of locally minimal (and generalized) solutions in the rest of the paper.

Proposition 2.9. *Let $\bar{x}$ and I_0 be given by Proposition 2.8. Let $\epsilon > 0$ and x_ε be a global minimizer for $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$. Then, up to subsequences, as $\varepsilon \to 0$,*

- *(i)* $U_\varepsilon(t, x_\varepsilon) \to U(t, \bar{x})$ *almost everywhere and in* L^1*;*
- *(ii)* $x_\varepsilon \to \bar{x}$ *uniformly;*
- *(iii)* $\dot{x}_\varepsilon \to \dot{\bar{x}}$ *in* L^2*;*
- *(iv)* $\dot{x}_\varepsilon \to \dot{\bar{x}}$ *almost everywhere;*
- *(v)* $\dfrac{\partial U_\varepsilon}{\partial t}(t, x_\varepsilon) \to \dfrac{\partial U}{\partial t}(t, \bar{x})$ *almost everywhere and in* L^1*.*

Proof. As we have already observed, for every $\varepsilon > 0$, the potential U_ε coincides with U on the sublevel $\{(t,x) : U(t,x) \le 1/\varepsilon\}$ and, by its definition, for every $(t,x) \in I_0 \times \mathbb{R}^k \setminus \Delta$

$$U_\varepsilon(t,x) \le U(t,x).$$

Therefore

$$\bar{\mathcal{A}}_\varepsilon(x, I_0) \le \bar{\mathcal{A}}(x, I_0)$$

for every $x \in H^1_{loc}\left(I_0, \mathbb{R}^k\right)$. It follows from (2.8) that

$$\bar{\mathcal{A}}_\varepsilon(x_\varepsilon, I_0) = \inf_{x \in H^1_{loc}} \bar{\mathcal{A}}_\varepsilon(x, I_0) \le \bar{\mathcal{A}}(\bar{x}, I_0) < +\infty, \tag{2.15}$$

which implies the boundedness of the family $\left\{\int_{I_0} |\dot{x}_\varepsilon|^2 + |x_\varepsilon - \bar{x}|^2\right\}_\varepsilon$. Hence we deduce the existence of a sequence $(x_{\varepsilon_n})_{\varepsilon_n} \subset (x_\varepsilon)_\varepsilon$ such that $(\dot{x}_{\varepsilon_n})_{\varepsilon_n}$ converges weakly in L^2 and uniformly to some limit $\tilde{x}$. In addition we observe that

$$\lim_{\varepsilon_n \to 0} U_{\varepsilon_n}(t, x_{\varepsilon_n}(t)) = U(t, \tilde{x}(t))$$

for every $t \in I_0$, regardless the finiteness of $U(t, \tilde{x}(t))$.
From (2.15) we also deduce the boundedness of the following integrals

$$\int_{I_0} U_{\varepsilon_n}(t, x_{\varepsilon_n})\, dt \le \bar{\mathcal{A}}_{\varepsilon_n}(x_{\varepsilon_n}, I_0) < +\infty.$$

and therefore, since the sequence $(U_{\varepsilon_n}(t, x_{\varepsilon_n}))_{\varepsilon_n}$ is positive, by applying Fatou's Lemma one deduces that

$$\int_{I_0} U(t, \tilde{x}) \le \liminf \int_{I_0} U_{\varepsilon_n}(t, x_{\varepsilon_n}) < +\infty.$$

Hence from the weak semicontinuity of the norm in L^2 (the sequence $(\dot{x}_{\varepsilon_n})_{\varepsilon_n}$ converges weakly in L^2 to $\tilde{x}$) we obtain the inequalities

$$\bar{\mathcal{A}}(\tilde{x}, I_0) \leq \liminf \bar{\mathcal{A}}_{\varepsilon_n}(x_{\varepsilon_n}, I_0) \leq \bar{\mathcal{A}}(\bar{x}, I_0)$$

which contradict Proposition 2.8, unless $\tilde{x} = \bar{x}$ and

$$\liminf \bar{\mathcal{A}}_{\varepsilon_n}(x_{\varepsilon_n}, I_0) = \bar{\mathcal{A}}(\bar{x}, I_0). \tag{2.16}$$

Therefore we deduce the L^2-convergence of the sequence $(\dot{x}_{\varepsilon_n})_{\varepsilon_n}$ and its convergence almost everywhere to $\dot{\bar{x}}$, up to subsequences. From (2.16) it follows also that

$$\lim_{\varepsilon_n \to 0} \int_{I_0} U_{\varepsilon_n}(t, x_{\varepsilon_n}) = \int_{I_0} U(t, \bar{x}). \tag{2.17}$$

From the convergence almost everywhere of $(U_{\varepsilon_n}(t, x_{\varepsilon_n}))_{\varepsilon_n}$ together with (2.17) we conclude its convergence in L^1 to $U(t, \bar{x})$.

We now turn to the convergence of the sequence $(\varphi_n(t))_{\varepsilon_n} = \left(\frac{\partial U_{\varepsilon_n}}{\partial t}(t, x_{\varepsilon_n}) \right)_{\varepsilon_n}$. To this aim, we observe that condition (U1) together with (2.11) imply the following chain of inequalities

$$\begin{aligned}
\left| \frac{\partial U_{\varepsilon_n}}{\partial t}(t, x_{\varepsilon_n}(t)) \right| &= \dot{\eta}_{\varepsilon_n}\left(U(t, x_{\varepsilon_n}(t))\right) \left| \frac{\partial U}{\partial t}(t, x_{\varepsilon_n}(t)) \right| \\
&\leq C_1 \dot{\eta}_{\varepsilon_n}\left(U(t, x_{\varepsilon_n}(t))\right) \left(U(t, x_{\varepsilon_n}(t)) + 1\right) \\
&\leq C_1 \left(\eta_{\varepsilon_n}\left(U(t, x_{\varepsilon_n}(t))\right) + 1\right) \\
&= C_1 \left(U_{\varepsilon_n}(t, x_{\varepsilon_n}(t)) + 1\right).
\end{aligned}$$

We already know that $U_{\varepsilon_n}(t, x_{\varepsilon_n}(t))$ converges in L^1. This implies the finiteness almost everywhere of $(\varphi_n(t))_{\varepsilon_n}$ and hence its almost everywhere convergence is due to the uniform convergence of $(x_{\varepsilon_n})_{\varepsilon_n}$. We obtain the L^1 convergence of $(\varphi_n(t))_{\varepsilon_n}$ to $\frac{\partial U}{\partial t}(t, \bar{x})$ from the Dominated Convergence Theorem. □

2.3 Conservation laws

Now the sequence of solutions to the regularized problems are used to prove the conservation of the energy for locally minimal solutions.

Proposition 2.10. *Let $\bar{x}$ and I_0 be given by Proposition 2.8. Then the energy associated to $\bar{x}$*

$$h\colon\ I_0 \to \mathbb{R}, \qquad h(t) := K(\dot{\bar{x}}(t)) - U(t, \bar{x}(t)) \tag{2.18}$$

is of class $W^{1,1}$ on I_0 and its weak derivative is

$$\dot{h}(t) = \frac{\partial U}{\partial t}(t, \bar{x}).$$

Proof. Let $(x_\varepsilon)_\varepsilon$ be the sequence of global minimizers for the corresponding functionals $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ convergent to $\bar{x}$ whose existence is proved in Proposition 2.9. Let h_ε be the energy associated to x_ε, that is

$$h_\varepsilon\colon\ I_0 \to \mathbb{R}, \qquad h_\varepsilon(t) := K(\dot{x}_\varepsilon(t)) - U_\varepsilon(t, x_\varepsilon(t)) + \frac{1}{2}|\bar{x}(t) - x_\varepsilon(t)|^2 \tag{2.19}$$

From Proposition 2.9 we immediately deduce that the sequence $(h_\varepsilon)_\varepsilon$ converges pointwise to $h(\bar{x})$; moreover from (2.15) and (2.19) we have

$$\int_{I_0} |h_\varepsilon(t)|\, dt \leq \bar{\mathcal{A}}_\varepsilon\left(x_\varepsilon, I_0\right) < \bar{\mathcal{A}}\left(\bar{x}, I_0\right).$$

From the Dominated Convergence Theorem we obtain that the sequence $(h_\varepsilon)_\varepsilon$ converges in L^1 to the integrable function h.

We still have to prove that h admits weak derivative. To this end, let us consider a test function $\varphi \in \mathcal{C}_0^\infty(I_0)$; we can write

$$\begin{aligned}\int_{I_0} h(t)\dot{\varphi}(t)\,dt &= \lim_{\varepsilon\to 0}\int_{I_0} h_\varepsilon(t)\dot{\varphi}(t)\,dt \\ &= \lim_{\varepsilon\to 0} -\int_{I_0}\frac{\partial U_\varepsilon}{\partial t}(t,x_\varepsilon(t))\varphi(t)\,dt.\end{aligned}$$

In consequence of Proposition 2.9, the sequence $\left(\frac{\partial U_\varepsilon}{\partial t}(t,x_\varepsilon(t))\right)_\varepsilon$ converges to $\frac{\partial U}{\partial t}(t,\bar{x}(t))$ in L^1; then

$$\lim_{\varepsilon\to 0}\int_{I_0}\frac{\partial U_\varepsilon}{\partial t}(t,x_\varepsilon(t))\varphi(t)\,dt = \int_{I_0}\frac{\partial U}{\partial t}(t,\bar{x}(t))\varphi(t)\,dt, \qquad \forall \varphi\in\mathcal{C}_0^\infty(I_0) \tag{2.20}$$

and hence

$$\int_{I_0} h(t)\dot{\varphi}(t)\,dt = -\int_{I_0}\frac{\partial U}{\partial t}(t,\bar{x})\varphi(t)\,dt, \qquad \forall\varphi\in\mathcal{C}_0^\infty(I_0)$$

which means that $\frac{\partial U}{\partial t}(t,\bar{x})$ is the weak derivative of $h(\bar{x})$. □

The next corollary follows straightforwardly.

Corollary 2.11. *The energy associated to a locally minimal solution on the interval* (a,b) *is in the Sobolev space* $W^{1,1}_{loc}((a,b),\mathbb{R})$.

We now investigate the behavior of the moment of inertia of a locally minimal solution when a singularity occurs (see Definition 2.6). The results contained in Proposition 2.12 and Corollary 2.13 are the natural extension of the classical Lagrange–Jacobi inequality to locally minimal solutions (see [53]).

Proposition 2.12. *Let* $\bar{x}$ *be a locally minimal solution and* I_0 *be given by Proposition 2.8. Then*

$$\frac{1}{2}\int_{I_0} I(\bar{x}(t))\ddot{\varphi}(t)\,dt \geq \int_{I_0}\left[2h(\bar{x}(t)) + (2-\tilde{\alpha})U(t,\bar{x}(t)) - C_2\right]\varphi(t)\,dt \tag{2.21}$$

for every $\varphi\in\mathcal{C}_0^\infty(I_0,\mathbb{R})$, $\varphi(t)\geq 0$.

Proof. Let $(x_\varepsilon)_\varepsilon$ be the sequence of global minimizers for the corresponding functionals $\bar{\mathcal{A}}_\varepsilon(\cdot, I_0)$ convergent to $\bar{x}$ whose existence is proved in Proposition 2.9. When we compute the second derivative of the moment of inertia of x_ε we obtain

$$\begin{aligned}\frac{1}{2}\ddot{I}(x_\varepsilon(t)) &= |\dot{x}_\varepsilon(t)|^2 + \ddot{x}_\varepsilon(t)\cdot x_\varepsilon(t) \\ &= 2h_\varepsilon(t) + 2U_\varepsilon(t,x_\varepsilon(t)) - |\bar{x}(t) - x_\varepsilon(t)|^2 \\ &\qquad + \left[\nabla U_\varepsilon(t,x_\varepsilon(t)) + (x_\varepsilon(t) - \bar{x}(t))\right]\cdot x_\varepsilon(t) \\ &= 2h_\varepsilon(t) + 2U_\varepsilon(t,x_\varepsilon(t)) + \bar{x}(t)\cdot(x_\varepsilon(t)-\bar{x}(t)) \\ &\qquad + \dot{\eta}_\varepsilon\left(U(t,x_\varepsilon)\right)\nabla U(t,x_\varepsilon(t))\cdot x_\varepsilon(t)\end{aligned}$$

hence, by assumption (U2) on the potential U and inequality (2.11), it follows that

$$\begin{aligned}\frac{1}{2}\ddot{I}(x_\varepsilon(t)) &\geq 2h_\varepsilon(t) + 2U_\varepsilon(t,x_\varepsilon(t)) + \bar{x}(t)\cdot(x_\varepsilon(t)-\bar{x}(t)) \\ &\qquad - \dot{\eta}_\varepsilon\left(U(t,x_\varepsilon)\right)\left[\alpha U(t,x_\varepsilon) + C_2\right] \\ &\geq 2h_\varepsilon(t) + (2-\tilde{\alpha})U_\varepsilon(t,x_\varepsilon(t)) + \bar{x}(t)\cdot(x_\varepsilon(t)-\bar{x}(t)) - C_2\end{aligned} \tag{2.22}$$

for some $\tilde{\alpha}\in(0,2)$ and $C_2>0$. Therefore, since $x_\varepsilon\in\mathcal{C}^2(I_0)$, for every $\varphi\in\mathcal{C}_0^\infty(I_0,\mathbb{R})$, $\varphi(t)\geq 0$

$$\frac{1}{2}\int_{I_0} I(x_\varepsilon(t))\ddot{\varphi}(t)\,dt = \frac{1}{2}\int_{I_0}\ddot{I}(x_\varepsilon(t))\varphi(t)\,dt$$

and, from (2.22),

$$\frac{1}{2}\int_{I_0} I(x_\varepsilon(t))\ddot{\varphi}(t)\,dt \geq \int_{I_0} [2h_\varepsilon(t) + (2-\tilde{\alpha})U_\varepsilon(t,x_\varepsilon(t)) + \bar{x}(t)\cdot(x_\varepsilon(t)-\bar{x}(t)) - C_2]\,\varphi(t)\,dt.$$

We conclude by passing to the limit as $\varepsilon \to 0$ in (2.22) and using the L^1-convergences proved in Propositions 2.9 and 2.10. □

The next corollaries follow directly.

Corollary 2.13 (**Lagrange–Jacobi inequality**). *Let $\bar{x}$ be given by Proposition 2.8. Then the following inequality holds in the distributional sense*

$$\frac{1}{2}\ddot{I}(\bar{x}(t)) \geq 2h(t) + (2-\tilde{\alpha})U(t,\bar{x}(t)) - C_2, \qquad \forall t \in (a,b).$$

Corollary 2.14. *Let $\bar{x}$ be given by Proposition 2.8. Then its moment of inertia is convex on I_0 whenever $\bar{x}$ has a singularity in t_0 and δ_0 is small enough.*

Proof. Whenever ε and δ_0 are sufficiently small, the right hand side of inequality (2.22) is strictly positive, indeed $h_\varepsilon(t)$ is bounded, x_ε converges to $\bar{x}$ uniformly and $U_\varepsilon(t,x_\varepsilon(t))$ diverges to $+\infty$. Whenever ε is small enough we conclude that

$$\ddot{I}(x_\varepsilon(t)) > 0$$

and hence $I(x_\varepsilon)$ are strictly convex functions in a neighborhood of t_0. Since the sequence $I(x_\varepsilon)$ uniformly converges to $I(\bar{x})$ we conclude that also $I(\bar{x})$ is convex on the interval I_0. □

We now investigate the possibility that a sequence of singularities accumulates at the right bound of the interval (a,b); in this section we will suppose that $b < +\infty$.

Lemma 2.15. *Let $\bar{x}$ be given in Proposition 2.8, h be its energy defined in* (2.18) *and fix $\tau \in (a,b)$ be such that*

$$\lambda := \frac{2-\tilde{\alpha}}{2} - C_1(b-\tau) \tag{2.23}$$

is a strictly positive constant. Then there exists a constant $K > 0$ such that

$$\left|\int_\tau^t h(s)ds\right| \leq \left(\frac{2-\tilde{\alpha}}{2} - \lambda\right)\int_\tau^t U(s,\bar{x}(s))ds + K, \qquad \forall t \in (\tau,b). \tag{2.24}$$

Proof. Since h is absolutely continuous on every interval $[\tau,t] \subset (a,b)$ (Corollary 2.11) we have

$$|h(t)| \leq |h(\tau)| + \int_\tau^t |\dot{h}(\xi)|\,d\xi, \qquad \forall t \in (\tau,b).$$

From Proposition 2.10 and assumption (U1) we obtain

$$\begin{aligned} |h(t)| &\leq |h(\tau)| + \int_\tau^t \left|\frac{\partial U}{\partial \xi}(\xi,\bar{x}(\xi))\right|\,d\xi \\ &\leq |h(\tau)| + C_1\int_\tau^t (U(\xi,\bar{x}(\xi)) + 1)\,d\xi \end{aligned} \tag{2.25}$$

and integrating both sides of the inequality on the interval $[\tau,t]$

$$\int_\tau^t |h(s)|ds \leq |h(\tau)|(t-\tau) + C_1\frac{(t-\tau)^2}{2} + C_1\int_\tau^t ds\int_\tau^s U(\xi,\bar{x}(\xi))\,d\xi.$$

Since U is positive, the integral $\int_\tau^s U(\xi,\bar{x}(\xi))\,d\xi$ increases in the variable s, hence we conclude

$$\left|\int_\tau^t h(s)ds\right| \leq \int_\tau^t |h(s)|ds \leq K + C_1(b-\tau)\int_\tau^t U(\xi,\bar{x}(\xi))\,d\xi.$$

where $K := |h(\tau)|(t-\tau) + C_1(t-\tau)^2/2$. □

Lemma 2.16. *Let $\bar{x}$ be given in Proposition 2.8 and τ be chosen as in Lemma 2.15. Suppose that there exist $\delta, C > 0$ such that*

$$I(\bar{x}(t)) \leq C, \text{ for every } t \in (b-\delta, b)$$

and

$$\liminf_{t \to b^-} \dot{I}(\bar{x}(t)) \leq C. \tag{2.26}$$

Then there exists $\tau \in (a, b)$ such that

$$\int_\tau^b U(t, \bar{x}(t))\, dt < +\infty.$$

Proof. If b is not a singularity for $\bar{x}$, the assertion follows from assumption (2.8). Otherwise, it follows from (2.26) that there exists an increasing sequence $(t_n)_n$ such that

$$t_n \to b \text{ as } n \to +\infty \quad \text{and} \quad \dot{I}(\bar{x}(t_n)) \leq C, \forall n.$$

Now let N be an integer such that $t_N \in (b-\delta, b)$ and the constant λ defined in (2.23), with $\tau = t_N$, is strictly positive. Hence, for every index $n > N$,

$$2C \geq \dot{I}(\bar{x}(t_n)) - \dot{I}(\bar{x}(t_N)) = \int_{t_N}^{t_n} \ddot{I}(\bar{x}(t))\, dt.$$

Corollary 2.13 implies that

$$C \geq 2\int_{t_N}^{t_n} h(t)\, dt + (2-\tilde{\alpha})\int_{t_N}^{t_n} U(t, \bar{x}(t))\, dt - C_2(t_n - t_N).$$

We now apply Lemma 2.15 to deduce that

$$2\lambda \int_{t_N}^{t_n} U(t, \bar{x}(t))\, dt \leq C_2(t_n - t_N) + C + 2K. \tag{2.27}$$

Since $\lambda > 0$ is fixed, as $n \to +\infty$ the proof is completed. □

Corollary 2.17. *Let $\bar{x}$ be a generalized solution on (a, b). Suppose that*

$$\limsup_{t \to b^-} I(\bar{x}(t)) < +\infty, \qquad \text{and} \qquad \liminf_{t \to b^-} \dot{I}(\bar{x}(t)) < +\infty. \tag{2.28}$$

Then, if $-\infty < a < \tau < b < +\infty$ there hold

(i) $\displaystyle\int_\tau^b U(t, \bar{x}(t))\, dt < +\infty;$

(ii) $\displaystyle\left|\int_\tau^b h(t)\, dt\right| < +\infty;$

(iii) $\displaystyle\int_\tau^b K(\dot{\bar{x}}(t))\, dt < +\infty;$

(iv) $\|h\|_\infty < +\infty$ *on* $[\tau, b)$.

(v) $\displaystyle\lim_{t \to b^-} \bar{x}(t)$ *exists.*

Proof. We first prove the assertions in the case of locally minimal solutions. The boundedness of the first integral follows from the assumption of local boundedness of the action functional on a locally minimal trajectory, assumption (2.8), and from Lemma 2.16. Concerning the second one, we use Corollary 2.11 and inequality (2.24); *(iii)* follows straightforwardly from *(i)*, *(ii)* and the definition of the energy h. The boundedness of $\|h\|_\infty$ on

(a,b) follows from Corollary 2.11 and inequality (2.25). To deduce *(v)* it is sufficient to remark that, from *(iii)*, $\bar{x}$ is Hölder-continuous on (a,b).

In order to extend the proof to generalized solutions, we first remark that all the constants and bounds appearing in the proof above do not depend on the specific solution $\bar{x}$, but only on the potential and the limits in equations 2.28, and the total energy valued at single instant $h(\tau)$ of the interval. Hence the assertions *(i)*, *(ii)* still hold true when passing to pointwise limit such that the energy $h(\tau)$ is bounded. The other assertions then follow from the fist two. □

Remark 2.18. In Corollary 2.17 *(v)*, Von Zeipel's Theorem is proved, for generalized solutions, under the additional assumption (2.26). The proof will be completed in Section 4.

Remark 2.19. From inequality (2.25) we can easily understand that in Definition 2.5 the convergence of the energy of the approximating sequence of locally minimal solutions can be assumed only at one point.

3 Asymptotic estimates at total collisions

The purpose of this section is to deepen the analysis of the asymptotics of generalized solutions as they approach a total collision at the origin; to this aim, we introduce some further hypothesis on the potential U. Though we will perform all the analysis in a left neighborhood of the collision instant, the analysis concerning right neighborhoods is the exact analogue.

We recall that $\bar{x}$ has a *t*otal collision at the origin at $t = t^*$ if $\lim_{t\to t^*} \bar{x}(t) = x(t^*) = 0$. Since by our assumptions 0 belongs to the singular set Δ of the potential, assumption (U0) reads that a total collision instant is a singularity for $\bar{x}$.

The results proved in § 2.3 have some relevant consequences in the case of total collisions at the origin; in particular Corollary 2.14 now reads

Corollary 3.1. *Let $\bar{x}$ be given by Proposition 2.8. If $|\bar{x}(t_0)| = 0$, then there exists $\delta_0 > 0$ such that $I(\bar{x})$ is continuous on $I_0 = [t_0 - \delta_0, t_0 + \delta_0]$, it admits weak derivative almost everywhere, the function $\dot{I}(\bar{x})$ is monotone increasing and $\dot{I}(\bar{x}) \in BV(I_0)$. Furthermore the following inequalities hold in the distributional sense*

$$\begin{aligned} \ddot{I}(\bar{x}(t)) &> 0 & \forall t &\in \bar{I}_0 \\ \dot{I}(\bar{x}(t)) &< 0 & \forall t &\in (t_0 - \delta_0, t_0) \\ \dot{I}(\bar{x}(t)) &> 0 & \forall t &\in (t_0, t_0 + \delta_0). \end{aligned}$$

Furthermore, since $I(\bar{x}(t)) \geq 0$, and $I(\bar{x}(t^*)) = 0$ if and only if $\bar{x}$ has a total collision at the origin at $t = t^*$, from Corollary 2.14 one can deduce that, whenever a total collision occurs at $t = t_0$, no other total collisions take place in the interval I_0. Concerning the occurrence of total collision at the boundary of the interval (a,b), we argue as in Lemmata 2.15 and 2.16 and we use the convexity of the function I to deduce that also boundary total collisions are isolated. It is worthwhile noticing that this fact does not prevent, at this stage, the occurrence of infinitely many other singularities in a neighborhood of a total collision at the origin. We summarize these remarks in the next theorem.

Theorem 1. *Let $\bar{x}$ be a generalized solution for the dynamical system* (2.2). *Suppose that $-\infty < a < b < +\infty$ and that there exists $t_0 \in [a,b]$ such that $|\bar{x}(t_0)| = 0$. Then there exists $\delta > 0$ such that, for every $t \in (t_0 - \delta, t_0 + \delta) \cap [a,b]$, $t \neq t_0$, we have $|\bar{x}(t)| \neq 0$.*

In terms of the radial variable r Corollary 3.1 and Theorem 1 state that whenever $r(t_0) = 0$ for some $t_0 \in (a,b)$ (t_0 can coincide with a or b when finite) then there exists $\delta > 0$ such that

$$\begin{aligned} r(t) &\neq 0, & \dot{r}(t) &< 0, & \forall t &\in (t_0 - \delta, t_0) \\ r(t) &\neq 0, & \dot{r}(t) &> 0, & \forall t &\in (t_0, t_0 + \delta). \end{aligned} \tag{3.1}$$

We moreover rewrite the bounded energy function as

$$h(t) = \frac{1}{2}\left(\dot r^2 + r^2|\dot s|^2\right) - U(t, rs). \tag{3.2}$$

Similarly, denoting by $(x_\varepsilon)_\varepsilon$ the sequence of global minimizers for $\bar{\mathcal{A}}_\varepsilon(\cdot, [t_0-\delta, t_0+\delta])$ converging to the locally minimal collision solution $\bar x$ whose existence is proved in Proposition 2.9, we define, for every ε,

$$r_\varepsilon := |x_\varepsilon| \in \mathbb{R} \qquad \text{and} \qquad s := \frac{x_\varepsilon}{|x_\varepsilon|} \in \mathcal{E}$$

and we write the energy in (2.19) as

$$h_\varepsilon(t) = \frac{1}{2}\left(\dot r_\varepsilon^2 + r_\varepsilon^2|\dot s_\varepsilon|^2\right) - U_\varepsilon(t, r_\varepsilon s_\varepsilon) + \frac{1}{2}|rs - r_\varepsilon s_\varepsilon|^2.$$

Furthermore the approximating action functional and the corresponding Euler–Lagrange equations in the new variables are respectively

$$\bar{\mathcal{A}}_\varepsilon(r_\varepsilon s_\varepsilon, [t_0-\delta, t_0+\delta]) := \int_{t_0-\delta}^{t_0+\delta} \frac{1}{2}\left(\dot r_\varepsilon^2 + r_\varepsilon^2|\dot s_\varepsilon|^2\right) + U_\varepsilon(t, r_\varepsilon s_\varepsilon) + \frac{1}{2}|rs - r_\varepsilon s_\varepsilon|^2\, dt$$

and

$$\begin{aligned} -\ddot r_\varepsilon + r_\varepsilon|\dot s_\varepsilon|^2 + \nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon)\cdot s_\varepsilon - (rs - r_\varepsilon s_\varepsilon)\cdot s_\varepsilon = 0 \\ -2r_\varepsilon \dot r_\varepsilon \dot s_\varepsilon - r_\varepsilon^2 \ddot s_\varepsilon + r_\varepsilon \nabla_T U_\varepsilon(t, r_\varepsilon s_\varepsilon) - r_\varepsilon(rs - r_\varepsilon s_\varepsilon) = \mu_\varepsilon s_\varepsilon, \end{aligned} \tag{3.3}$$

where $\mu_\varepsilon = r_\varepsilon^2|\dot s_\varepsilon|^2 - r_\varepsilon(rs - r_\varepsilon s_\varepsilon)\cdot s_\varepsilon$ is the Lagrange multiplier due to the presence of the constraint $|s_\varepsilon|^2 = 1$ and the vector $\nabla_T U_\varepsilon(t, r_\varepsilon s_\varepsilon)$ is the tangent components to the ellipsoid $\mathcal{E}$ of the gradient $\nabla U_\varepsilon(t, r_\varepsilon s_\varepsilon)$. A similar approximation procedure will be implicitly done for generalized solutions.

To proceed with the analysis of the asymptotic behavior near total collisions at the origin we need some stronger conditions on the potential U when the radial variable r tends to 0. These additional conditions includes quasi–homogeneous potential and logarithmic ones, in the following analysis, however we will treat separately the two different cases.

3.1 Quasi-homogeneous potentials

In this section we impose some stronger assumptions on the behavior of the potential when $|x|$ is small. The following conditions are trivially satisfied by α-homogeneous potentials and mimic the behavior of combination of such homogeneous potentials:

(U2)$_{\mathbf{h}}$ There exist $\alpha \in (0, 2)$, $\gamma > l$ and $C_2 \geq 0$ such that

$$\nabla U(t, x)\cdot x + \alpha U(t, x) \geq -C_2|x|^\gamma U(t, x),$$

whenever $|x|$ is small.

Remark 3.2. (U2)$_{\mathrm{h}}$ implies (U2) (for small values of $|x|$); in fact, by choosing $2 > \tilde\alpha > \alpha > 0$, one obtains

$$\begin{aligned} \nabla U(t, x)\cdot x + \tilde\alpha U(t, x) &= \nabla U(t, x)\cdot x + \alpha U(t, x) + (\tilde\alpha - \alpha)U(t, x) \\ &\geq -C_2|x|^\gamma U(t, x) + (\tilde\alpha - \alpha)U(t, x), \end{aligned}$$

and the last term remains bounded below as $|x| \to 0$ since $\tilde\alpha - \alpha > 0$.

Furthermore we suppose the existence of a function $\tilde U$ defined and of class $\mathcal{C}^1$ on $(a, b) \times (\mathcal{E}\setminus\Delta)$ such that

$$\inf_{(a,b)\times(\mathcal{E}\setminus\Delta)} \tilde U(t, s) > 0 \quad \text{and} \quad \lim_{s\to\mathcal{E}\cap\Delta} \tilde U(t, s) = +\infty \qquad \text{uniformly in } t. \tag{3.4}$$

The potential U is then supposed to verify the following condition uniformly in the variables t and s (on the compact subsets of $(a, b) \times (\mathcal{E}\setminus\Delta)$):

(U3)$_{\mathbf{h}}$ $\lim_{r\to 0} r^\alpha U(t,x) = \tilde U(t,s)$.

Remark 3.3. In (U2)$_{\rm h}$ and (U3)$_{\rm h}$ the value of α must be the same. We shall refer to potentials satisfying such assumptions as quasi–homogeneous (cf. [19]).

Lemma 3.4. *Let $\bar x$ be a generalized solution, let $t_0 \in (a,b]$ be a total collision instant and let δ be given in Theorem 1. Then, for every $\alpha' \in (\alpha, 2)$, we have*

$$\int_{t_0-\delta}^{t_0} -r^{\alpha'}\frac{\dot r}{r} U(t,rs)\,dt < +\infty,$$

where $\alpha \in (0,2)$ is the constant fixed in assumption (U2)$_{\rm h}$.

Proof. We consider the function

$$\Gamma_{\alpha'}(t) := r^{\alpha'}\left(\frac12 r^2|\dot s|^2 - U(t,rs)\right), \quad \alpha' \in (\alpha,2);$$

Replacing in (3.2) we have

$$\Gamma_{\alpha'}(t) = h(t)r^{\alpha'} - \frac12 \dot r^2 r^{\alpha'} \le h(t) r^{\alpha'};$$

since h is bounded (see Corollary 2.17, *(iv)*) and r tends to 0, we conclude that the function $\Gamma_{\alpha'}$ is bounded above on the interval $[t_0-\delta, t_0]$. We consider the corresponding functions (still bounded above) for the approximating problems:

$$\begin{aligned}\Gamma_{\alpha',\varepsilon}(t) &= r_\varepsilon^{\alpha'}\left(\frac12 r_\varepsilon^2|\dot s_\varepsilon|^2 - U_\varepsilon(t,r_\varepsilon s_\varepsilon) + \frac12|rs - r_\varepsilon s_\varepsilon|^2\right)\\ &= h_\varepsilon(t) r_\varepsilon^{\alpha'} - \frac12 \dot r_\varepsilon^2 r_\varepsilon^{\alpha'} \le h_\varepsilon(t) r_\varepsilon^{\alpha'},\end{aligned}$$

and we observe that the sequence $(\Gamma_{\alpha',\varepsilon})_\varepsilon$ converges almost everywhere and L^1 to $\Gamma_{\alpha'}$, as $\varepsilon \to 0$. We compute the derivative of $\Gamma_{\alpha',\varepsilon}(t)$ with respect to time as

$$\begin{aligned}\frac{d}{dt}\Gamma_{\alpha',\varepsilon}(t) &= \frac{2+\alpha'}{2} r_\varepsilon^{1+\alpha'}\dot r_\varepsilon |\dot s_\varepsilon|^2\\ &\quad + r_\varepsilon^{2+\alpha'}\dot s_\varepsilon\cdot\ddot s_\varepsilon + \alpha' r_\varepsilon^{\alpha'-1}\dot r_\varepsilon\left[\frac12|rs - r_\varepsilon s_\varepsilon|^2 - U_\varepsilon(t,r_\varepsilon s_\varepsilon)\right] \qquad (3.5)\\ &- r_\varepsilon^{\alpha'}\left[\frac{\partial U_\varepsilon}{\partial t}(t,r_\varepsilon s_\varepsilon) + \nabla U_\varepsilon(t,r_\varepsilon s_\varepsilon)(\dot r_\varepsilon s_\varepsilon + r_\varepsilon\dot s_\varepsilon)\right] + r_\varepsilon^{\alpha'}(rs - r_\varepsilon s_\varepsilon)\frac{d}{dt}(rs - r_\varepsilon s_\varepsilon).\end{aligned}$$

Now we multiply the Euler–Lagrange equation $(3.3)_2$ by $\dot s_\varepsilon$ to obtain (we recall that $\nabla_T U_\varepsilon(t,r_\varepsilon s_\varepsilon)\cdot\dot s_\varepsilon = \nabla U_\varepsilon(t,r_\varepsilon s_\varepsilon)\cdot\dot s_\varepsilon$ since s_ε and $\dot s_\varepsilon$ are orthogonal)

$$r_\varepsilon^2\ddot s_\varepsilon\cdot\dot s_\varepsilon = -2r_\varepsilon\dot r_\varepsilon|\dot s_\varepsilon|^2 + r_\varepsilon\nabla U_\varepsilon(t,r_\varepsilon s_\varepsilon)\cdot\dot s_\varepsilon - r_\varepsilon rs\cdot s_\varepsilon. \qquad (3.6)$$

Replacing (3.6) in (3.5) we have

$$\begin{aligned}\frac{d}{dt}\Gamma_{\alpha',\varepsilon}(t) = &-\frac{2-\alpha'}{2} r^{1+\alpha'}\dot r_\varepsilon|\dot s_\varepsilon|^2 - \alpha' r_\varepsilon^{\alpha'-1}\dot r_\varepsilon U_\varepsilon(t,r_\varepsilon s_\varepsilon) - r_\varepsilon^{\alpha'}\frac{\partial U_\varepsilon}{\partial t}(t,r_\varepsilon s_\varepsilon)\\ &- r_\varepsilon^{\alpha'-1}\dot r_\varepsilon\nabla U_\varepsilon(t,r_\varepsilon s_\varepsilon)\cdot(r_\varepsilon s_\varepsilon) - r_\varepsilon^{\alpha'+1} rs\cdot s_\varepsilon\\ &+ \frac{\alpha'}{2} r_\varepsilon^{\alpha'-1}\dot r_\varepsilon|rs - r_\varepsilon s_\varepsilon|^2 + r_\varepsilon^{\alpha'}(rs - r_\varepsilon s_\varepsilon)\frac{d}{dt}(rs - r_\varepsilon s_\varepsilon). \qquad (3.7)\end{aligned}$$

We now combine assumptions (U1), (U2)$_{\rm h}$ and (2.11) to obtain the following inequalities

$$\begin{aligned}-r_\varepsilon^{\alpha'}\frac{\partial U_\varepsilon}{\partial t}(t,r_\varepsilon s_\varepsilon) &= -r_\varepsilon^{\alpha'}\dot\eta_\varepsilon(U(t,r_\varepsilon s_\varepsilon))\frac{\partial U}{\partial t}(t,r_\varepsilon s_\varepsilon)\\ &\ge -C_1 r_\varepsilon^{\alpha'}\dot\eta_\varepsilon(U(t,r_\varepsilon s_\varepsilon))\left(U_\varepsilon(t,r_\varepsilon s_\varepsilon)+1\right)\\ &\ge -C_1 r_\varepsilon^{\alpha'}\left(\eta_\varepsilon(U(t,r_\varepsilon s_\varepsilon))+1\right)\\ &\ge -C_1 r_\varepsilon^{\alpha'}\left(U_\varepsilon(t,r_\varepsilon s_\varepsilon)+1\right),\end{aligned}$$

$$
\begin{aligned}
-r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}\nabla U_\varepsilon(t,r_\varepsilon s_\varepsilon)\cdot(r_\varepsilon s_\varepsilon) &= -r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}\dot\eta_\varepsilon(U(t,r_\varepsilon s_\varepsilon))\nabla U(t,r_\varepsilon s_\varepsilon)\cdot(r_\varepsilon s_\varepsilon)\\
&\geq -r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}\dot\eta_\varepsilon(U(t,r_\varepsilon s_\varepsilon))U(t,r_\varepsilon s_\varepsilon)\left[-\alpha - C_2 r_\varepsilon^\gamma\right]\\
&\geq r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}U_\varepsilon(t,r_\varepsilon s_\varepsilon)\left[\alpha + C_2 r_\varepsilon^\gamma\right].
\end{aligned}
$$

Finally, by replacing in (3.7), we obtain

$$
\frac{d}{dt}\Gamma_{\alpha',\varepsilon}(t) \geq \Psi_{\alpha',\varepsilon}(t)
$$

where

$$
\begin{aligned}
\Psi_{\alpha',\varepsilon}(t) = &-\frac{2-\alpha'}{2}r_\varepsilon^{1+\alpha'}\dot r_\varepsilon|\dot s_\varepsilon|^2 - (\alpha'-\alpha)r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}U_\varepsilon(t,r_\varepsilon s_\varepsilon) - C_1 r_\varepsilon^{\alpha'}\left(U_\varepsilon(t,r_\varepsilon s_\varepsilon)+1\right)\\
&+ C_2 r_\varepsilon^{\alpha'+\gamma}\frac{\dot r_\varepsilon}{r_\varepsilon}U_\varepsilon(t,r_\varepsilon s_\varepsilon) - r_\varepsilon^{\alpha'+1} rs\cdot s_\varepsilon\\
&+ \frac{\alpha'}{2}r_\varepsilon^{\alpha'-1}\dot r_\varepsilon|rs - r_\varepsilon s_\varepsilon|^2 + r_\varepsilon^{\alpha'}(rs - r_\varepsilon s_\varepsilon)\frac{d}{dt}(rs - r_\varepsilon s_\varepsilon).
\end{aligned}
$$

Since $\gamma > 0$ and $r_\varepsilon \to 0$ as $t \to t_0$, for every $\varepsilon > 0$, there exists a positive $\lambda_\varepsilon \leq (\alpha'-\alpha)/2$ such that

$$
C_2 r_\varepsilon^{\alpha'+\gamma}U_\varepsilon(t,r_\varepsilon s_\varepsilon) \leq \lambda_\varepsilon r_\varepsilon^{\alpha'}U_\varepsilon(t,r_\varepsilon s_\varepsilon)
$$

whenever δ is small enough; furthermore, since $-\frac{2-\alpha'}{2}r_\varepsilon^{1+\alpha'}\dot r_\varepsilon|\dot s_\varepsilon|^2$ is positive, we have

$$
\begin{aligned}
\Psi_{\alpha',\varepsilon}(t) \geq &-(\alpha'-\alpha-\lambda_\varepsilon)r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}U_\varepsilon(t,r_\varepsilon s_\varepsilon) - C_1 r_\varepsilon^{\alpha'}\left(U_\varepsilon(t,r_\varepsilon s_\varepsilon)+1\right)\\
&- r_\varepsilon^{\alpha'+1} rs\cdot s_\varepsilon + \frac{\alpha'}{2}r_\varepsilon^{\alpha'-1}\dot r_\varepsilon|rs - r_\varepsilon s_\varepsilon|^2 + r_\varepsilon^{\alpha'}(rs - r_\varepsilon s_\varepsilon)\frac{d}{dt}(rs - r_\varepsilon s_\varepsilon).
\end{aligned}
$$

Therefore, for every ε, the function $\Psi_{\alpha',\varepsilon}$ is larger than the sum of a positive term

$$
-(\alpha'-\alpha-\lambda_\varepsilon)r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}U_\varepsilon(t,r_\varepsilon s_\varepsilon),
$$

an integrable term

$$
-C_1 r_\varepsilon^{\alpha'}\left(U_\varepsilon(t,r_\varepsilon s_\varepsilon)+1\right)
$$

and a remainder

$$
-r_\varepsilon^{\alpha'+1} rs\cdot s_\varepsilon + \frac{\alpha'}{2}r_\varepsilon^{\alpha'-1}\dot r_\varepsilon|rs - r_\varepsilon s_\varepsilon|^2 + r_\varepsilon^{\alpha'}(rs - r_\varepsilon s_\varepsilon)\frac{d}{dt}(rs - r_\varepsilon s_\varepsilon)
$$

converging uniformly to $-r^{\alpha'+2}$ as ε tends to 0.

We can then conclude that, for every ε

$$
\begin{aligned}
\Gamma_{\alpha',\varepsilon}(t_0) - \Gamma_{\alpha',\varepsilon}(t_0-\delta) \geq &-\int_{t_0-\delta}^{t_0}(\alpha'-\alpha-\lambda_\varepsilon)r_\varepsilon^{\alpha'}\frac{\dot r_\varepsilon}{r_\varepsilon}U_\varepsilon(t,r_\varepsilon s_\varepsilon)\,dt\\
&- C_1\int_{t_0-\delta}^{t_0} r_\varepsilon^{\alpha'}\left(U_\varepsilon(t,r_\varepsilon s_\varepsilon)+1\right)dt\\
+\int_{t_0-\delta}^{t_0}\left[-r_\varepsilon^{\alpha'+1} rs\cdot s_\varepsilon + \frac{\alpha'}{2}r_\varepsilon^{\alpha'-1}\dot r_\varepsilon|rs - r_\varepsilon s_\varepsilon| + r_\varepsilon^{\alpha'}(rs - r_\varepsilon s_\varepsilon)\frac{d}{dt}(rs - r_\varepsilon s_\varepsilon)\right]dt.
\end{aligned}
$$

The right hand side of the last inequality is bounded above because of the boundedness of $\Gamma_{\alpha',\varepsilon}$. Passing to the limit as $\varepsilon \to 0$, from Proposition 2.9 and the boundedness above of the function $\Gamma_{\alpha'}$ it follows that

$$
\begin{aligned}
\Gamma_{\alpha'}(t_0) - \Gamma_{\alpha'}(t_0-\delta) \geq &-\int_{t_0-\delta}^{t_0}(\alpha'-\alpha-\lambda)r^{\alpha'}\frac{\dot r}{r}U(t,rs)\,dt\\
&- C_1\int_{t_0-\delta}^{t_0} r^{\alpha'}\left(U(t,rs)+1\right)dt - \int_{t_0-\delta}^{t_0} r^{\alpha'+2}\,dt,
\end{aligned}
$$

where $\lambda \leq (\alpha'-\alpha)/2$ is the limit, up to subsequences, of the bounded sequence $(\lambda_\varepsilon)_\varepsilon$. Now we recall that, by Lemma 2.16, $U(t,rs)$ is integrable; this fact implies the integrability of the function $r^{\alpha'}U(t,rs) - r^{\alpha'+2}$ and hence the existence of a constant K such that

$$\int_{t_0-\delta}^{t_0} -r^{\alpha'}\frac{\dot r}{r}U(t,rs)dt \leq K < +\infty.$$

□

Lemma 3.5 (**Monotonicity Formula).** *Let $\bar x$ be a generalized solution, let $t_0 \in (a,b]$ be a total collision instant and let $\delta > 0$ be the constant obtained in Theorem 1. Then the function*

$$\Gamma_\alpha(t) := r^\alpha\left[\frac{1}{2}r^2|\dot s|^2 - U(t,rs)\right] \tag{3.8}$$

is bounded on $[t_0-\delta, t_0)$ and

$$\begin{aligned}\Gamma_\alpha(t) \geq \Gamma_\alpha(t_0-\delta) &- \int_{t_0-\delta}^{t}\frac{2-\alpha}{2}r^{1+\alpha}\dot r|\dot s|^2\,d\xi \\ &- C_1\int_{t_0-\delta}^{t} r^\alpha\left(U(\xi,rs)+1\right)\,d\xi + C_2\int_{t_0-\delta}^{t} r^{\alpha+\gamma}\frac{\dot r}{r}U(\xi,rs)\,d\xi\end{aligned} \tag{3.9}$$

where $t\in[t_0-\delta,t_0]$.

Proof. Replacing in (3.2) the expression of the function Γ_α we have

$$\Gamma_\alpha(t) = h(t)r^\alpha - \dot r^2 r^\alpha \leq h(t)r^\alpha;$$

since h is bounded (see Corollary 2.17) and r tends to 0, we conclude that the function Γ_α is bounded above. Using the same approximation arguments described in Lemma 3.4, we obtain (3.9). From Lemma 3.4 we deduce the integrability of the negative function $r^{\alpha+\gamma}\frac{\dot r}{r}U(t,rs)$. Hence, since $-\frac{2-\alpha}{2}r^{1+\alpha}\dot r|\dot s|^2$ is positive and both $r^\alpha U(t,rs)$ and $r^{\alpha+\gamma}\frac{\dot r}{r}U(t,rs)$ are integrable (Lemma 3.4), the boundedness below of the function Γ_α follows from (3.9). □

Corollary 3.6. *In the same setting of Lemma 3.5 we have* $\displaystyle\int_{t_0-\delta}^{t_0} -r^{1+\alpha}\dot r|\dot s|^2 < +\infty.$

Proof. It follows from the boundedness above of the function Γ_α and inequality (3.9) since the terms $-C_1\int_{t_0-\delta}^{t} r^\alpha\left(U(t,rs)+1\right)dt$ and $C_2\int_{t_0-\delta}^{t} r^{\alpha+\gamma}\frac{\dot r}{r}U(t,rs)\,dt$ are negative. □

Lemma 3.7. *Let $\varphi(t) := -\dot r(t)r^{\alpha/2}(t)$, $t\in[t_0-\delta,t_0]$. Then there exist two constants depending on α, $c_{1,\alpha}\leq c_{2,\alpha}$, such that for all $t\in[t_0-\delta,t_0]$*

$$c_{1,\alpha} \leq \varphi(t) \leq c_{2,\alpha}.$$

Proof. Since the energy function h is bounded (see Corollary 2.17, *(iv)*) and we assume that r tends to 0 as t tends to t_0, the function $r^\alpha h(t) = \dfrac{1}{2}\varphi^2(t) + \Gamma_\alpha(t)$ is also bounded and by Lemma 3.5 we can deduce that $\varphi(t) = \sqrt{2\left[r^\alpha h(t) - \Gamma_\alpha(t)\right]}$ is bounded below and above by a pair of constants $0 \leq c_{1,\alpha}\leq c_{2,\alpha}$ on the interval $[t_0-\delta,t_0]$. □

Corollary 3.8. *In the same setting of Lemma 3.5 we have* $\displaystyle\lim_{t\to t_0}\int_{t_0-\delta}^{t}\frac{1}{r^{\alpha/2+1}} = +\infty.$

Proof. We can write the boundedness above of the function φ (proved in Lemma 3.7) as

$$-\frac{\dot r}{r} \leq \frac{c_{2,\alpha}}{r^{\alpha/2+1}}, \quad t\in[t_0-\delta,t_0]. \tag{3.10}$$

Integrating inequality (3.10) on the interval $[t_0-\delta,t]$, when $t\to t_0$, we obtain

$$\lim_{t\to t_0} c_{2,\alpha}\int_{t_0-\delta}^{t}\frac{d\xi}{r^{\alpha/2+1}} \geq \lim_{t\to t_0}\int_{t_0-\delta}^{t} -\frac{\dot r}{r}d\xi = \log r(t_0-\delta) - \lim_{t\to t_0}\log r(t) = +\infty$$

since r tends to 0 as $t\to t_0$. □

Lemma 3.9. *The lower bound $c_{1,\alpha}$ of the function φ defined in Lemma 3.7 can be chosen strictly positive, that is $c_{1,\alpha}>0$.*

Proof. We start proving an estimate above of the derivative of the function φ. With this purpose we consider the approximating sequence $(\varphi_\varepsilon)_\varepsilon$ where

$$\varphi_\varepsilon(t) = -\dot r_\varepsilon(t) r_\varepsilon^{\alpha/2}(t)$$

and, for every $\varepsilon>0$, we compute the first derivative of the smooth function φ_ε and we use the Euler–Lagrange equation $(3.3)_1$ for the approximating problem to obtain

$$\begin{aligned}\dot\varphi_\varepsilon(t) &= -\frac{\alpha}{2}r_\varepsilon^{\alpha/2-1}\dot r_\varepsilon^2 - r_\varepsilon^{\alpha/2}\ddot r_\varepsilon\\ &= -\frac{\alpha}{2}r_\varepsilon^{\alpha/2-1}\dot r_\varepsilon^2 - r_\varepsilon^{\alpha/2+1}|\dot s_\varepsilon|^2 - r_\varepsilon^{\alpha/2-1}\nabla U_\varepsilon(t,r_\varepsilon s_\varepsilon)\cdot(r_\varepsilon s_\varepsilon) + r_\varepsilon^{\alpha/2}(rs - r_\varepsilon s_\varepsilon)\cdot s_\varepsilon.\end{aligned}$$

Arguing as in the proof of Lemma 3.4 we use assumptions $(\mathrm{U2})_\mathrm{h}$ and (2.11) to deduce

$$\begin{aligned}\dot\varphi_\varepsilon(t) &\leq r_\varepsilon^{\alpha/2-1}\left[-\frac{\alpha}{2}\dot r_\varepsilon^2 - r_\varepsilon^2|\dot s_\varepsilon|^2 + (\alpha + C_2 r_\varepsilon^\gamma)U_\varepsilon(t,r_\varepsilon s_\varepsilon) + (rs - r_\varepsilon s_\varepsilon)\cdot(r_\varepsilon s_\varepsilon)\right]\\ &= \frac{1}{r_\varepsilon^{\alpha/2+1}}\Big[\frac{2-\alpha}{2}\varphi_\varepsilon^2(t) - 2r_\varepsilon^\alpha h_\varepsilon(t) - (2-\alpha)r_\varepsilon^\alpha U_\varepsilon(t,r_\varepsilon s_\varepsilon) + C_2 r_\varepsilon^{\alpha+\gamma}U_\varepsilon(t,r_\varepsilon s_\varepsilon)\\ &\qquad + r_\varepsilon^\alpha(rs - r_\varepsilon s_\varepsilon)\cdot(rs)\Big]\end{aligned}$$

and then, for every $\in (t_0-\delta,t_0)$,

$$\begin{aligned}\varphi_\varepsilon(t) \leq \varphi_\varepsilon^0 + \int_{t_0-\delta}^{t}\frac{1}{r_\varepsilon^{\alpha/2+1}}\Big[&\frac{2-\alpha}{2}\varphi_\varepsilon^2(\xi) - 2r_\varepsilon^\alpha h_\varepsilon(\xi) - (2-\alpha)r_\varepsilon^\alpha U_\varepsilon(\xi,r_\varepsilon s_\varepsilon)\\ &+ C_2 r_\varepsilon^{\alpha+\gamma}U_\varepsilon(\xi,r_\varepsilon s_\varepsilon) + r_\varepsilon^\alpha(rs - r_\varepsilon s_\varepsilon)\cdot(rs)\Big]d\xi\end{aligned}$$

where $\varphi_\varepsilon^0 = \varphi_\varepsilon(t_0-\delta)$. As $\varepsilon\to 0$, from Proposition 2.9 we have

$$\begin{aligned}&\varphi(t) \leq \varphi^0\\ &+ \int_{t_0-\delta}^{t}\frac{1}{r^{\alpha/2+1}}\left[\frac{2-\alpha}{2}\varphi^2(\xi) - 2r^\alpha h_\varepsilon(\xi) - (2-\alpha)r^\alpha U(\xi,rs) + C_2 r^{\alpha+\gamma}U(\xi,rs)\right]d\xi,\end{aligned}$$

where $\varphi^0 = \varphi(t_0-\delta)$. Since $\gamma>0$ there exists $\lambda\in(0,2-\alpha)$, such that

$$\varphi(t) \leq \varphi^0 + \int_{t_0-\delta}^{t}\frac{1}{r^{\alpha/2+1}}\left[\frac{2-\alpha}{2}\varphi^2(\xi) + C(\xi) - (2-\alpha-\lambda)r^\alpha U(\xi,rs)\right]d\xi,$$

where $C(t)$ is such that $|C(t)|\to 0$ as $t\to t_0$ and $2r^\alpha h(t)\leq C(t)$ on $[t_0-\delta,t_0]$. Furthermore, the uniform convergence assumed in condition $(\mathrm{U3})_\mathrm{h}$ implies that, denoting by $\tilde U_0$ the minimal value assumed by $\tilde U$ on the ellipsoid $\mathcal{E}$, there exist two positive constants $k_1,k_2>0$ such that

$$\varphi(t) \leq \varphi^0 + \int_{t_0-\delta}^{t}\frac{k_1}{r^{\alpha/2+1}}\left(\varphi^2(\xi) - k_2\tilde U_0\right)d\xi$$

whenever δ is sufficiently small. We will conclude showing that necessarily $\varphi^2(t)\geq k_2\tilde U_0$ and then choosing $c_{1,\alpha} := \sqrt{k_2\tilde U_0} > 0$.

By the sake of contradiction we suppose the existence of $\hat{t}$ such that $\varphi^2(\hat{t}) < k_2\tilde{U}_0$; then $\varphi^2 - k_2\tilde{U}_0 < 0$ in a neighborhood of $\hat{t}$ and

$$\varphi(t) \leq \varphi(\hat{t}) + \int_{\hat{t}}^{t} \frac{k_1}{r^{\alpha/2+1}} \left(\varphi^2(\xi) - k_2\tilde{U}_0\right) d\xi < \varphi(\hat{t})$$

for every $t \in (\hat{t}, t_0)$. We deduce the existence of a strictly positive constant $\hat{k}$ such that, for every $t \in (\hat{t}, t_0)$,

$$\varphi(t) - \varphi(\hat{t}) \leq -\hat{k} \int_{\hat{t}}^{t} \frac{d\xi}{r^{\alpha/2+1}}$$

Since the right hand side tends to $-\infty$ as t approaches t_0 (see Corollary 3.8), the last inequality contradicts the boundedness of the function φ. □

Corollary 3.10. *There exist two strictly positive constants* $0 < k_{1,\alpha} \leq k_{2,\alpha}$ *such that*

$$k_{1,\alpha}(t_0 - t)^{\frac{2}{\alpha+2}} \leq r(t) \leq k_{2,\alpha}(t_0 - t)^{\frac{2}{\alpha+2}},$$

whenever $t \in [t_0 - \delta_0, t_0]$.

Proof. The statement follows from Lemmata 3.7 and 3.9 with $k_{i,\alpha} := \left(\frac{\alpha+2}{2} c_{i,\alpha}\right)^{\frac{2}{\alpha+2}}$, $i = 1, 2$. □

Corollary 3.11. *There exists* $b > 0$ *such that*

$$\lim_{t \to t_0^-} \Gamma_\alpha(t) = -b \qquad \textit{and} \qquad \lim_{t \to t_0^-} \dot{r}^2 r^\alpha = 2b.$$

Proof. Since Γ_α is bounded and inequality (3.9) holds, Γ_α admits a limit when t tends to t_0 from the right. We call this limit $-b \in \mathbb{R}$. Since $\Gamma_\alpha(r, s) = h(t)r^\alpha - \frac{1}{2}\dot{r}^2 r^\alpha$, the energy h is bounded and r tends to 0 as $t \to t_0$ we conclude that $\dot{r}^2 r^\alpha$ converges to $2b$ and, using Lemma 3.9 we deduce that $b > 0$. □

Theorem 2. *Let $\bar{x}$ be a generalized solution for the dynamical system* (2.2)*, let $t_0 \in (a, b)$ (if $b < +\infty$ t_0 can coincide with b) be a total collision instant and let $\delta > 0$ be the constant obtained in Theorem 1. Let r, s be the new variables defined in (2.1); if the potential U satisfies assumptions (U0), (U1), (U2)$_h$, (U3)$_h$ then the following assertions hold*

(a) $\lim_{t \to t_0^-} r^\alpha U(t, rs) = b$*, where b is the strictly positive constant introduced in Corollary 3.11;*

(b) there is a positive constant K such that, as t tends to t_0,

$$r(t) \sim [K(t_0 - t)]^{\frac{2}{2+\alpha}}$$
$$\dot{r}(t) \sim -\frac{2K}{2+\alpha}[K(t_0 - t)]^{\frac{-\alpha}{2+\alpha}};$$

(c) $\lim_{t \to t_0^-} |\dot{s}(t)|(t_0 - t) = 0$*;*

(d) for every real positive sequence $(\lambda_n)_n$, such that $\lambda_n \to 0$ as $n \to +\infty$ we have

$$\lim_{n \to +\infty} |s(t_0 - \lambda_n) - s(t_0 - \lambda_n t)| = 0, \quad \forall t > 0.$$

Remark 3.12. Condition *(a)* of Theorem 2 together with assumptions (U3)$_h$ on U and (3.4) on $\tilde{U}$ imply that, if $\bar{x}$ is generalized solution and $|\bar{x}(t_0)| = 0$, then there exist $\delta > 0$ such that, for every $t \in (t_0 - \delta, t_0)$, $\bar{x}(t) \notin \Delta$, i.e., in a (left) neighborhood of the total collision instant no other collision is allowed: neither total nor partial. As a consequence, in such a neighborhood, the generalized solution $\bar{x}$ satisfies the dynamical system (2.2) and the corresponding variables (r, s) verify the Euler–Lagrange equations (2.6).

Proof of Theorem 2. We begin by proving statement *(a)*. The boundedness of the function Γ_α together with inequality (3.9) imply the integrability of the function $r^{\alpha+1}\dot r|\dot s|^2$ on the interval $[t_0-\delta, t_0]$ (see Corollary 3.6). Furthermore, since the integral of $\dot r/r$ on the same interval diverges to $-\infty$ we conclude that

$$\liminf_{t\to t_0^-} r^{\alpha+2}|\dot s|^2 = 0$$

and from (3.8) together with Corollary 3.11

$$\liminf_{t\to t_0^-} r^\alpha U(t, rs) = b. \tag{3.11}$$

It remains to prove that also $\limsup_{t\to t_0^-} r^\alpha U(t,rs) = b$. Suppose, for the sake of contradiction, the existence of a strictly positive ε such that

$$\limsup_{t\to t_0^-} r^\alpha U(t, rs) = b + 3\varepsilon. \tag{3.12}$$

Using assumption $(\mathrm{U3})_{\mathrm{h}}$ we have that (3.11) and (3.12) are respectively equivalent to

$$\liminf_{t\to t_0^-} \tilde U(t,s) = b \quad \text{ and } \quad \limsup_{t\to t_0^-} \tilde U(t,s) = b+3\varepsilon$$

and Corollary 3.11 implies the existence of t_ε such that $\Gamma_\alpha(t) \geq -b-\varepsilon/2$ whenever $t\in(t_\varepsilon, t_0]$. We can then define the set

$$\mathcal{U} := \left\{ t\in(t_\varepsilon, t_0) : \tilde U(t, s(t)) \geq b+\varepsilon \right\}.$$

We define two non-empty subsets of the ellipsoid $\mathcal{E}$ as

$$A := \left\{ s(t) : \tilde U(t,s(t)) \leq b+\varepsilon \right\} \quad \text{ and } \quad B := \left\{ s(t) : \tilde U(t, s(t)) \geq b + 2\varepsilon \right\};$$

since $\varepsilon > 0$ the quantity

$$d := \mathrm{dist}(A,B) = \inf_{s_1\in A, s_2\in B} |s_1 - s_2|$$

is strictly positive and there exists a sequence $(t_n)_{n\geq 0} \subset [t_0-\delta, t_0]$, such that

$$\begin{aligned}
&t_n \to t_0 \text{ as } n\to+\infty \\
&s(t_{2k}) \in \partial A \quad \text{and} \quad s(t_{2k+1}) \in \partial B \quad \text{ for every } k\in\mathbb{N} \\
&b+\varepsilon \leq \tilde U(t, s(t)) \leq b+2\varepsilon, \quad \text{for every } t\in(t_{2k}, t_{2k+1}) \text{ and } k\in\mathbb{N}.
\end{aligned}$$

Hence $(t_{2k}, t_{2k+1}) \subset \mathcal{U}$, for every k, and from the definition of the function Γ_α in (3.8) we have that

$$r^{\alpha+2}|\dot s|^2 \geq \varepsilon \text{ in the intervals } (t_{2k}, t_{2k+1}). \tag{3.13}$$

We now estimate the integral on (t_{2k}, t_{2k+1}) of the integrable (on $[t_0-\delta, t_0]$) function $r^{\alpha+1}\dot r|\dot s|^2$ using (3.13) and Corollary 3.10

$$\begin{aligned}
\int_{t_{2k}}^{t_{2k+1}} -\frac{\dot r}{r} r^{\alpha+2}|\dot s|^2 dt \geq \varepsilon \int_{t_{2k}}^{t_{2k+1}} -\frac{\dot r}{r} dt = \varepsilon \log\frac{r(t_{2k})}{r(t_{2k+1})} \\
\geq \frac{2\varepsilon}{2+\alpha}\log\frac{c_{1,\alpha}(t_0-t_{2k})}{c_{2,\alpha}(t_0-t_{2k+1})}.
\end{aligned} \tag{3.14}$$

On the other hand, using Hölder inequality, we have

$$d^2 \leq |s(t_{2k+1}) - s(t_{2k})| \leq \left(\int_{t_{2k}}^{t_{2k+1}} |\dot s| dt\right)^2 \leq \int_{t_{2k}}^{t_{2k+1}} -r^{\alpha+2}\frac{\dot r}{r}|\dot s|^2 dt \int_{t_{2k}}^{t_{2k+1}} \frac{dt}{-r^{\alpha+1}\dot r} \tag{3.15}$$

and from Lemma 3.7 and Corollary 3.10, we obtain

$$\begin{aligned}\int_{t_{2k}}^{t_{2k+1}} \frac{dt}{-r^{\alpha+1}\dot r} &= \int_{t_{2k}}^{t_{2k+1}} \frac{1}{-r^{\alpha/2}\dot r}\frac{1}{r^{\alpha/2+1}}dt \\ &\leq \frac{2}{2+\alpha}\frac{1}{c_{1,\alpha}^2}\int_{t_{2k}}^{t_{2k+1}}\frac{dt}{t_0-t} = \frac{2}{2+\alpha}\frac{1}{c_{1,\alpha}^2}\log\frac{t_0-t_{2k}}{t_0-t_{2k+1}}.\end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) we obtain

$$\int_{t_{2k}}^{t_{2k+1}} -r^{\alpha+2}\frac{\dot r}{r}|\dot s|^2 dt \geq \frac{2+\alpha}{2}d^2 c_{1,\alpha}^2\left[\log\frac{t_0-t_{2k}}{t_0-t_{2k+1}}\right]^{-1}. \tag{3.17}$$

From the estimates (3.14) and (3.17) we deduce

$$\int_{t_{2k}}^{t_{2k+1}} -r^{\alpha+2}\frac{\dot r}{r}|\dot s|^2 dt \geq \frac{\varepsilon}{2+\alpha}\log\frac{c_{1,\alpha}(t_0-t_{2k})}{c_{2,\alpha}(t_0-t_{2k+1})} + \frac{2+\alpha}{4}d^2c_{1,\alpha}^2\left[\log\frac{t_0-t_{2k}}{t_0-t_{2k+1}}\right]^{-1}.$$

Summing on the index k and recalling that the positive function $-\dot r r^{\alpha+1}|\dot s|^2$ has a finite integral on $[t_0-\delta, t_0]$ (Corollary 3.6) we have

$$\begin{aligned}+\infty > \int_{t_0-\delta}^{t_0} -\dot r r^{\alpha+1}|\dot s|^2 dt &> \sum_{k\geq 0}\int_{t_{2k}}^{t_{2k+1}} -\dot r r^{\alpha+1}|\dot s|^2 dt \\ &\geq \frac{\varepsilon}{2+\alpha}\sum_{k\geq 0}\log\frac{c_{1,\alpha}(t_0-t_{2k})}{c_{2,\alpha}(t_0-t_{2k+1})} + \frac{2+\alpha}{4}d^2c_{1,\alpha}^2\sum_{k\geq 0}\left[\log\frac{t_0-t_{2k}}{t_0-t_{2k+1}}\right]^{-1}.\end{aligned} \tag{3.18}$$

Since $c_{2,\alpha}/c_{1,\alpha}$ is bounded (see Lemma 3.9), for the last term in (3.18) to be finite it is necessary that

$$\lim_{k\to+\infty}\frac{t_0-t_{2k}}{t_0-t_{2k+1}} = \frac{c_{2,\alpha}}{c_{1,\alpha}} \quad \text{and} \quad \lim_{k\to+\infty}\frac{t_0-t_{2k}}{t_0-t_{2k+1}} = +\infty. \tag{3.19}$$

This is a contradiction, hence we conclude that

$$\limsup_{t\to t_0} r^\alpha U(t, rs) = b$$

and, after replacing the value in (3.8),

$$\lim_{t\to t_0} r^{\alpha+2}|\dot s|^2 = 0. \tag{3.20}$$

To prove *(b)*, from Corollary 3.11 we obtain

$$\lim_{t\to t_0^-}\frac{r(t)^{\alpha/2+1}}{(\alpha/2+1)(t_0-t)} = \lim_{t\to t_0^-} -r(t)^{\alpha/2}\dot r(t) = \sqrt{2b};$$

we then conclude by defining $K := \dfrac{2+\alpha}{2}\sqrt{2b}$. The second estimate follows directly.
Part *(c)* directly follows from (3.20) and *(b)*.
We conclude by proving statement *(d)*. If $t = 1$ there is nothing to prove. Suppose $t > 0$, $t \neq 1$, and consider a sequence $(\lambda_n)_n$, $\lambda_n \to 0$; let N be such that $\lambda_n < \delta/\max(1, t)$, $\forall n \geq N$. Whenever $t > 1$, for every $n \geq N$, we have

$$t_0 - \delta < t_0 - \lambda_n t < t_0 - \lambda_n < t_0$$

and

$$\begin{aligned}|s(t_0-\lambda_n) - s(t_0-\lambda_n t)| &\leq \int_{t_0-\lambda_n t}^{t_0-\lambda_n}|\dot s| du \\ &\leq \left(\int_{t_0-\lambda_n t}^{t_0-\lambda_n} r^{1+\alpha/2}|\dot s|^2 du\right)^{1/2}\left(\int_{t_0-\lambda_n t}^{t_0-\lambda_n}\frac{du}{r^{1+\alpha/2}}\right)^{1/2}\end{aligned}$$

It is not restrictive to suppose $t > 1$: indeed, when $t \in (0,1)$, we obtain an equivalent estimate by permuting the integration bounds. From Corollary 3.6 and Lemmata 3.7 and 3.9 we obtain

$$+\infty > \int_{t_0-\delta}^{t_0} r^{1+\alpha}\dot{r}|\dot{s}|^2 du \geq \int_{t_0-\delta}^{t_0} c_{1,\alpha} r^{1+\alpha/2}|\dot{s}|^2 du.$$

Then, since the constant $c_{1,\alpha}$ is strictly positive, we have

$$\lim_{n\to+\infty} \int_{t_0-\lambda_n t}^{t_0-\lambda_n} r^{1+\alpha/2}|\dot{s}|^2 du = 0.$$

Moreover, as n tends to $+\infty$, the second integral $\int_{t_0-\lambda_n t}^{t_0-\lambda_n} r^{-(1+\alpha/2)} < +\infty$; indeed both integration bounds tend to t_0 and the asymptotic estimate proved in *(b)* holds. Hence, as $\lambda_n \to 0$

$$\begin{aligned}\lim_{n\to+\infty} \int_{t_0-\lambda_n t}^{t_0-\lambda_n} \frac{du}{r^{1+\alpha/2}} &= \lim_{n\to+\infty} \left[\int_{t_0-\lambda_n t}^{t_0-\lambda_n} C\frac{du}{(t_0-u)} + o(1)\right] \\ &= C \lim_{n\to+\infty} [\log(\lambda_n) - \log(\lambda_n t) + o(1)] = -C\log t\end{aligned}$$

that is bounded since t is fixed and $C = \left[\frac{\sqrt{2b}(\alpha+2)}{2}\right]^{-(\alpha+2)/2}$. □

Theorem 3. *In the same setting of Theorem 2, assume that the potential U verifies the further assumption*

(U4)$_\mathbf{h}$ $\lim_{r\to 0} r^{\alpha+1}\nabla_T U(t,x) = \nabla_T \tilde{U}(t,s)$.

Then

$$\lim_{t\to t_0} \text{dist}\left(\mathcal{C}^b, s(t)\right) = \lim_{t\to t_0} \inf_{\bar{s}\in\mathcal{C}^b} |s(t)-\bar{s}| = 0,$$

where $\mathcal{C}^b$ is the set of central configurations for $\tilde{U}$ at level b, namely the subset of critical points of the restriction of $\tilde{U}$ to the ellipsoid $\mathcal{E}$:

$$\mathcal{C}^b := \left\{s : \tilde{U}(t_0,s) = b, \nabla_T\tilde{U}(t_0,s) = 0\right\}. \tag{3.21}$$

Remark 3.13. When U is homogeneous, as in the classical keplerian potential, then $\tilde{U}$ is simply the restriction of U on $\mathcal{E}$ and Theorem 3 asserts that the angular component s of the motion tends to a set of central configurations.

Proof. Since in *(a)* of Theorem 2 we have already proved that $\lim_{t\to t_0} \tilde{U}(t,s(t)) = b$, it remains to show that

$$\lim_{t\to t_0^-} |\nabla_T\tilde{U}(t,s(t))| = 0$$

that, using condition (U4)$_\text{h}$, is equivalent to

$$\lim_{t\to t_0^-} r^{\alpha+1}|\nabla_T U(t,rs)| = 0.$$

We now consider the Euler–Lagrange equation $(2.6)_2$ multiplied by r^α

$$-2r^{\alpha+1}\dot{r}\dot{s} - r^{\alpha+2}\ddot{s} + r^{\alpha+1}\nabla_T U(t,rs) = r^{\alpha+2}|\dot{s}|^2 s;$$

since $r^{\alpha+1}\dot{r}\dot{s} = r^{\alpha/2+1}\dot{r}r^{\alpha/2}\dot{s}$ is the product of a bounded term with an infinitesimal one (see equation (3.20) and Lemma 3.7), while $|r^{\alpha+2}|\dot{s}|^2 s| = r^{\alpha+2}|\dot{s}|^2$ tends to 0 for (3.20), we claim that

$$\lim_{t\to t_0^-} r^{\alpha+2}\ddot{s} = 0. \tag{3.22}$$

We perform the time rescaling (cf. McGehee's change of coordinates in 6.1)

$$\tau = \int_{t_0-\delta}^{t} \frac{d\xi}{r^{\alpha/2+1}} \tag{3.23}$$

which maps the interval $[t_0-\delta, t_0)$ into $[0, +\infty)$ (see Corollary 3.8). If the prime $'$ denotes the derivative with respect to the new variable τ, then (3.22) is equivalent to

$$\lim_{\tau\to+\infty} s''(\tau) = 0 \tag{3.24}$$

and the limit (3.20) reads simply

$$\lim_{\tau\to+\infty} |s'(\tau)|^2 = 0. \tag{3.25}$$

Suppose now, for the sake of contradiction, that there exists a sequence $(\tau_n)_n$ such that $\tau_n \to +\infty$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} \nabla_T \tilde{U}(\tau_n, s(\tau_n)) = \lim_{n\to+\infty} s''(\tau_n) = \sigma$$

for some $\sigma \neq 0$. Since the ellipsoid $\mathcal{E}$ is compact, up to subsequences, $(s(\tau_n))_n$ converges to some $\bar{s}$. Furthermore, from Theorem 2 we know that $\tilde{U}(\tau_n, s(\tau_n))$ tends to the finite limit b as $n \to +\infty$, hence $(t_0, \bar{s})$ is a regular point both for $\tilde{U}$ and for $\nabla_T \tilde{U}$. We moreover remark that, since the limit (3.25) holds, for every fixed positive constant $h > 0$, there holds

$$s(\tau) \to \bar{s}, \qquad \text{uniformly on } [\tau_n, \tau_n + h], \text{ for every } n$$

and also

$$\sup_{\tau\in[\tau_n,\tau_n+h]} |\nabla_T \tilde{U}(\tau, s(\tau)) - \sigma| \to 0, \qquad \text{as } n \to +\infty.$$

We can then compute

$$\begin{aligned} s'(\tau_n + h) - s'(\tau_n) &= \int_{\tau_n}^{\tau_n+h} s''(\tau)d\tau \\ &= \int_{\tau_n}^{\tau_n+h} \nabla_T \tilde{U}(\tau, s(\tau))d\tau + o(1) \\ &= h\sigma + o(1) \qquad \text{as } n \to +\infty. \end{aligned}$$

We obtain the contradiction

$$0 = \lim_{n\to+\infty} |s'(\tau_n + h) - s'(\tau_n)| = h|\sigma| \neq 0.$$

□

3.2 Logarithmic potentials

Aim of this section is to extend the asymptotic estimates of Theorem 2 to potentials having logarithmic singularities. We follow the same scheme and we still work in a left neighborhood of a total collision instant t_0, $(t_0 - \delta, t_0)$. The main differences concern the monotonicity formulæ (Lemmata 3.16 and 3.17).

In this setting, we suppose the existence of a continuous function

$$M\colon (a, b) \to \mathbb{R} \quad \text{such that } \dot{M}(t) \text{ is bounded on } (t_0 - \delta, t_0) \tag{3.26}$$

and we replace conditions $(U2)_h$ and $(U3)_h$ with

$(U2)_l$ There exist $\gamma > 0$ and $C_2 \geq 0$ such that

$$\nabla U(t, x) \cdot x + M(t) \geq -C_2|x|^\gamma U(t, x),$$

whenever $|x|$ is small.

(U3)$_{\mathbf{l}}$ $\lim_{|x|\to 0}[U(t,x)+M(t)\log|x|]=\tilde U(t,s)$, uniformly in t,

where $\tilde U$, as in the quasi–homogeneous case, is of class $\mathcal{C}^1$ on $(a,b)\times(\mathcal{E}\smallsetminus\Delta)$ and verifies (3.4).

Remark 3.14. (U2)$_{\mathrm{l}}$ implies (U2) (for small value of $|x|$) for every $\tilde\alpha\in(0,2)$.

Remark 3.15. From Corollary 2.17 and assumption (U3)$_{\mathrm{l}}$ it follows that the positive function $-M(t)\log|x|+\tilde U(t,s)$ is integrable in a neighborhood of a total collision at the origin.

We now prove the analogue of Lemmata 3.4 and 3.5 in the setting of logarithmic–type potentials.

Lemma 3.16. *Let $\bar x$ be a generalized solution, let $t_0\in(a,b]$ be a total collision instant and let δ be given in Theorem 1. Let γ be the positive exponent appearing in (U2)$_h$, then*

$$\int_{t_0-\delta}^{t_0}-r^\gamma\frac{\dot r}{r}U(t,rs)dt<+\infty. \tag{3.27}$$

Proof. We define the functions

$$\Gamma_{log}(r,s):=\frac12 r^2|\dot s|^2-[U(t,rs)+M(t)\log r] \tag{3.28}$$

and

$$\tilde\Gamma_{log}(r,s):=r^\gamma\Gamma_{log}; \tag{3.29}$$

since

$$\tilde\Gamma_{log}(r,s)=r^\gamma\left[h(t)-\frac12\dot r^2-M(t)\log r\right]\le r^\gamma h(t)-r^\gamma M(t)\log r,$$

then $\tilde\Gamma_{log}$ is bounded above, indeed h is bounded, M continuous and, since $\gamma>0$, $\lim_{r\to0}r^\gamma\log r=0$. We now proceed exactly as in the proof of Lemma 3.4: we omit here the approximation argument and we formally compute the time derivative of $\tilde\Gamma_{log}$

$$\frac{d}{dt}\tilde\Gamma_{log}(r,s)=\gamma r^{\gamma-1}\dot r\Gamma_{log}(r,s)+r^\gamma\frac{d}{dt}\Gamma_{log}(r,s).$$

Using the Euler–Lagrange equation $(2.6)_2$, we obtain

$$\frac{d}{dt}\Gamma_{log}(r,s)=-r\dot r|\dot s|^2-\frac{\partial U}{\partial t}(t,rs)-\frac{\dot r}{r}\nabla U(t,rs)\cdot(rs)-\dot M(t)\log r-M(t)\frac{\dot r}{r}.$$

From assumptions (U1) and (U2)$_{\mathrm{l}}$ we deduce that

$$\frac{d}{dt}\Gamma_{log}(r,s)\ge -r\dot r|\dot s|^2-C_1\left(U(t,rs)+1\right)+C_2r^\gamma\frac{\dot r}{r}U(t,rs)-\dot M(t)\log r \tag{3.30}$$

and then

$$\begin{aligned}\frac{d}{dt}\tilde\Gamma_{log}(r,s)\ge{}&-\frac{2-\gamma}{2}r^{\gamma+1}\dot r|\dot s|^2-\gamma r^\gamma\frac{\dot r}{r}U(t,rs)-\gamma r^\gamma\frac{\dot r}{r}M(t)\log r\\&-C_1r^\gamma U(t,rs)-C_1r^\gamma+C_2r^{2\gamma}\frac{\dot r}{r}U(t,rs)-\dot M(t)r^\gamma\log r.\end{aligned} \tag{3.31}$$

The first term in (3.31) is positive, since (3.1) holds; moreover, since r tends to 0 as t approaches t_0, there exist $\varepsilon\in(0,\gamma)$ and $\delta_0\in(0,\delta]$ such that

$$-\gamma r^\gamma\frac{\dot r}{r}U(t,rs)+C_2r^{2\gamma}\frac{\dot r}{r}U(t,rs)\ge-(\gamma-\varepsilon)r^\gamma\frac{\dot r}{r}U(t,rs)\ge0 \tag{3.32}$$

on $(t_0-\delta_0, t_0)$. The remaining terms in (3.31) are integrable functions, indeed the last term $\dot{M}(t)r^\gamma \log r$ is bounded as r tends to 0 (see (3.26)), $r^\gamma U \le U$ and U is integrable and we have the following estimate

$$-\gamma r^\gamma \frac{\dot{r}}{r} M(t)\log r \ge -\gamma r^{\gamma-1}\dot{r}\log r \max_{t\in[t_0-\delta,t_0]} M(t)$$

and

$$\int_{t_0-\delta}^{t_0} \gamma r^{\gamma-1}\dot{r}\log r dt = -r_0^\gamma \log r_0 + \int_{t_0-\delta}^{t_0} r^{\gamma-1}\dot{r} dt = r_0^\gamma\left(-\log r_0 + \frac{1}{\gamma}\right) < +\infty$$

where $r_0 = r(t_0-\delta)$. Hence the right hand side of (3.31) is the sum of an integrable function with a positive one; since the $\tilde{\Gamma}_{log}(r,s)$ is bounded above from (3.32) we have the estimate in (3.27). □

Lemma 3.17 (Monotonicity Formula). *The function Γ_{log} defined in (3.28) is bounded on $[t_0-\delta, t_0]$.*

Proof. We consider the expression of the derivative of Γ_{log} with respect to the time variable computed in (3.30). Using Lemma 3.16, the integrability of the function U and Remark 3.15 we deduce the boundedness below (in a left neighborhood of t_0) of the function Γ_{log} being the right hand side of (3.30) the sum of a positive function with an integrable one.

To prove the boundedness above of Γ_{log} we cannot use the boundedness of the energy function, indeed in this case we can just estimate $\Gamma_{log}(r,s) + M(t)\log r = h(t) - \frac{1}{2}\dot{r}^2$. By the sake of contradiction suppose that Γ_{log} diverges to $+\infty$ as t tends to t_0; since $U(t,rs)+M(t)\log r$ converges uniformly to $\tilde{U}(t,s)$ as t tends to t_0 and $\tilde{U}(t,s)$ is a positive function, if Γ_{log} diverges to $+\infty$

$$\exists t_1 \in (t_0-\delta, t_0) \text{ such that } \forall\, t \in (t_1, t_0), \quad r^2|\dot{s}|^2 > \max_{t\in[t_0-\delta,t_0]} M(t). \tag{3.33}$$

From assumption (3.33) we have

$$\begin{aligned}\int_{t_0-\delta}^{t_0} -\frac{\dot{r}}{r}\left(r^2|\dot{s}|^2 - M(t)\right)dt &= \int_{t_0-\delta}^{t_1} -\frac{\dot{r}}{r}\left(r^2|\dot{s}|^2 - M(t)\right)dt + \int_{t_1}^{t_0} -\frac{\dot{r}}{r}\left(r^2|\dot{s}|^2 - M(t)\right)dt \\ &\ge \text{constant} - \lim_{t\to t_0}\log r(t) = +\infty.\end{aligned} \tag{3.34}$$

We now define the function

$$\Omega_{log}(r,s) := \Gamma_{log}(r,s) + M(t)\log r = h(t) - \frac{1}{2}\dot{r}^2$$

that is bounded above. When we compute its derivative with respect to the time variable we obtain the sum of a positive function with an integrable one (we use assumption (3.33) and Lemma 3.16), indeed

$$\begin{aligned}\frac{d}{dt}\Omega_{log}(r,s) &= \frac{d}{dt}\Gamma_{log}(r,s) + \dot{M}(t)\log r + M(t)\frac{\dot{r}}{r} \\ &\ge -\frac{\dot{r}}{r}\left[r^2|\dot{s}|^2 - M(t)\right] - C_1\left(U(t,rs)+1\right) + C_2 r^\gamma \frac{\dot{r}}{r} U(t,rs).\end{aligned}$$

We can then conclude the boundedness of Ω_{log} on the interval $[t_0-\delta, t_0]$ and from the estimate on its derivative we have

$$\int_{t_0-\delta}^{t_0} -\frac{\dot{r}}{r}\left(r^2|\dot{s}|^2 - M(t)\right)dt < +\infty$$

that contradicts (3.34). We conclude that the function Γ_{log} is also bounded above. □

Corollary 3.18. *As t tends to t_0 the limit of the function Γ_{log} exists finite and*

$$\lim_{t\to t_0^+} -\frac{\dot r^2}{2\log r} = M_0$$

where $M_0 := M(t_0)$.

Proof. We argue as in the proof of Corollary 3.11 to show that the function Γ_{log} has a finite limit as t tends to t_0. Since $\Gamma_{log} = h(t) - \frac{1}{2}\dot r^2 - M(t)\log r$, we conclude dividing by $\log r$ using the boundedness of the function h. □

Theorem 4. *Let $\bar x$ be a generalized solution for the dynamical system (2.2) and let $t_0 \in (a,b)$ (in the case $b < +\infty$, t_0 can coincide with b) be a total collision instant. Let r, s be the new variables defined in (2.1); if the potential U satisfies assumptions (U0), (U1), (U2)$_l$, (U3)$_l$ then the following assertions hold*

(a) $\lim_{t\to t_0^-} [U(t, rs) + M(t)\log r] = -\lim_{t\to t_0^-} \Gamma_{log}(r,s) = b$*;*

(b) as t tends to t_0,

$$r(t) \sim (t_0 - t)\sqrt{-2M_0\log(t_0 - t)}$$
$$\dot r(t) \sim -\sqrt{-2M_0\log(t_0 - t)};$$

(c) $\lim_{t\to t_0^-} |\dot s(t)|(t_0 - t)\sqrt{-2M_0\log(t_0 - t)} = 0$*;*

(d) for every real positive sequence $(\lambda)_n$, such that $\lambda_n \to 0$ as $n \to +\infty$ we have

$$\lim_{n\to+\infty} |s(t_0 - \lambda_n) - s(t_0 - \lambda_n t)| = 0, \quad \forall t > 0.$$

Proof. (a) The proof is essentially the same of for Theorem 2.
(b) From Corollary 3.18 we deduce that

$$\dot r(t) \sim -\sqrt{-2M_0\log r(t)} \quad \text{as } t \text{ tends to } t_0.$$

We define $R(t) := (t_0 - t)\sqrt{-2M_0\log(t_0 - t)}$ and we remark that, as t tends to t_0

$$-\log R(t) = -\log(t_0 - t) - \log\left(\sqrt{-2M_0\log(t_0 - t)}\right) \sim -\log(t_0 - t)$$

and

$$\dot R(t) = -\sqrt{-2M_0\log(t_0 - t)} + \frac{M_0}{\sqrt{-2M_0\log(t_0 - t)}} \sim -\sqrt{-2M_0\log R(t)}.$$

Our aim is then to prove that the function $r(t)$ is asymptotic to $R(t)$ as t tends to t_0. We define the following functions

$$f(\xi) := -\sqrt{-2M_0\log\xi} \quad \text{and} \quad \Phi(\xi) := \int_0^\xi \frac{d\eta}{f(\eta)}, \quad \xi \in (0,1]$$

and we remark that $\Phi(0) = 0$ and Φ is a strictly decreasing function on $[0,1]$. Moreover

$$\dot r(t) \sim f(r(t)), \quad \dot R(t) \sim f(R(t)) \text{ as } t \text{ tends to } t_0$$

or equivalently

$$\lim_{t\to t_0} \frac{d}{dt}\Phi(r(t)) = \lim_{t\to t_0} \frac{d}{dt}\Phi(R(t)) = 1.$$

Since the function $\Phi(\xi)$ decreases in ξ and $r(t)$, $R(t) > 0$ decreases in t (when we stay close to the collision instant) we have that the functions $\Phi(r(t))$ and $\Phi(R(t))$ are negative on $(t_0-\delta_0, t_0)$, vanishes at t_0 (since $r(t_0) = R(t_0) = 0$) and increase in the variable t. Furthermore fixed $\bar{t} < t_0$, the following property holds

$$\begin{aligned} \frac{d}{dt}\Phi(r(t)) \leq 1 \leq \frac{d}{dt}\Phi(R(t)),\ \forall t \in (\bar{t}, t_0) \quad &\Rightarrow \quad \Phi(r(t)) \geq \Phi(R(t)),\ \forall t \in (\bar{t}, t_0) \\ &\Rightarrow \quad r(t) \leq R(t),\ \forall t \in (\bar{t}, t_0). \end{aligned} \tag{3.35}$$

For every $\epsilon > 0$, we consider the functions

$$\begin{aligned} R_\varepsilon^+(t) &:= (1+\varepsilon)R(t), \\ R_\varepsilon^-(t) &:= (1-\varepsilon)R(t). \end{aligned}$$

Since $\dot{R}(t) \sim f(R(t))$, we deduce that in a left neighborhood of t_0

$$\begin{aligned} \dot{R}_\varepsilon^+(t) = (1+\varepsilon)\dot{R}(t) \leq \left(1+\frac{\varepsilon}{2}\right) f(R(t)) \leq \left(1+\frac{\varepsilon}{2}\right) f(R_\varepsilon^+(t)), \\ \dot{R}_\varepsilon^-(t) = (1-\varepsilon)\dot{R}(t) \geq \left(1-\frac{\varepsilon}{2}\right) f(R(t)) \geq \left(1-\frac{\varepsilon}{2}\right) f(R_\varepsilon^-(t)), \end{aligned} \tag{3.36}$$

indeed

$$f(R(t)) = -\sqrt{-2M_0\log(R(t))} \leq -\sqrt{-2M_0\log(1+\varepsilon) - 2M_0\log(R(t))} = f(R_\varepsilon^+(t))$$

and similarly

$$f(R(t)) \geq -\sqrt{-2M_0\log(1-\varepsilon) - 2M_0\log(R(t))} = f(R_\varepsilon^-(t)).$$

From (3.36) we then obtain

$$\frac{d}{dt}\Phi(R_\varepsilon^+(t)) \geq 1 + \frac{\varepsilon}{2} \quad \text{and} \quad \frac{d}{dt}\Phi(R_\varepsilon^-(t)) \leq 1 - \frac{\varepsilon}{2}. \tag{3.37}$$

Moreover, since $\dot{r}(t) \sim f(r(t))$, again in a left neighborhood of t_0 we have that

$$\left(1+\frac{\varepsilon}{2}\right) f(r(t)) \leq \dot{r}(t) \leq \left(1-\frac{\varepsilon}{2}\right) f(r(t)) \tag{3.38}$$

and dividing (3.38) for the negative function $f(r(t))$ and comparing the resulting inequalities with (3.37) we have

$$\frac{d}{dt}\Phi(R_\varepsilon^-(t)) \leq \frac{d}{dt}\Phi(r(t)) \leq \frac{d}{dt}\Phi(R_\varepsilon^+(t)).$$

From (3.35) we deduce that, in a neighborhood of the collision instant t_0, the following chain of inequalities holds

$$(1-\varepsilon) \leq \frac{r(t)}{R(t)} \leq (1+\varepsilon).$$

The second estimate follows directly.
(c) From the result proved in *(a)* we have that $\lim_{t\to t_0} r|\dot{s}| = 0$; we conclude using *(b)*.
(d) As in the proof of Theorem 2, if $t = 1$ there is nothing to prove. We then chose $t > 0$, $t \neq 1$, a sequence $(\lambda_n)_n$, $\lambda_n \to 0$ and N, sufficiently large, such that $\lambda_n < \delta/\max(1,t)$, $\forall n \geq N$. We then obtain

$$\begin{aligned} |s(t_0-\lambda_n) - s(t_0-\lambda_n t)| &\leq \int_{t_0-\lambda_n t}^{t_0-\lambda_n} |\dot{s}(u)|du \\ &\leq \left(\int_{t_0-\lambda_n t}^{t_0-\lambda_n} -r(u)\dot{r}(u)|\dot{s}(u)|^2 du\right)^{\frac{1}{2}} \left(\int_{t_0-\lambda_n t}^{t_0-\lambda_n} -\frac{du}{r(u)\dot{r}(u)}\right)^{\frac{1}{2}}. \end{aligned}$$

The boundedness of the Γ_{log} and the estimate on its derivative in (3.30) imply the boundedness of the integral $\int_0^{t_0} r\dot{r}|\dot{s}|^2$ and then

$$\lim_{n\to+\infty} \int_{t_0-\lambda_n t}^{t_0-\lambda_n} -r(u)\dot{r}(u)|\dot{s}(u)|^2 du = 0.$$

Moreover, as n tends to $+\infty$, from *(b)* and *(c)* we have $r(u)\dot{r}(u) \sim -2M_0(t_0-u)\log(t_0-u)$, hence

$$\lim_{n\to+\infty}\int_{t_0-\lambda_n t}^{t_0-\lambda_n}\frac{du}{r(u)\dot{r}(u)} = \frac{1}{M_0}\lim_{n\to+\infty}\log\frac{\log\lambda_n t}{\log\lambda_n} = 0.$$

The proof is now complete. □

The behavior of the angular part is conserved also for logarithmic potential and the following result can be proved following the proof of Theorem 3.

Theorem 5. *In the same setting of Theorem 4, assuming furthermore that the potential U verifies*

(U4)l $\lim_{r\to 0} r\nabla_T U(t,x) = \nabla_T \tilde{U}(t,s),$

then there holds

$$\lim_{t\to t_0}\operatorname{dist}\left(\mathcal{C}^b, s(t)\right) = \lim_{t\to t_0}\inf_{\bar{s}\in\mathcal{C}^b}|s(t)-\bar{s}| = 0$$

where $\mathcal{C}^b$ is the central configuration subset defined in (3.21).

4 Partial collisions

This section is devoted to the study of the singularities which are not total collision at the origin. At first we shall prove the existence of a limiting configuration for bounded trajectories, that is the Von Zeipel's Theorem (stated on page 6). This fact allows the reduction from partial to total collisions through a change of coordinates. To carry on the analysis we shall extend the clustering argument proposed by McGehee in [37] to prove the Von Zeipel's Theorem. To this aim we need to introduce some further assumptions on the potential U and its singular set Δ. More precisely we suppose that

$$\Delta = \bigcup_{\mu\in\mathcal{M}} V_\mu, \tag{4.1}$$

where the V_μ's are distinct linear subspaces of $\mathbb{R}^k$ and $\mathcal{M}$ is a finite set; observe that the set Δ is a cone, as required on page 4. We endow the family of the V_μ's with the inclusion partial ordering and we assume the family to be closed with respect to intersection (thus we are assuming that $\mathcal{M}$ is a semilattice of linear subspaces of $\mathbb{R}^k$: it is the intersection semilattice generated by the arrangement of maximal subspaces V_μ's). With each $\xi\in\Delta$ we associate

$$\mu(\xi) = \min\{\mu \;:\; \xi\in V_\mu\} \qquad \text{i.e.,} \qquad V_{\mu(\xi)} = \bigcap_{\xi\in V_\mu} V_\mu.$$

Fixed $\mu\in\mathcal{M}$ we define the set of collision configurations satisfying

$$\Delta_\mu = \{\xi\in\Delta : \mu(\xi)=\mu\}$$

and we observe that this is an open subset of V_μ and its closure $\overline{\Delta_\mu}$ is V_μ. We also notice that the map $\xi\to\dim(V_{\mu(\xi)})$ is lower semicontinuous.

We denote by p_μ the orthogonal projection onto V_μ and we write

$$x = p_\mu(x) + w_\mu(x),$$

where, of course, $w_\mu = \mathbb{I} - p_\mu$.

We assume that, near the collision set, the potential depends, roughly, only on the projection orthogonal to the collision set: more precisely we assume

(U5) For every $\xi\in\Delta$, there is $\varepsilon>0$ such that

$$U(t,x) - U(t, w_{\mu(\xi)}(x)) = W(t,x) \in \mathcal{C}^1\left((a,b)\times B_\varepsilon(\xi)\right),$$

where $B_\varepsilon(\xi) = \{x \;:\; |x-\xi|<\varepsilon\}$.

Theorem 6. *Let $\bar{x}$ be a generalized solution for the dynamical system* (2.2) *on the bounded interval* (a, b). *Suppose that the potential U satisfies assumptions (U0), (U1), (U5), and $(U2)_h$, $(U3)_h$, $(U4)_h$ (or $(U2)_l$, $(U3)_l$, $(U4)_l$).*
If $\bar{x}$ is bounded on the whole interval (a, b) then

(a) $\bar{x}$ has a finite number of singularities which are collisions (the Von Zeipel's Theorem holds).

(b) Furthermore, if $t^* \in \bar{x}^{-1}(\Delta)$ is a collision instant, x^* the limit configuration of $\bar{x}$ as t tends to t^* and $\mu^* = \mu(x^*) \in \mathcal{M}$, then $r_{\mu^*} = |w_{\mu^*}(\bar{x})|$, $s_{\mu^*} = w_{\mu^*}(\bar{x})/r_{\mu^*}$ and $U_{\mu^*} = U(t, w_{\mu^*}(\bar{x}))$ satisfy the asymptotic estimates given in Theorems 2 and 3 (or Theorems 4 and 5 when $(U2)_l$, $(U3)_l$ and $(U4)_l$ hold).

Proof. Let $\bar{x}$ be a generalized solution with a singularity at $t = t^*$ (see Definition 2.6) and Δ^* its ω-limit set, that is

$$\Delta^* = \{x^* : \exists (t_n)_n \text{ such that } t_n \to t^* \text{ and } \bar{x}(t_n) \to x^*\}.$$

It is well known that the ω-limit of a bounded trajectory is a compact and connected set. From the Painlevé's Theorem (on page 5) we have the inclusion

$$\Delta^* \subset \Delta.$$

Von Zeipel's Theorem asserts that whenever $\bar{x}$ remains bounded as t approaches t^*, then the ω-limit set of $\bar{x}$ contains just one element, that is $\Delta^* = \{x^*\}$.

In view of Corollary 2.17, where we proved the Theorem in the case $\liminf_{t \to t^*} \dot{I}(\bar{x}(t)) < +\infty$, we are left with the case when

$$\lim_{t \to t^*} \dot{I}(\bar{x}(t)) = +\infty.$$

From this and our assumptions it follows that $I(\bar{x}(t))$ is a definitely increasing and bounded function. Hence it admits a limit

$$\lim_{t \to t^*} I(\bar{x}(t)) = I^*. \tag{4.2}$$

We perform the proof of Von Zeipel's Theorem in two steps.

Step 1. We suppose that $\mu(\Delta^*) = \{\mu^*\}$ for some $\mu^* \in \mathcal{M}$ and we show that $\Delta^* = \{x^*\}$.

As Δ^* is a compact and connected subset of V_{μ^*}, we have the following inclusions

$$\Delta^* \subset \Delta_{\mu^*} \subset V_{\mu^*}.$$

We consider the orthogonal projections

$$p(t) = p_{\mu^*}(\bar{x}(t)), \qquad w(t) = w_{\mu^*}(\bar{x}(t)).$$

Since we have assumed that $\mu(\Delta^*) = \{\mu^*\}$, then

$$\lim_{t \to t^*} w(t) = 0, \tag{4.3}$$

our aim is now to prove that

$$\lim_{t \to t^*} p(t) = x^*.$$

Projecting on V_{μ^*} the equations of motion, we obtain from (U5)

$$-\ddot{p} = p_{\mu^*}\left(\nabla U(t, \bar{x}(t))\right) = p_{\mu^*}\left(\nabla W(t, \bar{x}(t))\right) \tag{4.4}$$

where ∇W is globally bounded as $t \to t^*$. Indeed, fixed $\varepsilon > 0$, there exists $\delta > 0$ such that $\bar{x}(t) \in B_\varepsilon(\Delta^*)$ whenever $t \in (t^* - \delta, t^*)$, and from assumption (U5) and the compactness of $\Delta^* \subset \Delta_{\mu^*}$ we deduce the boundedness of the right hand side of (4.4). From this fact we easily deduce the existence of a limit for $(p(t))$ as t tends to t^*. A word of caution must be entered at this point. As $\bar{x}$ is a generalized solution to (2.2), the equation of motions are not available, because of the possible occurence of collisions, and therefore they can not be projected on V_{μ^*}. Nevertheless, exploiting the regularization method exposed in Section 2 and projecting the regularized equations, one can easily obtain the validity of (4.4) after passing to the limit.

Step 2. There always exists $\mu^* \in \mathcal{M}$ such that $\mu(\Delta^*) = \{\mu^*\}$.

Let μ^* be the element of $\mu(\Delta^*)$ associated with the subspace V_{μ^*} having *minimal dimension.* Since the function $\xi \to \dim(V_{\mu(\xi)})$ is lower semicontinuous, the minimality of the dimension has as a main implication that $\Delta_{\mu^*} \cap \Delta^*$ is compact. Hence the function ∇W appearing in (U5) can be though to be globally bounded in a neighborhood of $\Delta_{\mu^*} \cap \Delta^*$. In other words, when considering the orthogonal projections $p(t) = p_{\mu^*}(\bar{x}(t))$ and $w(t) = w_{\mu^*}(\bar{x}(t))$, as a major consequence of the minimality of the dimension μ^* we find the following implication:

$$\exists M > 0,\ \exists \varepsilon > 0 : |w(t)|^2 < \varepsilon \implies |p_{\mu^*}(\nabla W(t, \bar{x}))| \leq M. \tag{4.5}$$

We now compute the second derivative (with respect to the time t) of the function $|p(t)|^2$

$$\frac{d^2}{dt^2}|p(t)|^2 = 2\ddot{p}(t)\cdot p(t) + 2\dot{p}(t)\cdot\dot{p}(t) \geq -2p_{\mu^*}(\nabla W(t, \bar{x}(t)))\cdot p(t)$$

Thus, from the projected motion equation (4.4) and from (4.5) we infer

$$\exists K > 0,\ \exists \varepsilon > 0 : |w(t)|^2 < \varepsilon \implies \frac{d^2}{dt^2}|p(t)|^2 \geq -K. \tag{4.6}$$

We now argue by contradiction, supposing that $\mu(\Delta^*) \neq \{\mu^*\}$. Then

$$0 = \liminf_{t\to t^*} |w(t)|^2 < \limsup_{t\to t^*} |w(t)|^2. \tag{4.7}$$

Since, obviously, the total moment of inertia splits as

$$I(\bar{x}(t)) = |p(t)|^2 + |w(t)|^2,$$

from (4.7) and (4.2) we deduce that

$$I^* = \limsup_{t\to t^*} |p(t)|^2 > \liminf_{t\to t^*} |p(t)|^2 \tag{4.8}$$

and from (4.8) together with (4.6) we have

$$\exists K > 0,\ \exists \varepsilon > 0\ :\ |p(t)|^2 \geq I^* - \varepsilon \implies \frac{d^2}{dt^2}|p(t)|^2 \geq -K.$$

Let $(t_n^0)_n$ and $(t_n^*)_n$ be two sequences such that, fixed $\varepsilon > 0$

$$\begin{aligned}
&t_n^* < t_n^0 < t_{n+1}^* \quad \forall n \\
&t_n^0 \to t^* \quad t_n^* \to t^* \text{ as } n \to +\infty \\
&f(t_n^*) \to I^* \text{ as } n \to +\infty \text{ and } f'(t_n^*) = 0, \quad \forall n \\
&t_n^0 = \inf\{t > t_n^* : |p(t)|^2 \leq I^* - \varepsilon\}, \quad \forall n.
\end{aligned}$$

Hence $|p(t_n^0)|^2 - |p(t_n^*)|^2 = \dfrac{d}{dt^2}|p(\xi)|^2(t_n^0 - t_n^*)^2/2 \geq -K(t_n^0 - t_n^*)^2/2$ and then

$$-\varepsilon \geq \frac{-K}{2}(t_n^0 - t_n^*)^2 \quad \text{or} \quad (t_n^0 - t_n^*)^2 \geq \frac{2\varepsilon}{K}$$

in contradiction with the assumptions that both sequences $(t_n^0)_n$ and $(t_n^*)_n$ tend to the finite limit t^*. This concludes the proof of the Von Zeipel's Theorem. Next we prove isolatedness of collision instants.

To this aim, let us select $t^* \in \partial\left(\bar{x}^{-1}(\Delta)\right)$ a collision instant such that the dimension of $V_{\mu(\bar{x}(t^*))}$ is minimal among all dimensions of collision configurations $V_{\mu(\bar{x}(t))}$ in $(t^*-\delta, t^*+\delta)$ for some $\delta > 0$. As before, let us split the components of the trajectory $\bar{x}(t) = p(t) + w(t)$ on V_{μ^*} and its orthogonal complement.

Since μ^* is minimal (see (4.5)), we already know from the previous discussion that the equations of motion projected on the subspace V_{μ^*} (equation (4.4)) are not singular; on the other hand, by (U5), the trajectories in the orthogonal coordinates w are generalized solutions to a dynamical system of the form

$$-\ddot{w} = \nabla U(t, w) + \nabla W(t, p(t) + w). \tag{4.9}$$

Now, since $w(t)$ has a total collisions at the origin at t^*, we can apply the results of Section 3. More precisely, at first we deduce from Theorem 1 that t^* is isolated in the set of collisions Δ_{μ^*}; furthermore from Corollary 2.17 we deduce the boundedness of the action and the energy. Finally we conclude applying Theorems 2, 3 (or Theorems 4, 5 when $(U2)_1$, $(U3)_1$ and $(U4)_1$ hold) to the projection w. In particular from (a) in Theorem 2 (or Theorem 4) we obtain that every collision is isolated and hence, whenever the interval (a, b) is finite, the existence of a finite number of collisions. □

5 Absence of collisions for locally minimal path

As a matter of fact, solutions to the Newtonian n–body problem which are minimals for the action are, very likely, free of any collision. This was discovered in [48] for a class of periodic three–body problems and, since then, widely exploited in the literature concerning the variational approach to the periodic n–body problem. In general, the proof goes by the sake of the contradiction and involves the construction of a suitable variation that lowers the action in presence of a collision. A recent breakthrough in this direction is due of the neat idea, due to C. Marchal in [34], of averaging over a family of variations parameterized on a sphere. The method of averaged variations for Newtonian potentials has been developed and exposed in [10], and then extended to α–homogeneous potentials and various constrained minimization problems in [27]. This argument can be used in most of the known cases to prove that minimizing trajectories are collisionless. In this section we prove the absence of collisions for locally minimal solutions when the potentials have quasi–homogeneous or logarithmic singularities.

We consider separately the quasi–homogeneous and the logarithmic cases; indeed in the first case one can exploit the blow–up technique as developed in Section 7 of [27]; in § 5.1 we will just recall the main steps of this arguments. On the other hand, when dealing with logarithmic potentials, the blow–up technique is no longer available and we conclude proving directly some averaging estimates that can be used to show the nonminimality of large classes of colliding motions.

5.1 Quasi–homogeneous potentials

Let $\tilde{U}$ be the $\mathcal{C}^1$ function defined on $(a, b) \times (\mathcal{E} \setminus \Delta)$ introduced on page 14; we extend its definition on the whole $(a, b) \times (\mathbb{R}^k \setminus \Delta)$ in the following way

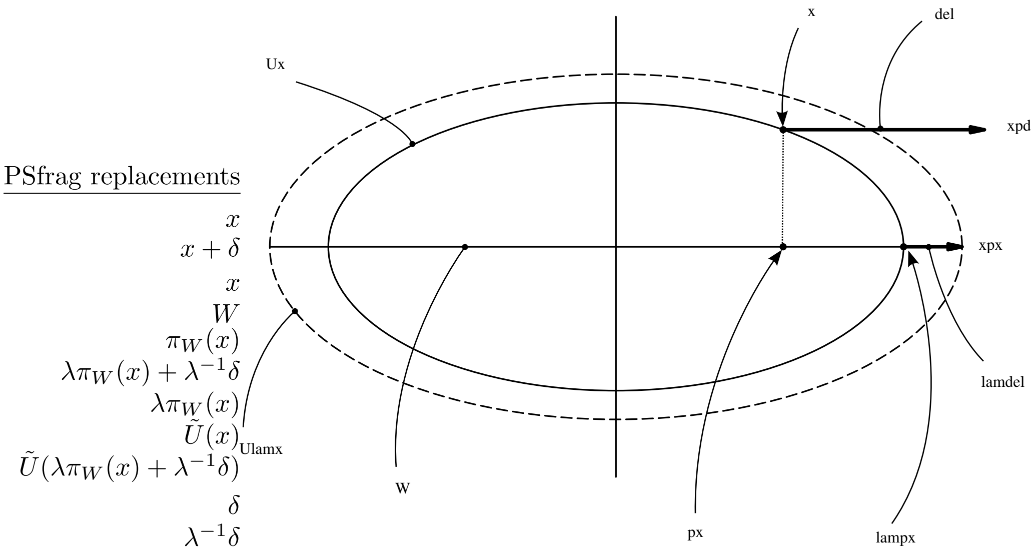
$$\tilde{U}(t, x) = |x|^{-\alpha} \tilde{U}(t, x/|x|).$$

Fixed t^* (in this section we will consider a locally minimal trajectory $\bar{x}$ with a collision at t^*) in this section, with an abuse of notation, we denote

$$\tilde{U}(x) = \tilde{U}(t^*, x). \tag{5.1}$$

Of course, the function $\tilde{U}$ is homogeneous of degree $-\alpha$ on $\mathbb{R}^k \setminus \Delta$.

Figure 1: Potential levels, with $\lambda = \left(\dfrac{\tilde U(\pi_W(x))}{\tilde U(x)} \right)^{1/\alpha} > 1$



Theorem 7. *In addition to (U0), (U1), (U2)$_h$, (U3)$_h$, (U4)$_h$, (U5), assume that, for a given $\xi \in \Delta$*

(U6) *there is a 2–dimensional linear subspace of $V^\perp_{\mu(\xi)}$, say W, where $\tilde U$ is rotationally invariant;*

(U7)$_\mathbf{h}$ *for every $x \in \mathbb{R}^k$ and $\delta \in W$ there holds*

$$\tilde U(x+\delta) \le \tilde U \left(\left(\frac{\tilde U(\pi_W(x))}{\tilde U(x)} \right)^{1/\alpha} \pi_W(x) + \left(\frac{\tilde U(x)}{\tilde U(\pi_W(x))} \right)^{1/\alpha} \delta \right)$$

where π_W denotes the orthogonal projection onto W.

Then generalized solutions do not have collisions at the configuration ξ at the time t^*.

Remark 5.1. Some comments on assumptions (U6) and (U7)$_\mathrm{h}$ are in order. Of course, as our potential $\tilde U$ is homogeneous of degree $-\alpha$ the function

$$\varphi(x) = \tilde U^{-1/\alpha}(x)$$

is a non negative, homogeneous of degree one function, having now Δ as zero set. In most of our applications φ will be indeed a quadratic form. Assume that φ^2 splits in the following way:

$$\varphi^2(x) = K|\pi_W(x)|^2 + \varphi^2(\pi_{W^\perp}(x))$$

for some positive constant K. Then (U6) and (U7)$_\mathrm{h}$ are satisfied. Indeed, denoting $w = \pi_W(x)$ and $z = x - w$ we have, for every $\delta \in W$,

$$\begin{aligned}
\varphi^2(x+\delta) &= K|w+\delta|^2 + \varphi^2(z) \\
&= K\left| \frac{\varphi(x)}{\varphi(w)} w + \frac{\varphi(w)}{\varphi(x)} \delta \right|^2 + K\frac{\varphi^2(z)}{\varphi^2(x)}|\delta|^2 \\
&\ge K\left| \frac{\varphi(x)}{\varphi(w)} w + \frac{\varphi(w)}{\varphi(x)} \delta \right|^2 \\
&= \varphi^2\left(\frac{\varphi(x)}{\varphi(w)} w + \frac{\varphi(w)}{\varphi(x)} \delta \right),
\end{aligned}$$

which is obviously equivalent to $(\mathrm{U7})_{\mathrm{h}}$. Therefore, we have the following Proposition.

Proposition 5.2. *Assume $\tilde{U}(x) = \mathcal{Q}^{-\alpha/2}(x)$ for some non negative quadratic form $\mathcal{Q}(x) = \langle Ax, x\rangle$. Then assumptions (U6) and $(U7)_h$ are satisfied whenever W is included in an eigenspace of A associated with a multiple eigenvalue.*

Remark 5.3. Given two potentials satisfying (U6) and $(\mathrm{U7})_{\mathrm{h}}$ for a common subspace W, their sum enjoys the same properties. On the other hand, if they do not admit a common subspace W, their sum does not satisfy (U6) and $(\mathrm{U7})_{\mathrm{h}}$.

Proof or Theorem 7. Let $\bar{x}(t)$ be a generalized solution with a collision at the time t^*, i.e. $\bar{x}(t^*) = \xi \in \Delta$; up to time-translation we assume that the collision instant is $t^* = 0$. Furthermore, using the same arguments needed in the proof of the Von Zeipel's Theorem in Section 4, we can suppose that $\xi = 0$. We consider the case of a boundary collision (interior collisions can be treated in a similar way). Then Theorem 2 ensures the existence of $\delta_0 > 0$ such that no other collision occurs in some interval $[0, \delta_0]$.

We consider the family of rescaled generalized solutions

$$\bar{x}^{\lambda_n}(t) := \lambda_n^{-\frac{2}{2+\alpha}} \bar{x}(\lambda_n t), \quad t \in [0, \delta_0/\lambda_n].$$

where $\lambda_n \to 0$ as $n \to +\infty$. From the asymptotic estimates of Theorem 3 we know that the angular part $(s(\lambda_n))_n$ converges, up to subsequences, to some central configuration $\bar{s}$, in particular $\bar{s}$ is in the ω–limit of $s(t)$.

For any $\bar{s}$ in the ω–limit of $s(t)$, a (right) blow-up of $\bar{x}$ in $t = 0$ is a path defined, for $t \in [0, +\infty)$, as

$$\bar{q}(t) := \zeta t^{\frac{2}{2+\alpha}}, \qquad \zeta = K\bar{s}, \tag{5.2}$$

where the constant $K > 0$ is determined by part *(b)* of Theorem 2. We note that the blow–up is a homothetic solution to the dynamical system associated with the homogeneous potential $\tilde{U}$ and that it has zero energy (the blow–up is parabolic). If $s(\lambda_n) \to \bar{s}$ as $n \to +\infty$, from Theorem 2, we obtain straightforwardly the pointwise convergence of $\bar{x}^{\lambda_n}$ to the blow up $\bar{q}$ and the H^1-boundedness of $\bar{x}^{\lambda_n}$ implies its uniform convergence on compact subsets of $[0, +\infty)$. Furthermore the convergence holds locally in the $H^1([0, +\infty))$–topology. Finally also the sequence $\dot{\bar{x}}^{\lambda_n}$ converges uniformly on every interval $[\varepsilon, T]$, with arbitrary $0 < \varepsilon < T$.

The following fact has been proven in [27], Proposition 7.9.

Lemma 5.4. *Let $\bar{x}$ be a locally minimizing trajectory with a total collision at $t = 0$ and let $\bar{q}$ be its blow–up in $t = 0$. Then $\bar{q}$ is a locally minimizing trajectory for the dynamical system associated with the homogeneous potential $\tilde{U}$ introduced in* (5.1).

We will conclude the proof showing that $\bar{q}$ cannot be a locally minimizing trajectory for the dynamical system associated with $\tilde{U}$. Following [27], we now introduce a class of suitable variations as follows:

Definition 5.5. The standard variation associated with $\delta \in \mathbb{R}^k \setminus \{0\}$ is defined as

$$v^\delta(t) = \begin{cases} \delta & \text{if } 0 \le |t| \le T - |\delta| \\ (T-t)\frac{\delta}{|\delta|} & \text{if } T - |\delta| \le |t| \le T \\ 0 & \text{if } |t| \ge T, \end{cases}$$

for some positive T.

We wish to estimate the action differential corresponding to a standard variation. To this aim we give the next definition.

Definition 5.6. The displacement potential differential associated with $\delta \in \mathbb{R}^k$ is defined as:

$$S(\zeta, \delta) = \int_0^{+\infty} \left(\tilde{U}(\zeta t^{2/(2+\alpha)} + \delta) - \tilde{U}(\zeta t^{2/(2+\alpha)}) \right) dt$$

where $\bar{q}(t) = \zeta t^{2/(2+\alpha)}$ is a blow-up of $\bar{x}$ in $t = 0$.

The quantity $S(\zeta,\delta)$ represents the *potential differential* needed for displacing the colliding trajectory originarily traveling along the ζ–direction to the point δ. It has been proven in [27] Proposition 9.2, that the function S represents the limiting behavior, as $\delta \to 0$, of the whole action differential:

$$\Delta\mathcal{A}^\delta := \int_{-\infty}^{+\infty} \left[K(\dot{\bar{q}} + \dot{v}^\delta) + \tilde{U}(\bar{q} + v^\delta) - K(\dot{\bar{q}}) - \tilde{U}(\bar{q})\right] dt.$$

Indeed, the fundamental estimate holds:

Lemma 5.7. *Let $\bar{q} = \zeta t^{2/(2+\alpha)}$ be a blow–up trajectory and v^δ any standard variation. Then, as $\delta \to 0$*

$$\Delta\mathcal{A}^\delta = |\delta|^{1-\alpha/2} S\left(\zeta, \frac{\delta}{|\delta|}\right) + O(|\delta|).$$

We observe that, from the homogeneity of $\tilde{U}$ it follows that

$$S(\lambda\xi, \mu\delta) = |\lambda|^{-1-\alpha/2} |\mu|^{1-\alpha/2} S(\xi,\delta) \tag{5.3}$$

(see [27, (8.2)]) and hence, if $\tilde{U}$ is invariant under rotations, the sign of S depends only on the angle between ξ and δ. To deal with the isotropic case (which is not the case here), the following function was introduced in [27]:

$$\Phi_\alpha(\vartheta) = \int_0^{+\infty} \frac{1}{\left(t^{\frac{4}{\alpha+2}} - 2\cos\vartheta t^{\frac{2}{\alpha+2}} + 1\right)^{\alpha/2}} - \frac{1}{t^{\frac{2\alpha}{\alpha+2}}} dt.$$

The value of $\Phi_\alpha(\vartheta)$ ranges from positive to negative values, depending on ϑ and α. Nevertheless, it is always negative, when averaged on a circle. Indeed, the following inequality was obtained in [27, Theorem 8.4].

Lemma 5.8. *For any $\alpha \in (0,2)$ there holds*

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi_\alpha(\vartheta) d\vartheta < 0.$$

This inequality will be a key tool in proving the following averaged estimate:

Lemma 5.9. *Assume (U6) and (U7)$_h$, then, if $\mathbb{S}$ is the unitary circle of W, for any $\zeta \in \mathbb{R}^k \setminus \{0\}$ the following inequality holds*

$$\int_{\mathbb{S}} S(\zeta,\delta) d\delta < 0 .$$

As a consequence,

$$\forall \zeta \in \mathbb{R}^k \setminus \{0\}\ \exists \delta = \delta(\zeta) \in \mathbb{S}\ :\ S(\zeta, \delta(\zeta)) < 0 .$$

Proof. As a first obvious application of Lemma 5.8 we obtain the assertion *for any* $\zeta \in W \setminus \{0\}$. Indeed, by (5.3) and (U6) we easily obtain

$$\zeta \in W\setminus\{0\} \quad \Longrightarrow \quad S(\zeta,\delta) = K\,|\zeta|^{-1-\alpha/2}\,\Phi_\alpha(\vartheta),$$

where K is a positive constant and ϑ denotes the angle between ζ and δ.

Now we prove the assertion for any $\zeta \neq 0$ in the configuration space. It follows from the homogeneity of $\tilde{U}$ that

$$\tilde{U}\left(\left(\frac{\tilde{U}(\pi_W(\zeta))}{\tilde{U}(\zeta)}\right)^{1/\alpha} \pi_W(\zeta)\right) = \tilde{U}(\zeta).$$

Hence $(U7)_h$ implies, for every $\delta \in \mathbb{S}$,

$$S(\zeta,\delta) \leq S\left(\left(\frac{\tilde{U}(\pi_W(\zeta))}{\tilde{U}(\zeta)}\right)^{1/\alpha} \pi_W(\zeta), \left(\frac{\tilde{U}(\zeta)}{\tilde{U}(\pi_W(\zeta))}\right)^{1/\alpha} \delta\right).$$

Hence (5.3) implies

$$S(\zeta,\delta) \leq \left(\frac{\tilde{U}(\zeta)}{\tilde{U}(\pi_W(\zeta))}\right)^{2/\alpha} S(\pi_W(\zeta),\delta),$$

and thus

$$\int_{\mathbb{S}} S(\zeta,\delta)\, d\delta \leq \left(\frac{\tilde{U}(\zeta)}{\tilde{U}(\pi_W(\zeta))}\right)^{2/\alpha} \int_{\mathbb{S}} S(\pi_W(\zeta,\delta))\, d\delta < 0 .$$

□

End of the Proof of Theorem 7. To conclude the proof, according with Lemma 5.9 we chose $\delta = \delta(\zeta) \in W \setminus \{0\}$ with the property that $S(\zeta, \delta(\zeta)/|\delta(\zeta)|) < 0$. As a consequence of Lemma 5.7, we can lower the value of the action of $\bar{q}$ by performing the standard variation $v^{\delta(\zeta)}$, provided the norm of $|\delta(\zeta)|$ is sufficiently small (in order to apply Lemma 5.7). Hence $\bar{q}$ can not be locally minimizing for the action. □

As we have already noticed, the class of potentials satisfying (U6) and $(U7)_h$ is not stable with respect to the sum of potentials. In order to deal with a class of potentials which is closed with respect to the sum, we introduce the following variant of Theorem 7.

Theorem 8. *In addition to (U0), (U1), $(U2)_h$, $(U3)_h$, $(U4)_h$, (U5), assume that $\tilde{U}$ has the form*

$$\tilde{U}(x) = \sum_{\nu=1}^{N} \frac{K_\nu}{(\mathrm{dist}(x, V_\nu))^\alpha}$$

where K_ν are positive constants and V_ν is a family of linear subspaces, with $\mathrm{codim}(V_\nu) \geq 2$*, for every $\nu = 1, \dots, N$. Then locally minimizing trajectories do not have collisions at the time t^*.*

Proof. Following the arguments of the proof of Theorem 7, the assertion will be proved once we show, as in Lemma 5.9, that, for every index ν, there holds

$$\int_{\mathbb{S}^{k-1}} S_\nu(\zeta,\delta)\, d\delta < 0,$$

where, of course, we denote

$$S_\nu(\zeta,\delta) = \int_0^{+\infty} \left(\mathrm{dist}(\zeta t^{2/(2+\alpha)} + \delta, V_\nu)^{-\alpha} - \mathrm{dist}(\zeta t^{2/(2+\alpha)}, V_\nu)^{-\alpha}\right) dt$$

and $\mathbb{S}^{k-1}$ is the unit sphere of the configuration space $\mathbb{R}^k$. This is an elementary consequence of Lemma 5.9 and the fact that the function $S_\nu(\zeta,\delta)$ only depends on the projection of ζ orthogonal to V_μ and has rotational invariance on $V_\nu^\perp$. Thus the integral of S_ν over the sphere is a positive multiple of its integral on any circle $\mathbb{S}$ orthogonal to V_ν. □

5.2 Logarithmic type potentials

In this section we prove the equivalent to Theorems 7 and 8 suitable for logarithmic type potentials. Concerning the quasi-homogeneous case we have seen that a crucial role is played by the construction of a blow–up function which minimizes a limiting problem. Before starting, let us highlight the reasons why, when dealing with logarithmic potentials, a blow-up limit can not exist. Indeed, the natural scaling should be $\bar{x}^{\lambda_n}(t) := \lambda_n^{-1}\bar{x}(\lambda_n t)$, which does not converge, since

$$\lim_{\lambda_n\to 0} \bar{x}^{\lambda_n}(t) = \lim_{\lambda_n\to 0} \frac{r(\lambda_n t)s(\lambda_n t)}{\lambda_n t\sqrt{-2M(0)\log(\lambda_n t)}} t\sqrt{-2M(0)\log(\lambda_n t)} = +\infty$$

for every $t > 0$. On the other, hand, looking at 5.2, the (right) blow–up should be, up to a change of time scale,

$$\bar{q}(t) := t\bar{s}, \quad i \in \mathbf{k}, \tag{5.4}$$

where $\bar{s}$ is a central configuration for the system limit of a sequence $s(\lambda_n)$ where $(\lambda_n)_n$ is such that $\lambda_n \to 0$. The blow up function defined in (5.4) is the pointwise limit of the normalized sequence

$$\bar{x}^{\lambda_n}(t) := \frac{1}{\lambda_n\sqrt{-2M(0)\log\lambda_n}}\bar{x}(\lambda_n t).$$

Unfortunately the path in (5.4) is not locally minimal for the limiting problem, indeed since, the sequence $(\ddot{\bar{x}}^{\lambda_n})_n$ converges to 0 as n tends to $+\infty$, the blow-up in (5.4) minimizes only the kinetic part of the action functional.

We shall overcome this difficulty by proving the averaged estimate in a direct way from the asymptotic estimates of Theorem 4 and assuming (5.6) on the potential U. As we have done for the quasi–homogeneous case, we extend the function $\tilde{U}$, introduced in assumption $(U3)_{\mathrm{l}}$, to the whole $(a, b) \times \mathbb{R}^k \smallsetminus \Delta$ in the natural way

$$\tilde{U}(t, x) = \tilde{U}(t, s) - M(t)\log|x|, \tag{5.5}$$

where M has been introduced in (3.26).

Theorem 9. *In addition to (U0), (U1), $(U2)_l$, $(U3)_l$, $(U4)_l$, (U5) assume the potential U to be of the form*

$$U(t, x) = \tilde{U}(t, x) + W(t, x) \tag{5.6}$$

where $\tilde{U}$ satisfies (5.5) *and W is a bounded $\mathcal{C}^1$ function on $(a, b) \times \mathbb{R}^k$. Furthermore assume that, for a given $\xi \in \Delta$, $\tilde{U}$ satisfies (U6) and*

$(\mathbf{U7})_{\mathbf{l}}$ *for every $x \in \mathbb{R}^k$ and $t \in (a, b)$ there holds*

$$\tilde{U}(t, x) = -\frac{1}{2}M(t)\log\left(|\pi_W x|^2 + \psi^2(\pi_{W^\perp}x)\right)$$

where π_W and $\pi_{W^\perp}$ denote the orthogonal projections onto W and $W^\perp$, ψ is $\mathcal{C}^1$ and homogeneous of degree 1.

Then locally minimizing trajectories do not have collisions at the configuration ξ at the time t^*.

Proof. As in the proof of Theorem 7, we consider a generalized solution $\bar{x}$ and we first reduce to the case of an isolated total collision at the origin occurring at the time $t = 0$. From Theorem 4 we deduce the existence of $\delta_0 > 0$ such that no other collision occur in $[-\delta_0, \delta_0]$, hence we perform a local variation on the trajectory of $\bar{x}$ that removes the collision and makes the action decrease.

Consider now the standard variation v^δ, defined at page 33, on the interval $[0, \delta_0]$ (i.e., in Definition 5.5 T is replaced by δ_0). Let $\Delta^\delta\mathcal{A}$ denote the difference

$$\Delta^\delta\mathcal{A} := \mathcal{A}(\bar{x} + v^\delta, [0, \delta_0]) - \mathcal{A}(\bar{x}, [0, \delta_0]);$$

generally speaking, this difference can be positive or negative, depending on the choice of δ. Our goal is to prove that, when *averaging over a suitable set of standard variations*, the action lowers. Hence $\Delta^\delta\mathcal{A}$ must be negative for at least one choise of δ and the path $\bar{x}$ can not be a local minimizer for the action.

We can write $\Delta^\delta \mathcal{A}$ as the sum of three terms

$$\Delta^\delta \mathcal{A} = \int_0^{\delta_0} \Delta^\delta \mathcal{K}(t)\, dt + \int_0^{\delta_0} \Delta^\delta \mathcal{U}(t)\, dt + \int_0^{\delta_0} \Delta^\delta \mathcal{W}(t)\, dt \tag{5.7}$$

where $\Delta^\delta \mathcal{K}(t)$, $\Delta^\delta \mathcal{U}(t)$ and $\Delta^\delta \mathcal{W}(t)$ are respectively the variations of the kinetic energy, of the singular potential $\tilde{U}$ and of the smooth part of the potential, W. More precisely since the first derivative of the function v^δ vanishes everywhere on $[0, \delta_0]$, except on $[\delta_0 - |\delta|, \delta_0]$, we compute

$$\Delta^\delta \mathcal{K}(t) := \begin{cases} 0, & \text{if } t \in [0, \delta_0 - |\delta|], \\ \dfrac{1}{2}(|\dot{\bar{x}} - \delta/|\delta||^2 - |\dot{\bar{x}}|^2) & \text{if } t \in [\delta_0 - |\delta|, \delta_0]. \end{cases} \tag{5.8}$$

Similarly

$$\Delta^\delta \mathcal{U}(t) := \tilde{U}(t, \bar{x} + v^\delta) - \tilde{U}(t, \bar{x}) \qquad \text{and} \qquad \Delta^\delta \mathcal{W}(t) := W(t, \bar{x} + v^\delta) - W(t, \bar{x}).$$

We now evaluate separately the mean values of the tree terms of $\Delta^\delta \mathcal{A}$ over the circle $S^{|\delta|}$ of radius $|\delta|$ in W.

Lemma 5.10. *There holds*

$$\frac{1}{2\pi|\delta|} \int_{S^{|\delta|}} \int_0^{\delta_0} (\Delta^\delta \mathcal{K} + \Delta^\delta \mathcal{W})\, dt\, d\delta = O(|\delta|). \tag{5.9}$$

Proof. From (5.8) we obtain

$$\int_0^{\delta_0} \Delta^\delta \mathcal{K}(t) dt = \int_{\delta_0 - |\delta|}^{\delta_0} \frac{1}{2}(|\dot{\bar{x}} - \delta/|\delta||^2 - |\dot{\bar{x}}|^2) dt = \frac{1}{2}\left(|\delta| - 2\int_{\delta_0 - |\delta|}^{\delta_0} \dot{\bar{x}}(t) \cdot \frac{\delta}{|\delta|} dt\right),$$

hence

$$\left|\int_0^{\delta_0} \Delta^\delta \mathcal{K}(t) dt\right| \leq O(|\delta|),$$

which does not depend on the circle $S^{|\delta|}$ where δ varies. Concerning the variation of the $\mathcal{C}^1$ function W we have

$$\begin{aligned} \left|\int_0^{\delta_0} \Delta^\delta \mathcal{W}(t)\, dt\right| &= \left|\int_0^{\delta_0 - |\delta|} \Delta^\delta \mathcal{W}(t)\, dt\right| + \left|\int_{\delta_0 - |\delta|}^{\delta_0} \Delta^\delta \mathcal{W}(t)\right| dt \\ &\leq W_1 |\delta|(\delta_0 - |\delta|) + 2W_2|\delta| = O(|\delta|), \end{aligned}$$

where W_1 is a bound for $\left|\frac{\partial W}{\partial x}(t, \bar{x} + \lambda v^\delta)\right|$, with $\lambda \in [0, 1]$ and $t \in [0, \delta_0 - |\delta|]$ while W_2 is an upper bound for $|W(t, x)|$. □

In order to estimates the variation of the potential part, $\Delta^\delta \mathcal{U}(t)$, we prove the next two technical lemmata. Let us start with recalling an equivalent version of the mean value property for the fundamental solution of the planar Laplace equation.

Lemma 5.11. *Fixed $z > 0$, for every $y \in \mathbb{R}$ such that $y \geq 2z$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log(y + 2z \cos \vartheta)\, d\vartheta = \log \frac{y + \sqrt{y^2 - 4z^2}}{2}.$$

Proof. Since $y \geq 2z$, then $\frac{y + \sqrt{y^2 - 4z^2}}{2} \geq z$. Let $x \in \mathbb{R}^2$ be such that $|x| = \frac{y + \sqrt{y^2 - 4z^2}}{2}$, then $y = (|x|^2 + z^2)/|x|$ and for every $\delta \in S^z$, where S^z is the circle of radius z, we have

$$\begin{aligned} |x + \delta|^2 &= |x|^2 + z^2 + 2z|x| \cos \vartheta \\ &= |x| \left(\frac{|x|^2 + z^2}{|x|} + 2z \cos \vartheta\right) = |x|(y + 2z \cos \vartheta). \end{aligned}$$

We have, as the logarithm is the fundamental solution to the Laplace equation on the plane,

$$\frac{1}{2\pi z}\int_{S^z} \log|x+\delta|^2\, d\delta = \max\{\log|x|^2, \log z^2\} = \begin{cases} \log|x|^2, & \text{if } |x| > z \\ \log z^2, & \text{if } |x| \le z. \end{cases} \quad . \tag{5.10}$$

Consequently, when computing

$$\begin{aligned}\int_{S^z} \log|x+\delta|^2 d\delta &= \int_{S^z} \log|x| d\delta + z\int_0^{2\pi} \log(y+2z\cos\vartheta)d\vartheta \\ &= 2\pi z\log|x| + z\int_0^{2\pi}\log(y+2z\cos\vartheta)d\vartheta,\end{aligned}$$

we find

$$2\pi z\log|x|^2 = 2\pi z\log|x| + z\int_0^{2\pi}\log(y+2z\cos\vartheta)d\vartheta.$$

We conclude replacing $|x| = \frac{y+\sqrt{y^2-4z^2}}{2}$. □

Now we consider the averages of the potential with respect to a circle in W (here we assume implicitly that $d \ge 3$).

Lemma 5.12. *Fixed $|\delta| > 0$, for every circle of radius $|\delta|$, $S^{|\delta|} \subset W$, for every $x \in \mathbb{R}^d$ and every $t \in [0, \delta_0]$, there holds*

$$\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}} \left(\tilde{U}(x+\delta) - \tilde{U}(x)\right) d\delta \le \begin{cases} 0 & (\textit{ if } |\pi_W x|^2 + \psi^2(\pi_{W^\perp}x) > |\delta|^2), \\ \frac{M(t)}{2}\log(|\pi_W x|^2 + \psi^2(\pi_{W^\perp}x)) - \log(|\delta|^2) & (\textit{otherwise}). \end{cases}$$

Proof. We consider the orthogonal decomposition of x, $x = \pi_W x + \pi_{W^\perp}x$, and we term $u := |\pi_W x|$ and $\varepsilon := \psi(\pi_{W^\perp}x)$. Since whenever $\delta \in W$ we have

$$|\pi_W(x+\delta)|^2 + \psi^2(\pi_{W^\perp}x) = u^2 + |\delta|^2 + 2u|\delta|\cos\vartheta + \varepsilon^2 \ge 0,$$

when $\cos\vartheta = -1$ we have $\frac{u^2+|\delta|^2+\varepsilon^2}{u|\delta|} \ge 2$ and, using Lemma 5.11 and equation (5.10), we compute

$$\begin{aligned}&\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}} \log(|\pi_W(x+\delta)|^2 + \psi^2(\pi_{W^\perp}x))\, d\delta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log(u^2+\varepsilon^2+|\delta|^2+2u|\delta|\cos\vartheta)\, d\vartheta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\log\left(\frac{u^2+\varepsilon^2+|\delta|^2}{u|\delta|} + 2\cos\vartheta\right) d\vartheta + \log(u|\delta|) \\ &= \log\left(\frac{u^2+\varepsilon^2+|\delta|^2+\sqrt{(u^2+\varepsilon^2+|\delta|^2)^2-4u^2|\delta|^2}}{2}\right) \\ &\ge \log\left(\frac{u^2+\varepsilon^2+|\delta|^2+\sqrt{(u^2+\varepsilon^2+|\delta|^2)^2-4u^2|\delta|^2-4\varepsilon^2|\delta|^2}}{2}\right) \\ &= \log\left(\frac{u^2+\varepsilon^2+|\delta|^2+|u^2+\varepsilon^2-|\delta|^2|}{2}\right) \\ &= \max\left(\log(|\pi_W x|^2+\psi^2(\pi_{W^\perp}x), \log(|\delta|^2)\right)\end{aligned}$$

and the assertion easily follows. □

Lemma 5.13. *Let S be the circle of radius $|\delta|$ on W; then, as $|\delta| \to 0$*

$$\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\int_0^{\delta_0} \Delta^\delta \mathcal{U}\, dt\, d\delta < -K|\delta|\sqrt{-\log|\delta|}, \qquad K > 0. \tag{5.11}$$

Proof. Let $S^{|\delta|}$ be the circle of radius $|\delta|$ on W, we apply Fubini-Tonelli's Theorem and we argue as in the proof of Lemma 5.12 to have

$$\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\int_0^{\delta_0}\Delta^\delta\mathcal{U}(t)dt\,d\delta = \int_0^{\delta_0}\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\tilde{U}(\bar{x}+v^\delta)-\tilde{U}(\bar{x})\,d\delta dt$$
$$= \frac{M^*}{2}\int_0^{\delta_0}\left\{-\max\left[\log(|\pi_W\bar{x}|^2+\psi^2(\pi_{W^\perp}\bar{x})),\log|v^\delta|^2\right]+\log(|\pi_W\bar{x}|^2+\psi^2(\pi_{W^\perp}\bar{x}))\right\}dt$$

where $M^* = \max_t|M(t)|$. We then straightforwardly deduce that, for every $S^{|\delta|}\subset W$

$$\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\int_0^{\delta_0}\Delta^\delta\mathcal{U}(t)dt\,d\delta<0.$$

In order to estimate more precisely this quantity, we observe that

$$\int_0^{\delta_0}\frac{1}{2\pi|v^\delta|}\int_{S^{|v^\delta|}}\tilde{U}(\bar{x}+v^\delta)-\tilde{U}(\bar{x})\,d\delta dt \le \int_A \log\frac{|\pi_W\bar{x}|^2+\psi^2(\pi_{W^\perp}\bar{x})}{|\delta|^2}\,dt \tag{5.12}$$

where

$$A := \left\{t\in[0,\delta_0-|\delta|] : |\pi_W\bar{x}|^2+\psi^2(\pi_{W^\perp}\bar{x})<|\delta|^2\right\}.$$

Furthermore, there exists a strictly positive constant C such that

$$Cr^2<|\pi_W x|^2+\psi^2(\pi_{W^\perp}x)<C^{-1}r^2$$

where, as usual, we denote $r^2=|\pi_W x|^2+|\pi_{W^\perp}x|^2$ the radius of x. The left inequality follows from Theorem 4 indeed the existence of a finite limit of $\tilde{U}(t,s(t))$ prevents the projection $|\pi_W x|^2$ and the function $\psi^2(\pi_{W^\perp}x)$ to be both infinitesimal with r^2. The right inequality follows from the continuity of ψ. From (5.12) and the asymptotic estimates of Theorem 4 we conclude that, as $|\delta|\to 0$

$$\begin{aligned}\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\int_0^{\delta_0}\Delta^\delta\mathcal{U}(t)dt\,d\delta &\le \int_{t:r(t)<|\delta|/\sqrt{C}}\log\frac{r^2(t)}{C|\delta|^2}\,dt\\ &\sim \int_0^{|\delta|/\sqrt{C}} 2\frac{\log(r/\sqrt{C}|\delta|)}{-\sqrt{-\log r}}dr\\ &< -2\int_0^{|\delta|/\sqrt{C}}\sqrt{-\log r}dr < -K|\delta|\sqrt{-\log|\delta|}\end{aligned}$$

for some positive K, since $-\sqrt{-\log r}$ is an increasing function on the interval $[0,|\delta|]$. □

End of the Proof of Theorem 7. Let $S^{|\delta|}$ be a circle in W with radius $|\delta|$ and $\Delta^\delta\mathcal{A}$ the variation of the action functional defined in (5.7), then from Lemmata 5.10 and 5.13 we conclude that, as $|\delta|$ tends to 0

$$\frac{1}{2\pi|\delta|}\int_{S^{|\delta|}}\Delta^\delta\mathcal{A}d\delta \le O(|\delta|)-K|\delta|\sqrt{-\log|\delta|}<0.$$

□

Of course, likewise to Theorem 8, there holds

Theorem 10. *In addition to (U0), (U1), (U2)$_l$, (U3)$_l$, (U4)$_l$, (U5), assume $\tilde{U}$ be of the form*

$$\tilde{U}(x) = -\sum_{\nu=1}^N K_\nu\log(\mathrm{dist}(x,V_\nu))$$

where K_ν are positive constants and V_ν is a family of linear subspaces, with $\mathrm{codim}(V_\nu)\ge 2$, *for every $\nu=1,\dots,N$. Then locally minimizing trajectories do not have collisions at the time t^*.*

5.3 Neumann boundary conditions and G–equivariant minimizers

As a final comment of this Section, we remark that, in our framework, the analysis allows to prove that minimizers to the fixed–ends (Bolza) problems are free of collisions: indeed all the variations of our class have compact support. However, other type of boundary conditions (generalized Neumann) can be treated in the same way. Indeed, consider a trajectory which is a (local) minimizer of the action among all paths satisfying the boundary conditions

$$x(0) \in X^0 \qquad x(T) \in X^1,$$

where X^0 and X^1 are two given linear subspaces of the configuration space. Consider a (locally) minimizing path $\bar{x}$: of course it has not interior collisions. In order to exclude boundary collisions we have to be sure that the class of variations preserve the boundary condition; this can be achieved by restricting to X^i the points δ appearing in the standard variations. Hence, to complete the averaging argument, one needs assumptions (U6) and $(\mathrm{U7})_{\mathrm{h}}$ or $(\mathrm{U7})_{\mathrm{l}}$ to be fulfilled also by the restriction of the potential to the boundary subspaces X^i. This point of view differs from that of [8], where the boundary subspaces can not be chosen arbitrarily in the configuration space. The argument in [8], already introduced in [27], does not involve any averaging on the boundary but relies upon a suitable choice of a standard variation whose projection is extremal.

The analysis of boundary conditions was a key point in the paper [27], where symmetric periodic trajectories where constructed by reflections about given subspaces. By Theorems 7 and 9 one can obtain the absence of collisions also for G–equivariant (local) minimizers, provided the group G satisfies the *Rotating Circle Property* introduced in [27] (see Example 6). Hence, existence of G–equivariant collisionless periodic solutions can be proved for the wide class of symmetry groups described in [27, 26, 4], for a much larger class of interacting potentials, including quasi–homogeneous and logarithmic ones. On the other hand, Theorems 8 and 10 can be applied to prove that G–equivariant minimals are collisionless for many relevant symmetry groups violating the rotating circle property, such as the groups of rotations in [25]; indeed, the idea of averaging on spheres having maximal dimension has been borrowed from that paper (cf. Example 7).

6 Examples and further remarks

We now discuss various examples of classes of potentials which fullfill our assumptions.

Example 1 (Homogeneous isotropic potentials)**.** The simplest example of function satisfying all our assumptions (U0), (U1), $(\mathrm{U2})_{\mathrm{h}}$, $(\mathrm{U3})_{\mathrm{h}}$, $(\mathrm{U4})_{\mathrm{h}}$, (U5), (U6) and $(\mathrm{U7})_{\mathrm{h}}$ is the α-homogeneous one-center problem:

$$U_\alpha(x) = \frac{1}{|x|^\alpha},$$

and its associated n–body problem:

$$U_\alpha(x) = \sum_{\substack{i<j \\ i,j=1}}^{n} \frac{m_i m_j}{|x_i - x_j|^\alpha}.$$

Assumptions (U0) and (U1) are trivially satisfied since U is positive, diverges to $+\infty$ when x approaches $\Delta = \{x \in \mathbb{R}^{nd} : x_i = x_j \text{ for some } i \neq j\}$, and does not depend on time. Furthermore in both (U2) and $(\mathrm{U2})_{\mathrm{h}}$ the equality is achieved with $\tilde{\alpha} = \alpha$ and $C_2 = 0$. Since U is homogeneous of degree $-\alpha$, in $(\mathrm{U3})_{\mathrm{h}}$ and $(\mathrm{U4})_{\mathrm{h}}$ the function $\tilde{U}$ coincides with U. (U5) and (U6) are trivially satisfied, while $(\mathrm{U7})_{\mathrm{h}}$ holds by virtue of Proposition 5.2

Example 2 (Logarithmic potentials)**.** Our results apply also to logarithmic singularities of type

$$U_{log}(x) = \sum_{\substack{i<j \\ i,j=1}}^{n} m_i m_j \log \frac{1}{|x_i - x_j|};$$

indeed (U2) is in this case satisfied for every value of $\tilde{\alpha}$ and $(\mathrm{U2})_{\mathrm{l}}$, $(\mathrm{U3})_{\mathrm{l}}$ and $(\mathrm{U4})_{\mathrm{l}}$ are verified with $C_2 = 0$.

Dynamical systems of type (2.2) with logarithmic interactions arise in the study of vortex flows in fluid mechanics, and, precisely, in the analysis of systems of n almost–parallel vortex filaments, under a linearized version of the LIA self–interaction assumption (see [30, 31]).

Example 3 (Anisotropic n–body potentials)**.** Consider potentials having the form

$$U(t,x) = \sum_{\substack{i<j \\ i,j=1}}^{n} U_{i,j}(t, x_i - x_j),$$

where the interaction potentials $U_{i,j}$ have a singularity at zero, of homogeneous or logarithmic type, but do depend on the angle. Typical examples are the Gutzwiller potentials [29]. Notice that the total potential satisfies assumptions (U0), (U1), and $(\mathrm{U2})_{\mathrm{h}}$, $(\mathrm{U3})_{\mathrm{h}}$, $(\mathrm{U4})_{\mathrm{h}}$ (or $(\mathrm{U2})_{\mathrm{l}}$, $(\mathrm{U3})_{\mathrm{l}}$, $(\mathrm{U4})_{\mathrm{l}}$) provided each of the $U_{i,j}$'s do. It not difficult to see that also (4.1) and (U5) hold (in the n–body case), while (U6) and $(\mathrm{U7})_{\mathrm{h}}$ or $(\mathrm{U7})_{\mathrm{l}}$ do not. Hence we can not exclude the presence of collisions for locally minimizing paths, though the results about isolatedness and the asymptotic estimates are still available. More generally, we can deal with potentials of the form

$$U_\alpha(rs) = r^{-\alpha}\tilde{U}(s),$$

where $\tilde{U} : \mathcal{E} \setminus \Delta \to \mathbb{R}$ is positive and admits an arbitrary singular set on the ellipsoid $\mathcal{E} = \{I = 1\}$, provided

$$\lim_{s \to \Delta} \tilde{U}(s) = +\infty\ .$$

It is worthwhile noticing that as a consequence of Theorem 2, a total collision trajectory will not interact, definitively, with the singularities of $\tilde{U}$.

The class of potentials satisfying our assumptions is clearly stable with respect to the addition of arbitrary perturbations of class $\mathcal{C}^1$. Therefore, we are mainly interested in the analysis of those perturbations which are singular themselves.

Example 4 (N–body potentials with time–varying masses)**.** Although the potentials in the previous examples do not depend on time, our assumptions allow an effective time–dependence of the potentials. For instance, we can choose positive and bounded $\mathcal{C}^1$ functions $m_i(t)$, $i = 1, \ldots, n$.

Obviously, the simplest example is the class of α-homogeneous n-body problem

$$U_\alpha(t,x) = \sum_{\substack{i<j \\ i,j=1}}^{n} \frac{m_i(t)m_j(t)}{|x_i - x_j|^\alpha}, \qquad 0 < \alpha < 2.$$

Assumptions (U0) and (U1) are trivially satisfied since U is positive, diverges to $+\infty$ when x approaches $\Delta = \{x \in \mathbb{R}^{nd} : x_i = x_j \text{ for some } i \neq j\}$, and does not depend on time. Furthermore in both (U2) and $(\mathrm{U2})_{\mathrm{h}}$ the equality is achieved with $\tilde{\alpha} = \alpha$ and $C_2 = 0$. Since U is homogeneous of degree $-\alpha$, in $(\mathrm{U3})_{\mathrm{h}}$ and $(\mathrm{U4})_{\mathrm{h}}$ the function $\tilde{U}$ coincides with U.

Example 5 (Quasi–homogeneous potentials)**.** We can also handle homogeneous perturbations of degree $-\beta$ of the potential U_α

$$U(x) = U_\alpha(x) + \lambda U_\beta(x) \qquad\qquad 0 < \beta < \alpha < 2.$$

Indeed, when $\lambda > 0$ condition $(\mathrm{U2})_{\mathrm{h}}$ is verified (with the strict inequality) with $\gamma = C_2 = 0$, while, when $\lambda < 0$, then (U2) holds, when $|x|$ is sufficiently small, with $C_2 = \alpha - \beta$ and $0 < \gamma < \alpha - \beta$.

As pointed out in [19] (where the case $\beta = 1$ and $\alpha > 1$ was treated), quasi–homogeneous potentials generalize classical potentials such as Newton, Coulomb, Birkhoff, Manev and many others. Therefore, the range of physical applications of quasi–homogeneous potentials spans from celestial mechanics and atomic physics to chemistry and crystallography. It is worthwhile noticing that the collision problem for quasi–homogeneous potentials exhibit an interesting and peculiar lack of regularity. Indeed, a classical framework for the study of collisions is given by the McGehee coordinates [36] (here and below we assume, for simplicity of notations, all the masses be equal to one):

$$\begin{aligned} r &= |x| = I^{1/2} \\ s &= \frac{x}{r} \\ v &= r^{\alpha/2}(y \cdot s) \\ u &= r^{\alpha/2}(y - y \cdot ss). \end{aligned}$$

After a reparametrization of the time–variable (see 3.23):

$$d\tau = r^{-1-\alpha/2}dt, \tag{6.1}$$

the equation of motions become (here $'$ denotes differentiation with respect to the new time variable τ):

$$\begin{aligned} r' &= rv \\ v' &= \frac{\alpha}{2}v^2 + |u|^2 - r^{\alpha-\beta}\lambda U_\beta(s) - \alpha U_\alpha(s) \\ s' &= u \\ u' &= \left(\frac{\alpha}{2} - 1\right) vu - |u|^2 s + r^{\alpha-\beta}\lambda \left(U_\beta(s)s - \nabla U_\beta(s)\right) + \alpha U_\alpha(s)s + \nabla U_\alpha(s)\ . \end{aligned}$$

The field depends on r in a non smooth manner, unless $\alpha - \beta \geq 1$ (this last condition was indeed assumed in [19]). Hence the flow *can not be continuously extended* to the total collision manifold $C = \{(r, s, v, u) \ : \ r = 0,\ \frac{1}{2}(|u|^2 + v^2) - 2U_\alpha = 0\}$. Another peculiar feature of this system is that the monotonicity of the variable v can not be ensured close to the collision manifold. As a consequence, the usual analysis of collision and near collision motions can not be extended to this case.

Example 6 (N–body potential reduced by a symmetry group satisfying the rotating circle property)**.** The paper [51] deals with minimal trajectories to the spatial $2N$–body problem under the *hip–hop symmetry*, where the configuration is constrained at all time to form a regular antiprism. This problem has three degrees of freedom and the reduced potential of a configuration generated by the point of coordinates $(u, \zeta) \in \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$ decomposes as

$$U(u, \zeta) = \frac{K(N)}{|u|^\alpha} + U_0(u, \zeta),$$

where

$$\begin{aligned} K(N) &= \sum_{k=1}^{N-1} \frac{1}{\sin^\alpha(\frac{k\pi}{N})}, \\ U_0(u, \zeta) &= \sum_{k=1}^{N} \frac{1}{\left(\sin^2\left(\frac{(2k-1)\pi}{2N}\right)|u|^2 + \zeta^2\right)^{\frac{\alpha}{2}}}, \end{aligned}$$

The first term comes from the interaction among points of the same N–agon and is singular at simultaneous partial collisions on the ζ–axis. The second term, $U_0(u, \zeta)$, comes from the interaction between the the upper and lower N–agons and is singular only at the origin. One easily verifies that all the assumptions are satisfied, including, again by Proposition 5.2, (U6) and $(\mathrm{U7})_\mathrm{h}$. In general, one easily verifies that, for a given symmetry group G of the N–body problem, it is equivalent to say that $\ker \tau$ has the rotating circle property and that the reduced potential verifies (U6) and $(\mathrm{U7})_\mathrm{h}$.

Example 7 (N–body potential reduced by a symmetry group not satisfying the rotating circle property)**.** Consider the symmetry groups generated by rotations introduced in [25]: the configuration is, at all time, an orbit of a group Y of rotations about given lines in the 3–dimensional space. When Y is a finite group, the reduced potential takes the form required in Theorem 8 and minimizers can be shown to be free of collision.

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